Mathematics 2011 (Outside Delhi) Set I

Time allowed : 3 Hours **Maximum marks : 100**

SECTION – A

1. Let A = {1, 2, 3}, B = {4, 5, 6, 7} and let f = {(1, 4), (2, 5), (3, 6)} be a function from A to B. State whether f is one-one or not. **[1]**

Solution : Given,

$$A = \{1, 2, 3\},$$
$$B = \{4, 5, 6, 7\}$$
$$f = \{(1, 4), (2, 5), (3, 6)\}$$

$f : A \to B$ is defined as

$\therefore\ f(1) = 4,\quad f(2) = 5,\ f(3) = 6.$

Different points of the domain have different f-image in the range.

$\therefore\ f$ is one-one. **Ans.**

2. What is the principal value of

$$\cos^{-1}\left(\cos\frac{2\pi}{3}\right) + \sin^{-1}\left(\sin\frac{2\pi}{3}\right)?$$ **[1]**

Solution : Given,

$$\cos^{-1}\left(\cos\frac{2\pi}{3}\right) + \sin^{-1}\left(\sin\frac{2\pi}{3}\right)$$

We know that principal value branch of $\cos^{-1}$ is $[0, \pi]$ and $\sin^{-1}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\therefore$ The principal value of

$$\cos^{-1}\left[\cos\frac{2\pi}{3}\right] + \sin^{-1}\left[\sin\frac{2\pi}{3}\right]$$
$$= \frac{2\pi}{3} + \sin^{-1}\left[\sin\left(\pi - \frac{2\pi}{3}\right)\right]$$
$$= \frac{2\pi}{3} + \sin^{-1}\left(\sin\frac{\pi}{3}\right)$$
$$= \frac{2\pi}{3} + \frac{\pi}{3} = \pi.$$ **Ans.**

3. Evaluate : $\begin{vmatrix} \cos 15^\circ & \sin 15^\circ \\ \sin 75^\circ & \cos 75^\circ \end{vmatrix}$. **[1]**

Solution : $\begin{vmatrix} \cos 15^\circ & \sin 15^\circ \\ \sin 75^\circ & \cos 75^\circ \end{vmatrix}$

$$= \cos 75^\circ.\cos 15^\circ - \sin 75^\circ.\sin 15^\circ$$
$$= \cos(75^\circ + 15^\circ)$$

$[\because \cos(A + B) = \cos A \cos B - \sin A \sin B]$

$$= \cos 90^\circ$$
$$= 0.$$ **Ans.**

4. If A =, $\begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$ **write A^{-1} in terms of A.** **[1]**

Solution : Here, $|A| = 2(-2) - 5(3)$

$$= -4 - 15 = -19 \neq 0$$

$\therefore A^{-1}$ exists

and $\text{adj } A = \begin{bmatrix} -2 & -3 \\ -5 & 2 \end{bmatrix}$

$$\therefore \quad A^{-1} = \frac{1}{|A|}.\text{adj } A$$
$$= \frac{-1}{19}\begin{bmatrix} -2 & -3 \\ -5 & 2 \end{bmatrix}$$
$$= \frac{1}{19}\begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix} = \frac{1}{19}A.$$ **Ans.**

5. If a matrix has 5 elements, write all possible orders it can have. **[1]**

Solution : Since a matrix of order $m \times n$ has mn elements therefore, to find all possible orders of a matrix with 5 elements, we have to fill all possible ordered pairs (m, n) of positive integers whose product is 5. Hence possible orders are 1×5 and 5×1. **Ans.**

6. Evaluate : $\int (ax + b)^3\, dx$ **[1]**

Solution : Let, $I = \int (ax + b)^3 .dx$

Let $t = ax + b$

Differentiating w.r.t. x, we get

$$\frac{dt}{dx} = a - 0$$
$$\Rightarrow \quad dx = \frac{dt}{a}$$
$$I = \int t^3 . \frac{dt}{a}$$
$$\therefore \quad I = \frac{1}{a}\int t^3 . dt$$
$$= \int t^3 . dt \frac{1}{a}$$
$$= \frac{1}{a}\left(\frac{t^4}{4}\right) + C$$
$$= \frac{1}{4a}t^4 + C$$
$$= \frac{1}{4a}(ax + b)^4 + C$$ **Ans.**

7. Evaluate : $\int \frac{dx}{\sqrt{1 - x^2}}$. **[1]**

Solution : Let $I = \int \frac{dx}{\sqrt{1 - x^2}}$.

$$\left[\because \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)+C\right]$$
$$= \sin^{-1} x + C. \qquad \textbf{Ans.}$$

8. Write the direction-cosines of the line joining the points (1, 0, 0) and (0, 1, 1). **[1]**

Solution : The d.r's of line joining points (1, 0, 0) and (0, 1, 1) are $0-1, 1-0, 1-0$

i.e. $-1, 1, 1$

$$\therefore \text{ D.C's are } \frac{-1}{\sqrt{1^2+1^2+1^2}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$$
$$= \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}. \qquad \textbf{Ans.}$$

9. Write the projection of the vector $\hat{i}-\hat{j}$ on the vector $\hat{i}+\hat{j}$. **[1]**

Solution : Let $\vec{a} = \hat{i}-\hat{j}$ and $\vec{b} = \hat{i}+\hat{j}$.

Now, projection of $\vec{a}$ on $\vec{b}$

$$= \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \frac{(\hat{i}-\hat{j}).(\hat{i}+\hat{j})}{\sqrt{1^2+1^2}} = \frac{1-1}{\sqrt{2}} = \frac{0}{\sqrt{2}} = 0. \qquad \textbf{Ans.}$$

10. Write the vector equation of a line given by $\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-6}{2}$. **[1]**

Solution : The given line is

$$\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2} \qquad ...(i)$$

The equation of line (i) comparing with

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$$

We have $x_1 = 5, y_1 = -4, z_1 = 6$ and $a = 3, b = 7, c = 2$

Fixed point vector,

$$\vec{a} = x_1\hat{i}+y_1\hat{j}+z_1\hat{k} = 5\hat{i}-4\hat{j}+6\hat{k}$$

Direction vector,

$$\vec{b} = a\hat{i}+b\hat{j}+c\hat{k} = 3\hat{i}+7\hat{j}+2\hat{k}$$

$\therefore$ Vector equation of the given line is

$$\vec{r} = \vec{a} + \lambda\vec{b}$$
$$\vec{r} = (5\hat{i}-4\hat{j}+6\hat{k}) + \lambda(3\hat{i}+7\hat{j}+2\hat{k}).$$

Ans.

SECTION — B

11. Let $f : R \to R$ be defined as $f(x) = 10x + 7$. Find the function $g : R \to R$ such that $gof = fog = I_R$. **[4]**

Solution : It is given that $f : R \to R$ is defined as

$$f(x) = 10x + 7$$

One-one

Let $f(x) = f(y)$, where $x, y \in R$

$\Rightarrow 10x + 7 = 10y + 7$

$\Rightarrow x = y$

$\therefore f$ is a one-one function.

Onto :

For $y \in R$, let $y = 10x + 7$

$$\therefore \quad x = \frac{y-7}{10} \in R$$

Therefore, for any $y \in R$, there exists $x = \frac{y-7}{10} \in R$ such that

$$f(x) = f\left(\frac{y-7}{10}\right) = 10\left(\frac{y-7}{10}\right)+7 = y-7+7 = y$$

$\therefore f$ is onto.

Therefore, f is one-one and onto.

Thus, f is an invertible function.

Let us define $f : R \to R$ as $g(y) = \frac{y-7}{10}$

Now we have,

$$gof(x) = g(f(x)) = g(10x+7)$$
$$= \frac{10x+7-7}{10} = x$$
$$fog(y) = f(g(y))$$
$$\Rightarrow \quad f\left(\frac{y-7}{10}\right) = 10\left(\frac{y-7}{10}\right)+7 = y-7+7 = y$$

$\therefore \quad gof = I_R$ and $fog = I_R$

Hence, the required function on $g : R \to R$ is defined as

$$g(y) = \frac{y-7}{10}. \qquad \textbf{Ans.}$$

OR

A binary operation * on the set {0, 1, 2, 3, 4, 5} is defined as :

$$a*b = \begin{cases} a+b, & \text{if } a+b<6 \\ a+b-6, & \text{if } a+b\geq 6 \end{cases}$$

Show that zero is the identity for this operation and each element '*a*' of the set is invertible with $6-a$, being the inverse of '*a*'.**

12. Prove that : $\tan^{-1}\left[\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right] = \frac{\pi}{4}-\frac{1}{2}\cos^{-1} x, -\frac{1}{\sqrt{2}} \leq x \leq 1$ **[4]**

Solution : L.H.S.

$$= \tan^{-1}\left[\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right]$$

**Answer is not given due to the change in present syllabus

Put $x = \cos\theta \Rightarrow \theta = \cos^{-1} x$ and $0 \le \theta \le \frac{3\pi}{4}$.

$$\therefore \quad \tan^{-1}\left[\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right]$$

$$= \tan^{-1}\left(\frac{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}\right)$$

$[\because 1 + \cos\theta = 2\cos^2(\theta/2)$ and $1-\cos\theta = 2\sin^2(\theta/2)]$

$$= \tan^{-1}\left[\frac{\sqrt{2}\cos\frac{\theta}{2}-\sqrt{2}\sin\frac{\theta}{2}}{\sqrt{2}\cos\frac{\theta}{2}+\sqrt{2}\sin\frac{\theta}{2}}\right]$$

$$= \tan^{-1}\left[\frac{\sqrt{2}\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)}{\sqrt{2}\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)}\right]$$

Inside the bracket divide numerator and denominator by $\cos\frac{\theta}{2}$, we get

$$= \tan^{-1}\left[\frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}\right]$$

$$= \tan^{-1}\left[\frac{\tan\frac{\pi}{4}-\tan\frac{\theta}{2}}{1+\tan\frac{\pi}{4}.\tan\frac{\theta}{2}}\right]$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right]$$

$$= \frac{\pi}{4}-\frac{\theta}{2}$$

$$= \frac{\pi}{4}-\frac{1}{2}\cos^{-1} x = \text{R.H.S.}$$

Hence Proved.

13. Using properties of determinants, solve the following for x :

$$\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0. \qquad [4]$$

Solution : L.H.S. $= \begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix}$

Applying $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$, we get

$$= \begin{vmatrix} 2 & 6 & 12 \\ 4 & 18 & 48 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix}$$

Taking 2 common from R_1 and R_2

$$= 2\times2\begin{vmatrix} 1 & 3 & 6 \\ 2 & 9 & 24 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix}$$

Applying $C_2 \to C_2 - 3C_1$; $C_3 \to C_3 - 2C_2$, we get

$$= 4\begin{vmatrix} 1 & 0 & 0 \\ 2 & 3 & 6 \\ x-8 & -x-3 & -x-10 \end{vmatrix}$$

Expanding along C_1, we get

$$= 4(-3x-30+6x+18)$$
$$= 4[3x-12] = 0$$
$$\Rightarrow \quad 3x = 12$$
$$\Rightarrow \quad x = 4.$$

Ans.

14. Find the relationship between 'a' and 'b' so that the function 'f' defined by :

$$f(x) = \begin{cases} ax+1, & \text{if} \quad x \le 3 \\ bx+3, & \text{if} \quad x > 3 \end{cases}$$

is continuous at $x = 3$. **[4]**

Solution : $\because f(x)$ is continuous at $x = 3$,

$$\therefore \quad f(3) = \lim_{x\to3^-} f(x) = \lim_{x\to3^+} f(x)$$
$$\Rightarrow \quad \lim_{x\to3^-}(ax+1) = \lim_{x\to3^+}(bx+3)$$
$$\Rightarrow \quad 3a+1 = 3b+3$$
$$\Rightarrow \quad 3a = 3b+2$$
$$\Rightarrow \quad a = b+\frac{2}{3}.$$

Ans.

OR

If $x^y = e^{x-y}$, show that $\frac{dy}{dx} = \frac{\log x}{\{\log(xe)\}^2}$.

Solution : We have,

$$x^y = e^{x-y}$$

Taking log on both sides, we get

$$\log x^y = \log e^{x-y}$$
$$y\log x = (x-y)\log e = x-y$$

$[\because \log e = 1]$

$$\Rightarrow \quad y\log x + y = x$$
$$\Rightarrow \quad y(1+\log x) = x$$
$$\therefore \quad y = \frac{x}{(1+\log x)}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(1+\log x)\frac{d}{dx}x - x\frac{d}{dx}(1+\log x)}{(1+\log x)^2}$$

$$= \frac{(1+\log x)1 - x\left(0+\frac{1}{x}\right)}{(1+\log x)^2}$$

$$= \frac{1+\log x - 1}{(1+\log x)^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{\log x}{(1+\log x)^2} = \frac{\log x}{(\log e + \log x)^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\log x}{[\log(xe)]^2}.$$

Hence Proved.

15. Prove that $y = \frac{4\sin\theta}{(2+\cos\theta)} - \theta$ is an increasing function in $\left[0, \frac{\pi}{2}\right]$. **[4]**

Solution : We have,

$$f(\theta) = \frac{4\sin\theta}{2+\cos\theta} - \theta$$

Differentiating w.r t. θ, we get

$$\therefore \quad f'(\theta) = \frac{(2+\cos\theta)(4\cos\theta) - 4\sin\theta(0-\sin\theta)}{(2+\cos\theta)^2} - 1$$

$$= \frac{8\cos\theta + 4\cos^2\theta + 4\sin^2\theta}{(2+\cos\theta)^2} - 1$$

$$= \frac{8\cos\theta + 4}{(2+\cos\theta)^2} - 1$$

$$[\because \sin^2\theta + \cos^2\theta = 1]$$

$$= \frac{8\cos\theta + 4 - (2+\cos\theta)^2}{(2+\cos\theta)^2}$$

$$= \frac{8\cos\theta + 4 - 4 - \cos^2\theta - 4\cos\theta}{(2+\cos\theta)^2}$$

$$= \frac{4\cos\theta - \cos^2\theta}{(2+\cos\theta)^2}$$

$$= \frac{\cos\theta(4-\cos\theta)}{(2+\cos\theta)^2}$$

For all $\theta \in \left[0, \frac{\pi}{2}\right]$, $\frac{\cos\theta(4-\cos\theta)}{(2+\cos\theta)^2} \geq 0$, as $\cos\theta$ is +ve

Hence, $f(\theta)$ is increasing in $\left[0, \frac{\pi}{2}\right]$.

Hence Proved.

OR

If the radius of a sphere is measured as 9 cm with an error of 0.03 cm, then find the approximate error in calculating its surface area.

Solution : Let r be the radius of sphere and Δr be the error in measuring the radius.

Then $r = 9$ cm, $\Delta r = 0.03$ cm.

Now surface area S of the sphere is

$$S = 4\pi r^2$$

Differentiating w.r. to r, we get

$$\frac{dS}{dr} = 8\pi r$$

$$\therefore \quad \Delta S \approx dS = \frac{dS}{dr}\Delta r$$

$$= 8\pi r . \Delta r$$

$$= 8\pi \times 9 \times 0.03$$

$$= 2.16\,\pi \text{ cm}^2.$$

This is the approximate error in calculating surface area. **Ans.**

16. If $x = \tan\left(\frac{1}{a}\log y\right)$, show that $(1+x^2)\frac{d^2y}{dx^2} + (2x-a)\frac{dy}{dx} = 0.$ **[4]**

Solution : Given,

$$x = \tan\left(\frac{1}{a}\log y\right)$$

$$\Rightarrow \quad \tan^{-1} x = \frac{1}{a}\log y$$

$$\Rightarrow \quad a\tan^{-1} x = \log y$$

Differentiating both sides w.r.t. x, we get

$$a\frac{1}{1+x^2} = \frac{1}{y}\frac{dy}{dx}$$

$$\Rightarrow \quad (1+x^2)\frac{dy}{dx} = ay$$

Again differentiating both sides w.r.t. x, we get

$$(1+x^2)\frac{d^2y}{dx^2} + \frac{dy}{dx}.2x = a\frac{dy}{dx}$$

$$\Rightarrow \quad (1+x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - a\frac{dy}{dx} = 0$$

$$\therefore \quad (1+x^2)\frac{d^2y}{dx^2} + (2x-a)\frac{dy}{dx} = 0.$$

Hence Proved.

17. Evaluate : $\int_0^{\pi/2} \frac{x+\sin x}{1+\cos x}dx.$ **[4]**

Solution : Let $I = \int_0^{\pi/2} \frac{x+\sin x}{1+\cos x}dx.$

$$= \int_0^{\pi/2} \frac{x + 2\sin\frac{x}{2}.\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}dx$$

$$\left[\because \sin x = 2\sin\frac{x}{2}.\cos\frac{x}{2} \text{ and } \cos x = 2\cos^2\frac{x}{2} - 1\right]$$

$$= \frac{1}{2}\int_0^{\pi/2} x.\sec^2\frac{x}{2}.dx + \int_0^{\pi/2}\tan\frac{x}{2}.dx$$

$$= \frac{1}{2}\left[\left\{x.\frac{\tan\frac{x}{2}}{\frac{1}{2}}\right\}_0^{\pi/2} - \int_0^{\pi/2}\frac{\tan\frac{x}{2}}{\frac{1}{2}}.dx\right] + \int_0^{\pi/2}\tan\frac{x}{2}.dx$$

$$= \left[x.\tan\frac{x}{2}\right]_0^{\pi/2} - \int_0^{\pi/2}\tan\frac{x}{2}.dx + \int_0^{\pi/2}\tan\frac{x}{2}.dx$$

$$= \frac{\pi}{2}.\tan\frac{\pi}{4}$$

$$= \frac{\pi}{2}\times 1 = \frac{\pi}{2}.$$ **Ans.**

18. Solve the following differential equation :

$$xdy - ydx = \sqrt{x^2+y^2}dx.$$ **[4]**

Solution : The given differential equation is

$$xdy - ydx = \sqrt{x^2+y^2}dx$$

Separate the variables, we get

$$x\,dy = (y+\sqrt{x^2+y^2})dx$$

$$\therefore \quad \frac{dy}{dx} = \frac{y+\sqrt{x^2+y^2}}{x} \quad ...(i)$$

This is a homogeneous differential equation.

Putting $y = vx$

$$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx} \quad ...(ii)$$

From equation (i) and (ii), we get

$$v + x\frac{dv}{dx} = \frac{vx+\sqrt{x^2+v^2x^2}}{x}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{vx+x\sqrt{1+v^2}}{x}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = v+\sqrt{1+v^2}$$

$$\Rightarrow \quad x\frac{dv}{dx} = v+\sqrt{1+v^2} - v = \sqrt{1+v^2}$$

$$\therefore \quad \frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x}$$

On integrating both sides, we get

$$\log\left|v+\sqrt{1+v^2}\right| = \log\,|x| + \log|C|$$

$$\Rightarrow \log\left|\frac{y}{x}+\sqrt{1+\frac{y^2}{x^2}}\right| = \log\,|Cx|$$

$$\Rightarrow \quad \frac{y+\sqrt{x^2+y^2}}{x} = Cx$$

$$\Rightarrow \quad y+\sqrt{x^2+y^2} = Cx^2.$$ **Ans.**

19. Solve the following differential equation :

$$(y+3x^2)\frac{dx}{dy} = x.$$ **[4]**

Solution : The given differential equation is

$$(y+3x^2)\frac{dx}{dy} = x$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{x}{y+3x^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y+3x^2}{x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x}+3x$$

$$\Rightarrow \quad \frac{dy}{dx}+\left(-\frac{1}{x}\right)y = 3x$$

This is a linear differential equation of the type

$$\frac{dy}{dx} + Py = Q,$$

Here, $P = -\frac{1}{x}$ and $Q = 3x$

Integrating factor (I.F.)

$$= e^{\int P dx}$$

$$= e^{\int \frac{-1}{x}dx}$$

$$= e^{-\log x}$$

$$= e^{\log(x)^{-1}} = \frac{1}{x}$$

Required solution is

$$y.(\text{I.F.}) = \int(\text{Q.(I.F.)})dx + C$$

$$\frac{y}{x} = \int\left(3x\frac{1}{x}\right)dx + C$$

$$\Rightarrow \quad \frac{y}{x} = \int 3\,dx + C$$

$$\Rightarrow \quad \frac{y}{x} = 3x + C$$

$$\therefore \quad y = 3x^2 + Cx.$$ **Ans.**

20. Using vectors, find the area of the triangle with vertices A(1, 1, 2), B(2, 3, 5) and C(1, 5, 5). [4]

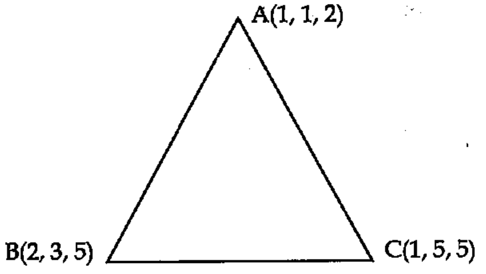
Solution : The vertices of triangle ABC are given as A(1, 1, 2), B(2, 3, 5) and C(1, 5, 5)

Let O be the origin of triangle

$$\therefore \quad \overrightarrow{OA} = \hat{i}+\hat{j}+2\hat{k}$$

$$\overrightarrow{OB} = 2\hat{i}+3\hat{j}+5\hat{k}$$

$$\vec{OC} = \hat{i}+5\hat{j}+5\hat{k}$$



$$\vec{AB} = \vec{OB}-\vec{OA}$$
$$= 2\hat{i}+3\hat{j}+5\hat{k}-\hat{i}-\hat{j}-2\hat{k}$$
$$= \hat{i}+2\hat{j}+3\hat{k}$$
$$\vec{AC} = \vec{OC}-\vec{OA}$$
$$= \hat{i}+5\hat{j}+5\hat{k}-\hat{i}-\hat{j}-2\hat{k}$$
$$= 4\hat{j}+3\hat{k}$$

Now, $\vec{AB}\times\vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 0 & 4 & 3 \end{vmatrix}$

$$= \hat{i}(6-12)-\hat{j}(3-0)+\hat{k}(4-0)$$
$$= -6\hat{i}-3\hat{j}+4\hat{k}$$

$\therefore \quad |\vec{AB}\times\vec{AC}| = \sqrt{(-6)^2+(-3)^2+(4)^2}$

$$= \sqrt{36+9+16}$$
$$= \sqrt{61}$$

$\therefore$ Area of $\Delta ABC = \frac{1}{2}|\vec{AB}\times\vec{AC}|$

$= \frac{1}{2}\sqrt{61}$ sq. units. **Ans.**

21. Find the shortest distance between the following lines whose vector equations are :

$$\vec{r} = (1-t)\hat{i}+(t-2)\hat{j}+(3-2t)\hat{k} \text{ and}$$
$$\vec{r} = (s+1)\hat{i}+(2s-1)\hat{j}-(2s+1)\hat{k}. \qquad [4]$$

Solution : Given equations are

$$\vec{r} = (\hat{i}-2\hat{j}+3\hat{k})+t(-\hat{i}+\hat{j}-2\hat{k}) \qquad ...(i)$$

and $\vec{r} = (\hat{i}-\hat{j}-\hat{k})+s(\hat{i}+2\hat{j}-2\hat{k}) \qquad ...(ii)$

On comparing equations (i) and (ii) with

$\vec{r} = \vec{a_1}+\lambda\vec{b_1}$ and $\vec{r} = \vec{a_2}+\mu\vec{b_2}$, we get

$\therefore \quad \vec{a_1} = \hat{i}-2\hat{j}+3\hat{k}, \vec{b_1} = -\hat{i}+\hat{j}-2\hat{k}$

and $\vec{a_2} = \hat{i}-\hat{j}-\hat{k}, \vec{b_2} = \hat{i}+2\hat{j}-2\hat{k}$.

$\therefore \vec{a_2}-\vec{a_1} = \hat{i}-\hat{j}-\hat{k}-\hat{i}+2\hat{j}-3\hat{k} = 0\hat{i}+\hat{j}-4\hat{k}$

Now, $\vec{b_1}\times\vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & -2 \\ 1 & 2 & -2 \end{vmatrix}$

$$= \hat{i}(-2+4)-\hat{j}(2+2) + \hat{k}(-2-1)$$
$$= 2\hat{i}-4\hat{j}-3\hat{k}$$

$\therefore \quad |\vec{b_1}\times\vec{b_2}| = \sqrt{(2)^2+(-4)^2+(-3)^2}$

$$= \sqrt{4+16+9} = \sqrt{29}$$

$\therefore$ The shortest distance between given lines is

$$\text{S.D.} = \left|\frac{(\vec{b_1}\times\vec{b_2}).(\vec{a_2}-\vec{a_1})}{|(\vec{b_1}\times\vec{b_2})|}\right|$$
$$= \left|\frac{(2\hat{i}-4\hat{j}-3\hat{k}).(0\hat{i}+\hat{j}-4\hat{k})}{\sqrt{29}}\right|$$
$$= \left|\frac{0-4+12}{\sqrt{29}}\right|$$
$$= \frac{8}{\sqrt{29}} \text{ units.}$$

Ans.

22. A random variable X has the following probability distribution :

X	0	1	2	3	4	5	6	7
P(X)	0	K	2K	2K	3K	K^2	$2K^2$	$7K^2+K$

Determine : **[4]**

(i) K

(ii) P(X < 3)

(iii) P(X > 6)

(iv) P(0 < X < 3).

Solution : It is known that the sum of probability distribution of variable is one.

(i) $\therefore \quad \Sigma P(X) = 1$

Therefore,

$$P(0)+P(1)+P(2)+P(3)+P(4)+P(5)+P(6)+P(7) = 1$$
$$0+K+2K+2K+3K+K^2+2K^2+7K^2+K = 1$$
$$\Rightarrow \quad 10K^2+9K-1 = 0$$
$$\Rightarrow \quad 10K^2+10K-K-1 = 0$$
$$\Rightarrow \quad 10K(K+1)-1(K+1) = 0$$
$$\Rightarrow \quad (K+1)(10K-1) = 0$$
$$\Rightarrow \quad K+1 = 0$$

$\Rightarrow \quad K = -1$

and $\quad 10K - 1 = 0$

$\Rightarrow \quad K = 1/10$

Here, K = – 1 is not possible as the possibility of an event is never negative,

$\therefore \quad K = \frac{1}{10}.$

Hence, probability distribution of X is

X	0	1	2	3	4	5	6	7
P(X)	0	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{1}{100}$	$\frac{2}{100}$	$\frac{7}{100}+\frac{1}{10}$

(ii) $P(X<3) = P(0)+P(1)+P(2)$

$$= 0+\frac{1}{10}+\frac{2}{10}=\frac{3}{10}$$

(iii) $P(X>6) = P(7) = \frac{7}{100}+\frac{1}{10}=\frac{17}{100}$

(iv) $P(0<X<3) = P(1)+P(2)$

$$= \frac{1}{10}+\frac{2}{10}=\frac{3}{10}.$$ **Ans.**

OR

Find the probability of throwing at most 2 sixes in 6 throws of a single die.

Solution : Here,

$$n = 6$$

$$p = P\text{ (getting 6)} = \frac{1}{6}.$$

$$\therefore q = 1-p = 1-\frac{1}{6}=\frac{5}{6}$$

P (getting at most 2 sixes)

$$= P(X \le 2)$$

$$= P(0)+P(1)+P(2)$$

$$= {}^6C_0\left(\frac{5}{6}\right)^{6-0}\left(\frac{1}{6}\right)^0 + {}^6C_1\left(\frac{5}{6}\right)^{6-1}\left(\frac{1}{6}\right)^1 + {}^6C_2\left(\frac{5}{6}\right)^{6-2}\left(\frac{1}{6}\right)^2$$

$$= \left(\frac{5}{6}\right)^6+\left(\frac{5}{6}\right)^5+15\left(\frac{5}{6}\right)^4\left(\frac{1}{6}\right)^2$$

$$= \left(\frac{5}{6}\right)^4\left[\left(\frac{5}{6}\right)^2+\frac{5}{6}+15\times\left(\frac{1}{6}\right)^2\right]$$

$$= \left(\frac{5}{6}\right)^4\left[\frac{25}{36}+\frac{5}{6}+15\times\frac{1}{36}\right]$$

$$= \left(\frac{5}{6}\right)^4\left[\frac{25+30+15}{36}\right]$$

$$= \left(\frac{5}{6}\right)^4\times\frac{70}{36}=\frac{35}{18}\left(\frac{5}{6}\right)^4$$

$$= \frac{7}{3}\left(\frac{5}{6}\right)^5.$$ **Ans.**

SECTION – C

23. Using matrices, solve the following system of equations:

$$4x+3y+2z = 60$$
$$x+2y+3z = 45$$
$$6x+2y+3z = 70.$$ **[6]**

Solution :

$$4x+3y+2z = 60,$$
$$x+2y+3z = 45,$$
$$6x+2y+3z = 70.$$

The equation of system can be written in matrix form

$$AX = B \quad ...(i)$$

$$\begin{bmatrix}4&3&2\\1&2&3\\6&2&3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}60\\45\\70\end{bmatrix}$$

Here, $A = \begin{bmatrix}4&3&2\\1&2&3\\6&2&3\end{bmatrix}$, $X = \begin{bmatrix}x\\y\\z\end{bmatrix}$ and $B = \begin{bmatrix}60\\45\\70\end{bmatrix}$

$$\therefore \quad |A| = \begin{vmatrix}4&3&2\\1&2&3\\6&2&3\end{vmatrix}$$

$$= 4(6-6)-3(3-18)+2(2-12)$$

$$= 0+45-20 = 25 \ne 0$$

$\therefore A^{-1}$ exists.

Cofactors of A,

$A_{11} = (6-6) = 0$, $A_{12} = -(3-18) = 15$,

$A_{13} = (2-12) = -10$

$A_{21} = -(9-4) = -5$, $A_{22} = (12-12) = 0$

$A_{23} = -(8-18) = 10$

$A_{31} = (9-4) = 5$, $A_{32} = -(12-2) = -10$

$A_{33} = (8-3) = 5$

$$\therefore \quad \text{adj } A = \begin{bmatrix}0&15&-10\\-5&0&10\\5&-10&5\end{bmatrix}^T$$

$$\text{adj } A = \begin{bmatrix}0&-5&5\\15&0&-10\\-10&10&5\end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{25}\begin{bmatrix}0&-5&5\\15&0&-10\\-10&10&5\end{bmatrix}$$

From (i), $\quad X = A^{-1}B$

$$\therefore \quad X = \frac{1}{25}\begin{bmatrix} 0 & -5 & 5 \\ 15 & 0 & -10 \\ -10 & 10 & 5 \end{bmatrix}\begin{bmatrix} 60 \\ 45 \\ 70 \end{bmatrix}$$

$$= \frac{1}{25}\begin{bmatrix} 0-225+350 \\ 900+0-700 \\ -600+450+350 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{25}\begin{bmatrix} 125 \\ 200 \\ 200 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 8 \end{bmatrix}$$

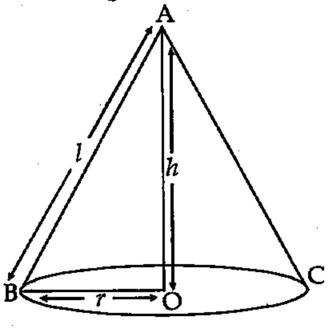
$\therefore \quad x = 5, y = 8, z = 8.$ **Ans.**

24. Show that the right circular cone of least curved surface and given volume has an altitude equal to $\sqrt{2}$ times the radius of the base. [6]

Solution : Let radius of cone $= r$

Height of cone $= h$

and slant height of cone $= l$



Curved surface area of cone

$$S = \pi r l \qquad \text{...(i)}$$

$$\therefore \quad S^2 = \pi^2 r^2 l^2$$

$$\Rightarrow \quad S^2 = \pi^2 r^2 (h^2 + r^2) \text{...(ii)}$$

$$[\because l^2 = r^2 + h^2]$$

Volume of cone

$$V = \frac{1}{3} r^2 h$$

$$\Rightarrow \quad 3V = \pi r^2 h$$

$$\therefore \quad h = \frac{3V}{\pi r^2} \qquad \text{...(iii)}$$

Putting the value of h in equation (ii), we get

$$S^2 = \pi^2 r^2 \left(\frac{9V^2}{\pi^2 r^4} + r^2\right)$$

$$= \pi^2 r^2 \left[\frac{9V^2 + \pi^2 r^6}{\pi^2 r^4}\right]$$

$$= \left[\frac{9V^2 + \pi^2 r^6}{r^2}\right]$$

$$= \frac{9V^2}{r^2} + \pi^2 r^4$$

Let $\quad S^2 = f(r)$

$$\therefore \quad f(r) = \frac{9V^2}{r^2} + \pi^2 r^4$$

Differentiating w.r.t. r, we get

$$f'(r) = -18\, V^2 r^{-3} + 4\pi^2 r^3 \quad \text{...(iv)}$$

For stationary points,

$$f'(r) = 0$$

$$\Rightarrow \quad -18V^2 r^{-3} + 4\pi^2 r^3 = 0$$

$$\Rightarrow \quad \frac{-18V^2}{r^3} = -4\pi^2 r^3$$

$$\Rightarrow \quad 4\pi^2 r^6 = 18V^2$$

$$\Rightarrow \quad r^6 = \frac{18V^2}{4\pi^2}$$

$$\Rightarrow \quad r^6 = \frac{9V^2}{2\pi^2}$$

$$\Rightarrow \quad r^3 = \frac{3V}{\sqrt{2}\pi}$$

$$\Rightarrow \quad r = \left(\frac{3V}{\sqrt{2}\pi}\right)^{1/3}$$

Again differentiating (iv), we get

$$f''(r) = 54V^2 r^{-4} + 12\pi^2 r^2$$

$$\therefore \text{ At} \quad r = \left(\frac{3V}{\sqrt{2}\pi}\right)^{1/3}$$

$$f''(r) = 54V^2 \left(\frac{3V}{\sqrt{2}\pi}\right)^{-4/3} + 12\pi^2 \left(\frac{3V}{\sqrt{2}\pi}\right)^2 > 0$$

$\therefore f(r)$ is minimum at

$$r = \left(\frac{3V}{\sqrt{2}\pi}\right)^{1/3}$$

$\therefore$ Surface area is minimum at

$$r^3 = \frac{3V}{\sqrt{2}\pi}$$

$$\therefore \quad V = \frac{\sqrt{2}\pi r^3}{3}$$

Putting the value of V in equation (iii), we get

$$h = \frac{\frac{3\sqrt{2}\pi r^3}{3}}{\pi r^2}$$

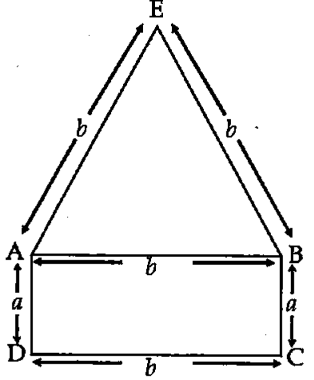
$$\Rightarrow \quad h = \sqrt{2} r$$

Hence, altitude is equal to $\sqrt{2}$ times the radius of base. **Hence Proved.**

OR

A window has the shape of a rectangle surmounted by an equilateral triangle. If the perimeter of the window is 12 m, find the dimensions of the rectangle that will produce the largest area of the window.

Solution : Let ABCD be a rectangle and let the side of an equilateral triangle be AB = b (length of the rectangle) and BC = a (width of the rectangle)



Total perimeter of the window = 12

$$\therefore \quad b + 2a + 2b = 12$$

$$\Rightarrow \quad 2a + 3b = 12$$

$$\Rightarrow \quad a = \frac{12-3b}{2} \quad \text{...(i)}$$

Now, area A of the window = Area of rectangle + Area of equilateral Δ

$$A = ab + \frac{\sqrt{3}}{4}b^2$$

$$= \left(\frac{12-3b}{2}\right)b + \frac{\sqrt{3}}{4}b^2 \quad \text{[using (i)]}$$

$$\therefore \quad A = 6b - \frac{3}{2}b^2 + \frac{\sqrt{3}}{4}b^2$$

Differentiating w.r.t. b, we get

$$\frac{dA}{db} = 6 - 3b + \frac{\sqrt{3}}{2}b$$

For maxima or minima,

$$\frac{dA}{db} = 0$$

$$\Rightarrow \quad \left(3 - \frac{\sqrt{3}}{2}\right)b = 6$$

$$\Rightarrow \quad b = \frac{12}{6-\sqrt{3}}$$

Also, $$\frac{d^2A}{db^2} = -3 + \frac{\sqrt{3}}{2} < 0$$

$\Rightarrow$ Area is maximum, when $b = \dfrac{12}{6-\sqrt{3}}$

From (i),

$$a = \frac{12-3b}{2} = 6 - \frac{3}{2}\cdot\frac{12}{6-\sqrt{3}}$$

$$a = \frac{18-6\sqrt{3}}{6-\sqrt{3}}.$$

Hence, the dimensions of the rectangle that will produce the largest area of the window are $\left(\dfrac{18-6\sqrt{3}}{6-\sqrt{3}}\right)$ and $\left(\dfrac{12}{6-\sqrt{3}}\right)$. **Ans.**

25. Evaluate : $\displaystyle\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$. **[6]**

Solution : Let,
$$I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$$

$$= \int_{\pi/6}^{\pi/3} \frac{dx}{1+\dfrac{\sqrt{\sin x}}{\sqrt{\cos x}}}$$

$$\Rightarrow \quad I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}\,dx}{\sqrt{\cos x}+\sqrt{\sin x}} \quad \text{...(i)}$$

$$\Rightarrow \quad I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)}}{\sqrt{\cos\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)}+\sqrt{\sin\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)}}dx$$

$$\left[\because \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\right]$$

$$\Rightarrow \quad I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}+\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}dx$$

$$\Rightarrow \quad I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}\,dx}{\sqrt{\sin x}+\sqrt{\cos x}} \quad \text{...(ii)}$$

On adding equations (i) and (ii), we get

$$2I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}+\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx$$

$$\Rightarrow \quad 2I = \int_{\pi/6}^{\pi/3} 1.dx$$

$$\Rightarrow \quad 2I = [x]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$\Rightarrow \quad 2I = \left[\frac{\pi}{3} - \frac{\pi}{6}\right]$$

$$\Rightarrow \quad 2I = \left[\frac{2\pi - \pi}{6}\right]$$

$$\Rightarrow \quad 2I = \frac{\pi}{6}$$

$$\therefore \quad I = \frac{\pi}{12}.$$

Ans.

OR

Evaluate : $\int \frac{6x+7}{\sqrt{(x-5)(x-4)}} dx.$

Solution : Let $I = \int \frac{6x+7}{\sqrt{(x-5)(x-4)}} dx$

$$= \int \frac{6x+7}{\sqrt{x^2-9x+20}}$$

Let $\quad 6x+7 = A\frac{d}{dx}(x^2-9x+20) + B$

$\therefore \quad 6x+7 = A(2x-9) + B$

Equating the coefficients of like terms from both sides, we get

$$2A = 6$$

$\Rightarrow \quad A = 3$

and $\quad -9A + B = 7$

$\Rightarrow \quad -9 \times 3 + B = 7$

$\therefore \quad B = 7 + 27 = 34$

$$\therefore \quad I = \int \frac{3(2x-9)}{\sqrt{x^2-9x+20}} dx + \int \frac{34}{\sqrt{x^2-9x+20}} dx$$

Putting $x^2 - 9x + 20 = t$

$\Rightarrow \quad (2x-9)dx = dt$ in first integral, we get

$$\therefore \quad I = \int \frac{3}{\sqrt{t}} dt + 34 \int \frac{dx}{\sqrt{\left(x-\frac{9}{2}\right)^2 + 20 - \frac{81}{4}}}$$

$$= 3\int t^{-1/2} dt + 34 \int \frac{dx}{\sqrt{\left(x-\frac{9}{2}\right)^2 - \frac{1}{4}}}$$

$$= 3\frac{t^{\frac{1}{2}}}{\frac{1}{2}} + 34\int \frac{dx}{\sqrt{\left(x-\frac{9}{2}\right)^2 - \left(\frac{1}{2}\right)^2}}$$

$$= 6\sqrt{t} + 34\log\left|\left(x-\frac{9}{2}\right) + \sqrt{\left(x-\frac{9}{2}\right)^2 - \left(\frac{1}{2}\right)^2}\right| + C$$

$$= 6\sqrt{x^2-9x+20} + 34\log\left|\left(x-\frac{9}{2}\right) + \sqrt{x^2-9x+20}\right| + C.$$

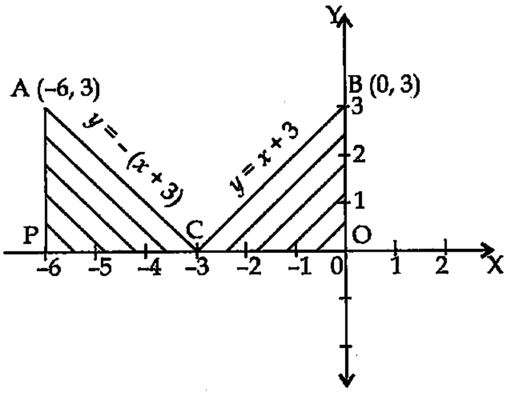
Ans.

26. Sketch the graph of $y = |x+3|$ and evaluate the area under the curve $y = |x+3|$ above x-axis and between $x = -6$ to $x = 0$. **[6]**

Solution : For drawing a sketch of the graph of $y = |x+3|$, we construct the following table of values

x	–6	–5	–3	–2	–1	0
y	3	2	0	1	2	3

Plot these points, a rough sketch is shown in the figure below.



Note that $y = |x+3|$

$$= \begin{cases} -(x+3) & \text{for } x \le -3 \\ x+3 & \text{for } x > 3 \end{cases}$$

So graph consists of two half lines meeting at $x = -3$.

Also $\int_{-6}^{0} |x+3| dx$ area enclosed between graph of $y = |x+3|$, the x-axis and the lines $x = -6, x = 0$

$= \text{area } \Delta APC + \text{area } \Delta COB$

$$= \int_{-6}^{-3} (-x-3)dx + \int_{-3}^{0} (x+3)dx$$

$$= \left[\frac{-x^2}{2} - 3x\right]_{-6}^{-3} + \left[\frac{x^2}{2} + 3x\right]_{-3}^{0}$$

$$= \left(-\frac{9}{2}+9\right) - (-18+18) + 0 - \left(\frac{9}{2}-9\right)$$

$$= \frac{9}{2}+\frac{9}{2} = 9 \text{ sq. units.}$$ **Ans.**

27. Find the distance of the point (–1, –5, –10), from the point of intersection of the line $\vec{r} = (2\hat{i} - \hat{j} + 2\hat{k}) + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$ and Plane $\vec{r}.(\hat{i} - \hat{j} + \hat{k}) = 5$. **[6]**

Solution : Equation of the line is

$$\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$$

$$\vec{r} = (2+3\lambda)\hat{i} + (-1+4\lambda)\hat{j} + (2+2\lambda)\hat{k} \quad ...(i)$$

The equation of the given plane is

$$\vec{r}.(\hat{i} - \hat{j} + \hat{k}) = 5 \quad ...(ii)$$

Since the point of intersection of the line and plane lies on the plane as well as on the line, from (i)

Any point on line is

$$= (2+3\lambda, -1+4\lambda, 2+2\lambda)$$

It lies on the plane (ii)

$(2 + 3\lambda)(1) + (-1 + 4\lambda)(-1) + (2 + 2\lambda)(1) = 5$

$\Rightarrow \quad (2 + 3\lambda + 1 - 4\lambda + 2 + 2\lambda) = 5$

$\Rightarrow \quad 5 + \lambda = 5$

$\Rightarrow \quad \lambda = 0$

Substituting the value of λ in (i), we get

$$\vec{r} = [2+3(0)]\hat{i} + [-1 + 4(0)]\hat{j} + [2+2(0)]\hat{k}$$

$$= 2\hat{i} - \hat{j} + 2\hat{k}$$

Thus, the point of intersection of the given line and plane is (2, –1, 2).

Now, the distance of the point (–1, –5, –10) from the point (2, –1, 2)

$$= \sqrt{(-1-2)^2 + (-5+1)^2 + (-10-2)^2}$$

$$= \sqrt{9+16+144}$$

$$= \sqrt{169} = 13.$$ **Ans.**

28. Given three identical boxes I, II and III each containing two coins. In box I, both coins are gold coins, in box II, both are silver coins and in box III, there is one gold and one silver coin. A person chooses a box at random and takes out a coin. If the coin is of gold, what is the probability that the other coin in the box is also of gold ? [6]

Solution : Let E_1 be box I is chosen, E_2 be box II is chosen and E_3 be box III be chosen and A be the coin drawn is of gold.

We have,

$$P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$$

$P(A/E_1)$ = Probability of drawing a gold coin from box I

$\therefore \quad P(A/E_1) = \frac{2}{2} = 1$

$P(A/E_2)$ = Probability of drawing a gold coin from box II

$\therefore \quad P(A/E_2) = 0$

$P(A/E_3)$ = Probability of drawing a gold coin from box III

$\therefore \quad P(A/E_3) = \frac{1}{2}$

$\therefore$ Probability that the other coin in the box is of gold = Probability that gold coin is drawn from the box I

$= P(E_1/A)$

$$= \frac{P(E_1)P(A/E_1)}{P(E_1)P(A/E_1)+P(E_2)P(A/E_2)+P(E_3)P(A/E_3)}$$

$$= \frac{\frac{1}{3}\times 1}{\frac{1}{3}\times 1+\frac{1}{3}\times 0+\frac{1}{3}\times\frac{1}{2}} = \frac{2}{3}.$$ **Ans.**

29. A merchant plans to sell two types of personal computer – a desktop model and a portable model that will cost ₹ 25,000 and ₹ 40,000 respectively. He estimates that the total monthly demand of computers will not exceed 250 units. Determine the number of units of each type of computers which the merchant should stock to get maximum profit if he does not want to invest more than ₹ 70 lakhs and his profit on the desktop model is ₹ 4,500 and on the portable model is ₹ 5,000. Make an L.P.P. and solve it graphically. [6]

Solution : Let the merchant stock x desktop computers and y portable computers. We construct the following table :

Type	Number	Cost per computer	Investment	Maximum (profit)
Desktop	x	₹25,000	₹25,000x	₹4,500x
Portable	y	₹40,000	₹40,000y	₹5,000y
	250		₹70,00,000	

∴ The LPP is

Maximize $Z = 4{,}500\,x + 5{,}000\,y$

Subject to constraints :

$$x + y \le 250$$

$$25{,}000\,x + 40{,}000\,y \le 70{,}00{,}000$$

$$\Rightarrow \quad 5x + 8y \le 1{,}400$$

and $\quad x \ge 0\ ,\ y \ge 0$

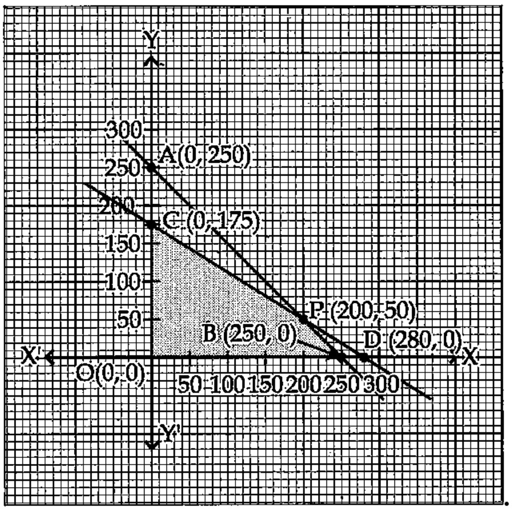
First we draw the lines AB and CD whose equations are

$$x + y = 250 \quad \text{...(i)}$$

	A	B
x	0	250
y	250	0

and $\quad 5x + 8y = 1400 \quad$ ...(ii)

	C	D
x	0	280
y	175	0



The feasible region is OBPCO which is shaded in the figure.

The vertices of feasible region are O(0, 0), B(250,0), P(200, 50) and C(0, 175).

P is the point of intersection of the lines.

$$x + y = 250$$

and $\quad 5x + 8y = 1400$

Solving these equation we get point P(200, 50).

∴ The value of objective function

$$Z = 4500\,x + 5000y$$

At these vertices are as follows :

Corner points	Maximize $Z = 4500x + 5000y$
At O(0, 0)	$Z = 0$
At B(250, 0)	$Z = 1125000$
At P(200, 50)	$Z = 1150000$ (maximum)
At C(0, 175)	$Z = 875000$

Hence, the profit is maximum at ₹11,50,000 when 200 desktop computers and 50 portable computers are stocked. **Ans.**

Mathematics 2011 (Outside Delhi) SET II

Time allowed : 3 hours **Maximum marks : 100**

Note : Except for the following questions, all the remaining questions have been asked in previous set.

SECTION — A

9. Evaluate : $\int \frac{(\log x)^2}{x} dx$ [1]

Solution : Let $I = \int \frac{(\log x)^2}{x} dx$

Putting $\log x = t \Rightarrow \frac{1}{x} dx = dt$

$$\therefore \quad I = \int t^2 dt$$

$$= \frac{t^3}{3} + C$$

$$= \frac{(\log x)^3}{3} + C. \qquad \textbf{Ans.}$$

10. Write a unit vector in the direction of the vector $\vec{a} = 2\hat{i} + \hat{j} + 2\hat{k}$. **[1]**

Solution : The given vector is

$$\vec{a} = 2\hat{i} + \hat{j} + 2\hat{k}$$

$$|\vec{a}| = \sqrt{(2)^2 + (1)^2 + (2)^2} = \sqrt{4+1+4} = \sqrt{9} = 3$$

Unit vector, $\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{2\hat{i}+\hat{j}+2\hat{k}}{3}$

$$= \frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}. \qquad \textbf{Ans.}$$

SECTION — B

19. Prove the following :

$$2\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \tan^{-1}\left(\frac{31}{17}\right) \qquad \textbf{[4]}$$

Solution : L.H.S.

$$= 2\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left[\frac{2.\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2}\right] + \tan^{-1}\frac{1}{7}$$

$$\left[\because 2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}\right]$$

$$= \tan^{-1}\left(\frac{1}{\frac{3}{4}}\right) + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\frac{4}{3} + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\left[\frac{\frac{4}{3}+\frac{1}{7}}{1-\frac{4}{3}.\frac{1}{7}}\right]$$

$$= \tan^{-1}\left(\frac{\frac{31}{21}}{\frac{17}{21}}\right)$$

$$= \tan^{-1}\left(\frac{31}{17}\right) = \text{R.H.S.}$$

Hence Proved.

20. Using properties of determinants, solve the following for x :

$$\begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0. \qquad \textbf{[4]}$$

Solution : The determinant is

$$\begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0$$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\begin{vmatrix} 3a-x & a-x & a-x \\ 3a-x & a+x & a-x \\ 3a-x & a-x & a+x \end{vmatrix} = 0$$

Taking $(3a-x)$ common from C_1

$$(3a-x)\begin{vmatrix} 1 & a-x & a-x \\ 1 & a+x & a-x \\ 1 & a-x & a+x \end{vmatrix} = 0$$

Applying $R_2 \to R_2 - R_1$; $R_3 \to R_3 - R_1$, we get

$$(3a-x)\begin{vmatrix} 1 & a-x & a-x \\ 0 & 2x & 0 \\ 0 & 0 & 2x \end{vmatrix} = 0$$

Expanding along C_1, we get

$$(3a-x)\,[1(2x.2x-0)] = 0$$

$$\Rightarrow \quad 4x^2(3a-x) = 0$$

$$\Rightarrow \quad 4x^2 = 0 \Rightarrow x = 0$$

and $\quad 3a - x = 0 \Rightarrow x = 3a$

Ans.

21. Evaluate : $\int_0^{\frac{\pi}{4}} \log(1+\tan x)dx$. **[4]**

Solution : Let $I = \int_0^{\frac{\pi}{4}} \log(1+\tan x)dx$

$$= \int_0^{\frac{\pi}{4}} \log\left(1+\tan\left(\frac{\pi}{4}-x\right)\right)dx$$

[using property $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$]

$$= \int_0^{\frac{\pi}{4}} \log\left(1+\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}\right)dx$$

$$= \int_0^{\frac{\pi}{4}} \log\left(1+\frac{1-\tan x}{1+\tan x}\right)dx$$

$$= \int_0^{\frac{\pi}{4}} \log\left(\frac{1+\tan x+1-\tan x}{1+\tan x}\right)dx$$

$$= \int_0^{\frac{\pi}{4}} \log\left(\frac{2}{1+\tan x}\right)dx$$

$$= \int_0^{\frac{\pi}{4}} \log 2\, dx - \int_0^{\frac{\pi}{4}} \log(1+\tan x)\,dx$$

$$= \log 2\,[x]_0^{\frac{\pi}{4}} - I$$

$$\Rightarrow \quad 2I = \log 2\left[\frac{\pi}{4}-0\right]$$

$$\Rightarrow \quad I = \frac{\pi}{8}\log 2.$$ **Ans.**

22. Solve the following differential equation :

$$xdy - (y+2x^2)dx = 0$$ **[4]**

Solution : Given,

$$xdy-(y+2x^2)dx = 0$$

$$\Rightarrow \quad x\,dy = (y+2x^2)dx$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y+2x^2}{x}$$

$$\Rightarrow \quad \frac{dy}{dx}-\frac{1}{x}y = 2x \quad \text{...(i)}$$

This is a linear differential equation of the form

$$\frac{dy}{dx}+Py = Q,$$

Here, $P = -\frac{1}{x}$ and $Q = 2x$

Integrating factor,

$$\text{I.F.} = e^{\int P dx}$$

$$= e^{\int\left(-\frac{1}{x}\right)} = e^{-\log x}$$

$$= e^{\log x^{-1}} = x^{-1} = \frac{1}{x}$$

∴ The solution is given by

$$y \times \text{I. F.} = \int(Q\times \text{I.F.})\,dx + C$$

$$\Rightarrow \quad \frac{y}{x} = \int 2x.\frac{1}{x}dx + C$$

$$\Rightarrow \quad y.\frac{1}{x} = 2x + C$$

$$\Rightarrow \quad y = 2x^2 + Cx.$$ **Ans.**

SECTION – C

28. Using matrices, solve the following system of equations : **[6]**

$$x + 2y + z = 7$$
$$x + 3z = 11$$
$$2x - 3y = 1$$

Solution : The given equations are

$$x+2y+z = 7$$
$$x+3z = 11$$
$$2x-3y = 1$$

The given system of equations can be written in matrix form

$$AX = B \quad \text{...(i)}$$

$$\begin{bmatrix}1 & 2 & 1\\1 & 0 & 3\\2 & -3 & 0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}7\\11\\1\end{bmatrix}$$

where, $A = \begin{bmatrix}1 & 2 & 1\\1 & 0 & 3\\2 & -3 & 0\end{bmatrix}$, $X = \begin{bmatrix}x\\y\\z\end{bmatrix}$, $B = \begin{bmatrix}7\\11\\1\end{bmatrix}$

Now, $$|A| = \begin{vmatrix}1 & 2 & 1\\1 & 0 & 3\\2 & -3 & 0\end{vmatrix}$$

$$= 1(0+9)-2(0-6) + 1(-3-0) = 21-3 = 18 \neq 0$$

∴ A^{-1} exists

Cofactors of A,

$A_{11} = 0+9 = 9, A_{12} = -(0-6) = 6, A_{21} = -(0+3) = -3$
$A_{13} = -3-0 = -3, \quad A_{22} = 0-2 = -2,$
$A_{23} = -(-3-4) = 7, \; A_{31} = 6-0 = 6,$
$A_{32} = -(3-1) = -2, \; A_{33} = 0-2 = -2$

$$\text{adj } A = \begin{bmatrix}9 & -3 & 6\\6 & -2 & -2\\-3 & 7 & -2\end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj } A$$

$$= \frac{1}{18}\begin{bmatrix}9 & -3 & 6\\6 & -2 & -2\\-3 & 7 & -2\end{bmatrix}$$

$$\therefore \quad X = A^{-1}B \quad \text{[from (i)]}$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \frac{1}{18}\begin{bmatrix}9 & -3 & 6\\6 & -2 & -2\\-3 & 7 & -2\end{bmatrix}\begin{bmatrix}7\\11\\1\end{bmatrix}$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \frac{1}{18}\begin{bmatrix}36\\18\\54\end{bmatrix} = \begin{bmatrix}2\\1\\3\end{bmatrix}$$

$$\Rightarrow \quad x = 2, y = 1, z = 3.$$ **Ans.**

29. Find the equation of the plane passing through the line of intersection of the planes $\vec{r}.(\hat{i}+\hat{j}+\hat{k}) = 1$ and $\vec{r}.(2\hat{i}+3\hat{j}-\hat{k})+4 = 0$ and parallel to x-axis. **[6]**

Solution : The given equation of planes are

$$\vec{r}.(\hat{i}+\hat{j}+\hat{k}) = 1$$

and $\vec{r}.(2\hat{i}+3\hat{j}-\hat{k})+4=0$

The cartesian equation of the planes are

$$x+y+z-1 = 0 \quad ...(i)$$

$$2x+3y-z+4 = 0 \quad ...(ii)$$

Equation of plane passing through the intersection of the plane (i) and (ii) is

$$(x+y+z-1)+\lambda(2x+3y-z+4)=0 \quad ...(iii)$$

$\Rightarrow \quad x+y+z-1+2\lambda x+3\lambda y-\lambda z+4\lambda=0$

$\Rightarrow \quad x+2\lambda x+y+3\lambda y+z-\lambda z-1+4\lambda=0$

$\Rightarrow \quad (1+2\lambda)x+(1+3\lambda)y+(1-\lambda)z-1+4\lambda=0$

$\therefore$ Dr's of the normal to the plane are $1+2\lambda$, $1+3\lambda$, $1-\lambda$

This plane is parallel to x–axis

$\therefore \quad (1+2\lambda)(1)+(1+3\lambda)(0)+(1-\lambda)(0)=0$

[$\because$ $d.r'.s$ of x–axis are 1, 0, 0]

$\Rightarrow \quad 1+2\lambda=0$

$\Rightarrow \quad \lambda=-\frac{1}{2}$

Putting the value of λ in (iii), we get

$$(x+y+z-1)-\frac{1}{2}(2x+3y-z+4)=0$$

$\Rightarrow \quad 2x+2y+2z-2-2x-3y+z-4=0$

$\Rightarrow \quad -y+3z-6=0$

$\therefore \quad y-3z+6=0.$

Ans.

Mathematics 2011 (Outside Delhi) SET III

Time allowed : 3 hours **Maximum marks : 100**

Note : Except for the following questions, all the remaining questions have been asked in previous sets.

SECTION – A

1. Evaluate : $\int \frac{e^{\tan^{-1}x}}{1+x^2}dx.$ **[1]**

Solution : Let $I = \int \frac{e^{\tan^{-1}x}}{1+x^2}dx$

Putting $\tan^{-1}x=t$

$\Rightarrow \quad \frac{1}{1+x^2}.dx = dt$

$\therefore \quad I = \int e^t\,dt = e^t+C$

$= e^{\tan^{-1}x}+C.$ **Ans.**

2. Write the angle between two vectors $\vec{a}$ and $\vec{b}$ magnitudes with $\sqrt{3}$ and 2 respectively having $\vec{a}.\vec{b}=\sqrt{6}$. **[1]**

Solution : Given,

$$\vec{a}.\vec{b} = \sqrt{6}, |\vec{a}|=\sqrt{3}, |\vec{b}|=2$$

Angle between $\vec{a}$ and $\vec{b}$ is

$$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}=\frac{\sqrt{6}}{\sqrt{3}\times 2}\times\frac{\sqrt{3}}{\sqrt{3}}$$

$$=\frac{\sqrt{18}}{3\times 2}=\frac{3\sqrt{2}}{3\times 2}$$

$\Rightarrow \quad \cos\theta = \frac{1}{\sqrt{2}}$

$\therefore \quad \theta = \cos^{-1}\frac{1}{\sqrt{2}} \Rightarrow \theta=\frac{\pi}{4}.$ **Ans.**

SECTION – B

11. Prove that : $\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(\frac{1}{8}\right)=\frac{\pi}{4}.$ **[4]**

Solution : L.H.S.

$$=\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(\frac{1}{8}\right)$$

$$=\tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{5}}{1-\frac{1}{2}.\frac{1}{5}}\right)+\tan^{-1}\left(\frac{1}{8}\right)$$

$$\left[\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$$

$$= \tan^{-1}\left(\frac{\frac{7}{10}}{\frac{9}{10}}\right)+\tan^{-1}\left(\frac{1}{8}\right)$$

$$= \tan^{-1}\left(\frac{7}{9}\right)+\tan^{-1}\left(\frac{1}{8}\right)$$

$$= \tan^{-1}\left(\frac{\frac{7}{9}+\frac{1}{8}}{1-\frac{7}{9}.\frac{1}{8}}\right)$$

$$= \tan^{-1}\left(\frac{\frac{65}{72}}{\frac{65}{72}}\right)$$

$$= \tan^{-1}(1) = \frac{\pi}{4} = \text{R.H.S.}$$

Hence Proved.

12. Using properties of determinants, solve the following for x:

$$\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0. \qquad [4]$$

Solution :

$$\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\begin{vmatrix} 3x+a & x & x \\ 3x+a & x+a & x \\ 3x+a & x & x+a \end{vmatrix} = 0$$

Taking $(3x + a)$ common from C_1, we get

$$(3x+a)\begin{vmatrix} 1 & x & x \\ 1 & x+a & x \\ 1 & x & x+a \end{vmatrix} = 0$$

Applying $R_2 \to R_2 - R_1$; and $R_3 \to R_3 - R_1$, we get

$$(3x+a)\begin{vmatrix} 1 & x & x \\ 0 & a & 0 \\ 0 & 0 & a \end{vmatrix} = 0$$

Expanding along C_1, we get

$(3x + a)\,[1(a^2 - 0)] = 0$

$\Rightarrow \quad a^2(3x + a) = 0$

$\Rightarrow \quad 3x + a = 0$

$\Rightarrow \quad x = -\frac{a}{3}.$ **Ans.**

13. Evaluate : $\int_0^1 \log\left(\frac{1}{x}-1\right)dx.$ **[4]**

Solution : Let $I = \int_0^1 \log\left(\frac{1-x}{x}\right)dx$ ...(i)

$$= \int_0^1 \log\left(\frac{1-(1-x)}{1-x}\right).dx$$

$$\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$= \int_0^1 \log\left(\frac{x}{1-x}\right).dx \qquad ...(ii)$$

Adding (i) and (ii), we get

$$2I = \int_0^1 \log\left(\frac{x}{1-x}.\frac{1-x}{x}\right).dx$$

$$= \int_0^1 \log 1.dx$$

$$= \int_0^1 .0\,dx = 0$$

$\therefore \quad I = 0$ **Ans.**

14. Solve the following differential equation :
$xdy + (y - x^3)dx = 0$ **[4]**

Solution : We have,

$x.dy + (y - x^3).dx = 0$

or $\quad x.\frac{dy}{dx}+(y-x^3) = 0$

or $\quad \frac{dy}{dx}+\frac{y}{x} = x^2$

This is a linear differential equation of the form

$$\frac{dy}{dx}+Py = Q,$$

Here, $\quad P = \frac{1}{x}$ and $Q = x^2$

$$\text{I.F.} = e^{\int P dx}$$

$$= e^{\int \frac{1}{x}dx} = e^{\log x} = x$$

$\therefore$ The solution is given by

$$y \times \text{I.F.} = \int Q.(\text{I.F.}).dx + C$$

$$yx = \int x^2.x.dx + C$$

$\Rightarrow \quad yx = \int x^3.dx + C$

$\Rightarrow \quad yx = \frac{x^4}{4}+C$

$\Rightarrow \quad y = \frac{x^3}{4}+\frac{C}{x}$ **Ans.**

SECTION – C

23. Using matrices, solve the following system of equations : [6]

$$x + 2y - 3z = -4$$
$$2x + 3y + 2z = 2$$
$$3x - 3y - 4z = 11$$

Solution : The given system of equations can be written in matrix form as

$$AX = B \quad ...(i)$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \\ 11 \end{bmatrix}$$

where $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} -4 \\ 2 \\ 11 \end{bmatrix}$

Now, $$|A| = \begin{vmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{vmatrix}$$

$$= 1(-12+6) - 2(-8-6) - 3(-6-9)$$

$$= -6 + 28 + 45 = 67 \neq 0$$

$\therefore A^{-1}$ exists.

For adj A, cofactors are

$A_{11} = -12 + 6 = -6$, $A_{12} = -(-8-6) = 14$,
$A_{13} = (-6-9) = -15$, $A_{21} = -(-8-9) = 17$,
$A_{22} = -4 + 9 = 5$, $A_{23} = -(-3-6) = 9$
$A_{31} = 4 + 9 = 13$, $A_{32} = -(2+6) = -8$,
$A_{33} = 3 - 4 = -1$

$$\therefore \quad \text{adj } A = \begin{bmatrix} -6 & 14 & -15 \\ 17 & 5 & 9 \\ 13 & -8 & -1 \end{bmatrix}^T$$

$$= \begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj } A$$

$$= \begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{bmatrix}$$

$\therefore \quad X = A^{-1}B$ [from (i)]

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{67}\begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{bmatrix}\begin{bmatrix} -4 \\ 2 \\ 11 \end{bmatrix}$$

$$= \frac{1}{67}\begin{bmatrix} 201 \\ -134 \\ 67 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

$\Rightarrow \quad x = 3, y = -2$ and $z = 1$. **Ans.**

24. Find the equation of the plane passing through the line of intersection of the planes $2x + y - z = 3$ and $5x - 3y + 4z + 9 = 0$ and parallel to the line $\frac{x-1}{2} = \frac{y-3}{4} = \frac{z-5}{5}$. [6]

Solution : Given planes are

$$2x + y - z - 3 = 0 \quad ...(i)$$
$$5x - 3y + 4z + 9 = 0 \quad ..(ii)$$

Any plane passing through the line of intersection of (i) and (ii) can be taken as

$$2x + y - z - 3 + \lambda(5x - 3y + 4z + 9) = 0$$
$$(2+5\lambda)x + (1-3\lambda)y + (-1+4\lambda)z - 3 + 9\lambda = 0 \quad ...(iii)$$

This plane is parallel to the line

$$\frac{x-1}{2} = \frac{y-3}{4} = \frac{z-5}{5}.$$

If $\quad 2(2+5\lambda) + 4(1-3\lambda) + 5(-1+4\lambda) = 0$

$\Rightarrow \quad 4 + 10\lambda + 4 - 12\lambda - 5 + 20\lambda = 0$

$\Rightarrow \quad 18\lambda + 3 = 0$

$\Rightarrow \quad \lambda = -\frac{1}{6}$

Substituting this value of λ in (iii), we get the required plane as

$$\left(2-\frac{5}{6}\right)x + \left(1+\frac{3}{6}\right)y + \left(-1-\frac{4}{6}\right)z - 3 - \frac{9}{6} = 0$$

$$\Rightarrow \quad \frac{7}{6}x + \frac{9}{6}y - \frac{10}{6}z - \frac{27}{6} = 0$$

$$\Rightarrow \quad 7x + 9y - 10z - 27 = 0.$$

Ans.

Mathematics 2011 (Delhi) SET I

Time allowed : 3 hours **Maximum marks : 100**

SECTION — A

1. State the reason for the relation R in the set {1, 2, 3} given by R = {(1, 2), (2, 1)} not to be transitive. [1]

Solution : In the case of transitive relation

If (a, b) and $(b, c) \in R$

$\Rightarrow \quad (a, c) \in R$

Here, (1, 2) and (2, 1) ∈ R but (1, 1) ∉ R.

So, R is not transitive. **Ans.**

2. Write the value of $\sin\left(\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right)$. [1]

Solution : We have, $\sin\left[\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right]$

$$= \sin\left[\frac{\pi}{3} - \sin^{-1}\left(-\sin\frac{\pi}{6}\right)\right]$$

$$= \sin\left[\frac{\pi}{3} - \sin^{-1}\left\{\sin\left(-\frac{\pi}{6}\right)\right\}\right]$$

$$= \sin\left(\frac{\pi}{3} + \frac{\pi}{6}\right) = \sin\frac{3\pi}{6} = \sin\frac{\pi}{2}$$

$$= 1.$$ **Ans.**

3. For a 2 × 2 matrix, A = $[a_{ij}]$, whose elements are given by $a_{ij} = \frac{i}{j}$, write the value of a_{12}. [1]

Solution : The order of the given matrix is 2 × 2. So,

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times 2}$$

$$a_{ij} = \frac{i}{j}$$

Put $i = 1$ and $j = 2$

$$\therefore \quad a_{12} = \frac{1}{2}.$$ **Ans.**

4. For what value of x, the matrix $\begin{bmatrix} 5-x & x+1 \\ 2 & 4 \end{bmatrix}$ is singular ? [1]

Solution : Matrix A is singular if $|A| = 0$

$$\Rightarrow \begin{vmatrix} 5-x & x+1 \\ 2 & 4 \end{vmatrix} = 0$$

$$\Rightarrow 4(5-x) - 2(x+1) = 0$$

$$\Rightarrow 20 - 4x - 2x - 2 = 0$$

$$\Rightarrow x = 3.$$ **Ans.**

5. Write A^{-1} for $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$. [1]

Solution : We know that,

$$A^{-1} = \frac{1}{|A|} \times \text{adj } A$$

$$\text{adj } A = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

and $|A| = 6 - 5 = 1$

$$A^{-1} = \frac{1}{1} \times \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}.$$ **Ans.**

6. Write the value of $\int \sec x(\sec x + \tan x)dx$. [1]

Solution : Let $I = \int \sec x(\sec x + \tan x)dx.$

$$= \int \sec^2 x\,dx + \int \sec x.\tan x\,dx$$

$$= \tan x + \sec x + C.$$ **Ans.**

7. Write the value of $\int \frac{dx}{x^2 + 16}$ [1]

Solution : Let $I = \int \frac{dx}{x^2 + (4)^2}$

$$= \frac{1}{4}\tan^{-1}\frac{x}{4} + C.$$ **Ans.**

8. For what value of 'a' the vectors $2\hat{i} - 3\hat{j} + 4\hat{k}$ and $a\hat{i} + 6\hat{j} - 8\hat{k}$ are collinear ? [1]

Solution : Let,

$\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$ and $\vec{b} = a\hat{i} + 6\hat{j} - 8\hat{k}$

For collinear vectors

$$\vec{b} = \lambda\vec{a}, \text{ here } \lambda = -2$$

$$a\hat{i} + 6\hat{j} - 8\hat{k} = \lambda(2\hat{i} - 3\hat{j} + 4\hat{k})$$

On comparing, we get

$$6 = -3\lambda$$

$$\Rightarrow \quad \lambda = -2$$

Also $\quad a = 2\lambda$

$$\Rightarrow \quad a = -4$$ **Ans.**

9. Write the direction cosines of the vector $-2\hat{i} + \hat{j} - 5\hat{k}$. [1]

Solution : Direction cosines of the vector

$-2\hat{i} + \hat{j} - 5\hat{k}$ are

$$\frac{-2}{\sqrt{(-2)^2+1^2+(-5)^2}}, \frac{1}{\sqrt{(-2)^2+1^2+(-5)^2}}, \frac{-5}{\sqrt{(-2)^2+1^2+(-5)^2}}$$

i.e., $\frac{-2}{\sqrt{30}}, \frac{1}{\sqrt{30}}, \frac{-5}{\sqrt{30}}$. **Ans.**

10. Write the intercept cut off by the plane $2x + y - z = 5$ on x-axis. **[1]**

Solution : Given equation of plane is

$$2x + y - z = 5 \quad ...(i)$$

Intercept on x-axis *i.e.,* $y = 0, z = 0$

$$2x + 0 - 0 = 5 \quad \text{[from (i)]}$$

$$\therefore \quad x = \frac{5}{2}.$$ **Ans.**

SECTION – B

11. Consider the binary operation * on the set {1, 2, 3, 4, 5} defined by $a * b = \min.\{a, b\}$. Write the operation table of the operation *.** **[4]**

12. Prove the following :

$$\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) = \frac{x}{2}, x \in \left(0, \frac{\pi}{4}\right)$$

[4]

Solution : L.H.S.

$$= \cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right); \quad x \in \left(0, \frac{\pi}{4}\right)$$

$$= \cot^{-1}\left(\frac{(\sqrt{1+\sin x}+\sqrt{1-\sin x})(\sqrt{1+\sin x}+\sqrt{1-\sin x})}{(\sqrt{1+\sin x}-\sqrt{1-\sin x})(\sqrt{1+\sin x}+\sqrt{1-\sin x})}\right)$$

$$= \cot^{-1}\left(\frac{1+\sin x+1-\sin x+2\sqrt{(1+\sin x)(1-\sin x)}}{(1+\sin x)-(1-\sin x)}\right)$$

$$= \cot^{-1}\left(\frac{2+2\sqrt{1-\sin^2 x}}{1+\sin x-1+\sin x}\right)$$

$$= \cot^{-1}\left(\frac{2(1+\cos x)}{2\sin x}\right)$$

$$= \cot^{-1}\left(\frac{2\cos^2 x/2}{2\sin x/2.\cos x/2}\right)$$

$$\left[\because 1+\cos x = 2\cos^2 x/2\right]$$

$$= \cot^{-1}\left(\frac{\cos x/2}{\sin x/2}\right)$$

**Answer is not given due to the change in present syllabus

$$= \cot^{-1}(\cot x/2) = x/2 = \text{R.H.S.}$$

Hence Proved.

OR

Find the value of $\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{x-y}{x+y}\right)$.

Solution : Given,

$$\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{x-y}{x+y}\right)$$

$$= \tan^{-1}\left[\frac{\frac{x}{y}-\left(\frac{x-y}{x+y}\right)}{1+\frac{x}{y}\left(\frac{x-y}{x+y}\right)}\right]$$

$$\left[\because \tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)\right]$$

$$= \tan^{-1}\left[\frac{(x^2+xy-xy+y^2)/(x+y)y}{(xy+y^2+x^2-xy)/(x+y)y}\right]$$

$$= \tan^{-1}\left[\frac{x^2+y^2}{x^2+y^2}\right] = \tan^{-1}(1)$$

$$= \tan^{-1}\left(\tan\frac{\pi}{4}\right) = \frac{\pi}{4}.$$ **Ans.**

13. Using properties of determinants, prove that

$$\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix} = 4a^2b^2c^2.$$ **[4]**

Solution : L.H.S.

$$= \begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix}$$

Taking a, b, c common from R_1, R_2 and R_3 respectively.

$$\Delta = abc\begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix}$$

Taking a, b, c common from C_1, C_2 and C_3 respectively.

$$\Delta = a^2b^2c^2\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2$

$$= a^2b^2c^2\begin{vmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix}$$

Expanding along C_1, we get

$a^2b^2c^2\,(2\times 2) = 4a^2b^2c^2 = $ R.H.S.

Hence Proved.

14. Find the value of '*a*' for which the function *f* defined as

$$f(x) = \begin{cases} a\sin\frac{\pi}{2}(x+1), & x\le 0 \\ \frac{\tan x - \sin x}{x^3}, & x>0 \end{cases}$$

is continuous at $x = 0$. **[4]**

Solution : L.H.L $= \lim_{x\to 0^-}\left\{a\sin\frac{\pi}{2}(x+1)\right\}$

$$= a\sin\frac{\pi}{2} = a\times 1 = a$$

$$\text{R.H.L} = \lim_{x\to 0^+}\left(\frac{\tan x-\sin x}{x^3}\right)$$

$$= \lim_{x\to 0^+}\frac{\sin x(1-\cos x)}{x^3.\cos x}$$

$$= \lim_{x\to 0^+}\frac{\sin x}{x}.\lim_{x\to 0^+}\frac{2\sin^2\frac{x}{2}}{x^2}.\lim_{x\to 0^+}\frac{1}{\cos x}$$

$$\left(\because \cos x = 1-2\sin^2\frac{x}{2}\right)$$

$$= \lim_{x\to 0^+}\frac{\sin x}{x}.\frac{1}{2}\lim_{x\to 0^+}\left[\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right]^2.\lim_{x\to 0^+}\frac{1}{\cos x}$$

$$= 1.\frac{1}{2}.1.1 = \frac{1}{2}$$

Since, $f(x)$ is continuous

$\therefore$ L.H.L = R.H.L.

$$a = \frac{1}{2}.$$ **Ans.**

15. Differentiate $x^{x\cos x} + \frac{x^2+1}{x^2-1}$ w.r.t. *x*. **[4]**

Solution : Given, $x^{x\cos x} + \frac{x^2+1}{x^2-1}$

Let $y = u + v$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx}+\frac{dv}{dx} \quad ...(i)$$

Here, $u = x^{x\cos x}$

Taking log of both sides, we get

$$\log u = x\cos x\log x$$

Differentiating w.r.t. *x*, we get

$$\frac{1}{u}\frac{du}{dx} = x\cos x\frac{d}{dx}(\log x) + \log x\,\frac{d}{dx}(x\cos x)$$

$$\Rightarrow \frac{d}{dx} = u\left[x\cos x\times\frac{1}{x}+\log x[x(-\sin x)+\cos x\times 1]\right]$$

$$= x^{x\cos x}[\cos x - x\log x\sin x + \log x\cos x] \quad ...(ii)$$

and $v = \frac{x^2+1}{x^2-1}$

Differentiating w.r.t. *x*, we get

$$\frac{d}{dx} = \frac{(x^2-1)\frac{d}{dx}(x^2+1)-(x^2+1)\frac{d}{dx}(x^2-1)}{(x^2-1)^2}$$

$$= \frac{(x^2-1)\times 2x-(x^2+1)\times 2x}{(x^2-1)^2}$$

$$= \frac{2x(x^2-1-x^2-1)}{(x^2-1)^2}$$

$$\Rightarrow \frac{d}{dx} = \frac{-4x}{(x^2-1)^2} \quad ...(iii)$$

From (i), (ii) and (iii),

$$\frac{dy}{dx} = x^{x\cos x}[\cos x - x\log x\sin x + \log x\cos x] - \left(\frac{4x}{(x^2-1)^2}\right).$$ **Ans.**

OR

If $x = a(\theta - \sin\theta)$, $y = a(1+\cos\theta)$ find $\frac{d^2y}{dx^2}$.

Solution : Given,

$$x = a(\theta - \sin\theta)$$

Differentiating w.r.t. θ, we get

$$\frac{dx}{d\theta} = a(1-\cos\theta) \quad ...(i)$$

and $y = a(1+\cos\theta)$

Differentiating w.r. t. θ, we get

$$\frac{dy}{d\theta} = a(-\sin\theta) \quad ...(ii)$$

$$\therefore \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a(-\sin\theta)}{a(1-\cos\theta)}$$

$$= \frac{-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}$$

$$\therefore \quad \frac{dy}{dx} = -\cot\frac{\theta}{2}$$

Differentiating w.r. t. x, we get

$$\frac{d^2y}{dx^2} = \frac{1}{2}\text{cosec}^2\frac{\theta}{2}.\frac{d\theta}{dx}$$

$$= \frac{1}{2\sin^2\frac{\theta}{2}}.\frac{1}{a(1-\cos\theta)}$$

$$= \frac{1}{2\sin^2\frac{\theta}{2}}.\frac{1}{2a\sin^2\frac{\theta}{2}}$$

$$= \frac{1}{4a}\text{cosec}^4\frac{\theta}{2}.$$ **Ans.**

16. Sand is pouring from a pipe at the rate of 12 cm³/s. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm? [4]

Solution : Let r be the radius, h be the height and V be the volume of the sand cone.

$$\therefore \quad \frac{dh}{dt} = 12\text{ cm}^3/\text{s and } h = \frac{1}{6}r$$

$$h = 4\text{ cm}$$

Volume of sand cone

$$V = \frac{1}{3}\pi r^2 h \Rightarrow \frac{dV}{dt} = \frac{d}{dt}\left(\frac{1}{3}\pi(6h)^2.h\right)$$

$$\Rightarrow \quad \frac{dh}{dt} = \frac{d}{dt}\left(\frac{1}{3}\pi\times 36\times h^3\right)$$

$$\therefore \quad \frac{dh}{dt} = \frac{3}{3}\times\pi h^2\times 36\times\frac{dh}{dt}$$

$$\Rightarrow \quad 12 = \pi(4)^2\times 36\times\frac{dh}{dt}$$

$$\left[\because \frac{dV}{dt} = 12\text{ cm}^3/\text{s}, h = 4\text{ cm}\right]$$

$$\Rightarrow \quad \frac{dh}{dt} = \frac{1}{48\pi}\text{cm/s.}$$ **Ans.**

OR

Find the points on the curve $x^2+y^2-2x-3=0$ at which the tangents are parallel to x–axis.

Solution : When the tangent is parallel to x–axis

$$\frac{dy}{dx} = 0$$

We have

$$x^2+y^2-2x-3 = 0 \quad \text{...(i)}$$

Differentiating w.r. t. x, we get

$$2x+2y\frac{dy}{dx}-2-0 = 0$$

$$\Rightarrow \quad 2y\frac{dy}{dx} = 2-2x$$

$$\Rightarrow \quad 2y\frac{dy}{dx} = 2(1-x)$$

$$\Rightarrow \quad y\times 0 = (1-x) \quad \left[\because \frac{dy}{dx} = 0\right]$$

$$\therefore \quad x = 1$$

Putting, $x = 1$ in equation (i), we get $y = \pm 2$

∴ The required points are (1, 2) and (1, –2).

Ans.

17. Evaluate : $\int\frac{5x+3}{\sqrt{x^2+4x+10}}dx.$ [4]

Solution : Let, $I = \int\frac{5x+3}{\sqrt{x^2+4x+10}}dx$

Let $\quad 5x+3 = A\frac{d}{dx}(x^2+4x+10)+B$

$$\Rightarrow \quad 5x+3 = A(2x+4)+B$$

$$\Rightarrow \quad 5x+3 = 2Ax+(4A+B)$$

Comparing the coefficient of x and constant on both sides, we get

$$2A = 5 \Rightarrow A = \frac{5}{2}$$

and $\quad 4A+B = 3$

$$\Rightarrow \quad 4\times\frac{5}{2}+B = 3$$

$$\Rightarrow \quad B = -7$$

$$I = \frac{5}{2}\int\frac{2x+4}{\sqrt{x^2+4x+10}}dx$$

$$-7\int\frac{1}{\sqrt{x^2+4x+10}}dx \quad \text{...(i)}$$

Putting $x^2+4x+10 = t$ in first term,

$$\Rightarrow \quad (2x+4)dx = dt$$

$$\therefore \quad I = \frac{5}{2}\int\frac{1}{\sqrt{t}}dt - 7\int\frac{1}{\sqrt{x^2+4x+10}}dx$$

$$= \frac{5}{2}\times 2\sqrt{t} - 7\frac{1}{\sqrt{(x+2)^2+(\sqrt{6})^2}}$$

$$= 5\sqrt{x^2+4x+10}$$

$$-7\log\left|(x+2)+\sqrt{(x+2)^2+(\sqrt{6})^2}\right|+C$$
$$= 5\sqrt{x^2+4x+10}\cdot$$
$$-7\log\left|(x+2)+\sqrt{x^2+4x+10}\right|+C.$$
Ans.

OR

Evaluate : $\int \frac{2x}{\sqrt{(x^2+1)(x^2+3)}}dx.$

Solution : Let, I

$$= \int \frac{2x}{\sqrt{(x^2+1)(x^2+3)}}dx$$

Putting $x^2 = t$

$\Rightarrow \quad 2x\,dx = dt$

$\therefore \quad I = \int \frac{1}{(1+t)(3+t)}dt$

Let $\frac{1}{(1+t)(3+t)} = \frac{A}{1+t}+\frac{B}{3+t}$

$$1 = A(3+t)+B(1+t)$$

Putting $t = -3$

$\Rightarrow \quad B = -\frac{1}{2}$

and putting $\quad t = -1$

$\Rightarrow \quad A = \frac{1}{2}$

$\therefore \quad \frac{1}{(1+t)(3+t)} = \frac{1/2}{1+t}-\frac{1/2}{3+t}$

$$I = \frac{1}{2}\int\frac{dt}{1+t}-\frac{1}{2}\int\frac{1}{3+t}dt$$
$$= \frac{1}{2}\log|1+t|-\frac{1}{2}\log|3+t|+C$$
$$= \frac{1}{2}\log|1+x^2|-\frac{1}{2}\log|3+x^2|+C$$
$$= \frac{1}{2}\log\left|\frac{1+x^2}{3+x^2}\right|+C.$$
Ans.

18. Solve the following differential equation :

$$e^x \tan y\,dx + (1-e^x)\sec^2 y.dy = 0 \qquad [4]$$

Solution : Given, $e^x \tan y\,dx + (1-e^x)\sec^2 y.dy = 0$

$\Rightarrow \quad e^x \tan y\,dx = -(1-e^x)\sec^2 y.dy$

$\Rightarrow \quad \frac{e^x}{1-e^x}dx = -\frac{\sec^2 y}{\tan y}dy$

Integrating both sides, we get

$$\int\frac{e^x}{1-e^x}dx = -\int\frac{\sec^2 y}{\tan y}dy$$

Putting $1-e^x = t \Rightarrow e^x.dx = -dt$

and $\quad \tan y = z \Rightarrow \sec^2 y.dy = dz$

$\Rightarrow \quad -\int\frac{dt}{t} = -\int\frac{dz}{z}$

$\Rightarrow \quad \log|t| = \log|z|+C$

$\therefore \quad \log|1-e^x| = \log|\tan y|+C.$ **Ans.**

19. Solve the following differential equation :

$$\cos^2 x\frac{dy}{dx}+y = \tan x. \qquad [4]$$

Solution : Given,

$$\cos^2 x\frac{dy}{dx}+y = \tan x$$

Dividing by $\cos^2 x$ on both sides, we get

$$\frac{dy}{dx}+\frac{y}{\cos^2 x} = \frac{\tan x}{\cos^2 x}$$
$$\frac{dy}{dx}+\sec^2 x.y = \tan x\sec^2 x \qquad ...(i)$$

This is a linear differential equation of the form

$$\frac{dy}{dx}+Py = Q$$

where $P = \sec^2 x$ and $Q = \tan x\sec^2 x$

$$\text{I.F.} = e^{\int P dx} = e^{\int\sec^2 x dx} = e^{\tan x}$$

$\therefore$ Solution is given by

$$y\times\text{I.F.} = \int\text{I.F.}\times Q.\,dx$$
$$y.e^{\tan x} = \int e^{\tan x}\times\tan x\sec^2 x\,dx$$

Putting $\tan x = t$

$\Rightarrow \quad \sec^2 x.\,dx = dt$

$\Rightarrow \quad y\times e^{\tan x} = \int e^t\times t.dt$

$$= t\int e^t dt-\int\left[\frac{d}{dt}(t)\int e^t.dt\right]dt$$

$\Rightarrow \quad y\,e^{\tan x} = t\times e^t - e^t + C$

$\Rightarrow \quad y\,e^{\tan x} = \tan x\,e^{\tan x}-e^{\tan x}+C$

$\Rightarrow \quad y.e^{\tan x} = e^{\tan x}[\tan x-1]+C$

$\therefore \quad y = (\tan x-1)+Ce^{-\tan x}.$

Ans.

20. Find a unit vector perpendicular to each of the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$, where $\vec{a} = 3\hat{i}+2\hat{j}+2\hat{k}$

and $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$. [4]

Solution : Given,

$$\vec{a} = 3\hat{i} + 2\hat{j} + 2\hat{k}$$

and $$\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$$

$$\therefore \quad \vec{a} + \vec{b} = 3\hat{i} + 2\hat{j} + 2\hat{k} + \hat{i} + 2\hat{j} - 2\hat{k}$$
$$= 4\hat{i} + 4\hat{j} + 0\hat{k}$$

and $$\vec{a} - \vec{b} = 3\hat{i} + 2\hat{j} + 2\hat{k} - \hat{i} - 2\hat{j} + 2\hat{k}$$
$$= 2\hat{i} + 0\hat{j} + 4\hat{k}$$

Now,

$$(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 4 & 0 \\ 2 & 0 & 4 \end{vmatrix}$$
$$= 16\hat{i} - 16\hat{j} + (-8)\hat{k}$$
$$= 16\hat{i} - 16\hat{j} - 8\hat{k}$$

$$\therefore |(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})| = \sqrt{(16)^2 + (-16)^2 + (-8)^2}$$
$$= \sqrt{256 + 256 + 64}$$
$$= \sqrt{576} = 24$$

$\therefore$ Required perpendicular unit vector

$$= \frac{(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})}{|(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})|}$$
$$= \frac{16\hat{i} - 16\hat{j} - 8\hat{k}}{24}$$
$$= \frac{16}{24}\hat{i} - \frac{16}{24}\hat{j} - \frac{8}{24}\hat{k} = \frac{2}{3}\hat{i} - \frac{2}{3}\hat{j} - \frac{1}{3}\hat{k}.$$ **Ans.**

21. Find the angle between the following pair of lines :

$$\frac{-x+2}{-2} = \frac{y-1}{7} = \frac{z+3}{-3}$$

and $$\frac{x+2}{-1} = \frac{2y-8}{4} = \frac{z-5}{4}$$

and check whether the lines are parallel or perpendicular. **[4]**

Solution : Given lines are $\frac{-x+2}{-2} = \frac{y-1}{7} = \frac{z+3}{-3}$

and $$\frac{x+2}{-1} = \frac{2y-8}{4} = \frac{z-5}{4}$$

Writing equation in standard form

$$\frac{-x+2}{-2} = \frac{y-1}{7} = \frac{z+3}{-3} \quad \text{...(i)}$$

and $$\frac{x+2}{-1} = \frac{y-2}{2} = \frac{z-5}{4} \quad \text{...(ii)}$$

Here, $$\vec{b_1} = 2\hat{i} + 7\hat{j} - 3\hat{k}$$
$$\vec{b_2} = -\hat{i} + 2\hat{j} + 4\hat{k}$$

Angle between lines is

$$\cos\theta = \frac{\vec{b_1}.\vec{b_2}}{|\vec{b_1}||\vec{b_2}|}$$

$$\vec{b_1} \cdot \vec{b_2} = (2\hat{i} + 7\hat{j} - 3\hat{k}).(-\hat{i} + 2\hat{j} + 4\hat{k})$$
$$= -2 + 14 - 12 = 0$$

$$|\vec{b_1}| = \sqrt{4 + 49 + 9} = \sqrt{62}$$
$$|\vec{b_2}| = \sqrt{1 + 4 + 16} = \sqrt{21}$$

$$\cos\theta = \frac{0}{\sqrt{62}.\sqrt{21}} = 0$$

$$\Rightarrow \quad \cos\theta = \cos\frac{\pi}{2}$$

$$\therefore \quad \theta = \frac{\pi}{2}.$$

For perpendicularity,

$$a_1 = 2, b_1 = 7, c_1 = -3$$
$$a_2 = -1, b_2 = 2, c_2 = 4$$

We know that

$$a_1 \times a_2 + b_1 \times b_2 + c_1 \times c_2$$
$$= 2 \times (-1) + 7 \times 2 - 3 \times 4$$
$$= -14 + 14 = 0$$

Hence, the lines are perpendicular. **Ans.**

22. Probabilities of solving a specific problem independently by A and B are $\frac{1}{2}$ and $\frac{1}{3}$ respectively. If both try to solve the problem independently, find the probability that (i) the problem is solved (ii) exactly one of them solves the problem. **[4]**

Solution : Given,

$$P(A) = \frac{1}{2}$$

and $$P(B) = \frac{1}{3}$$

$$\therefore \quad P(\bar{A}) = 1 - P(A)$$
$$= 1 - \frac{1}{2} = \frac{1}{2}$$

$$P(\bar{B}) = 1 - P(B)$$

$$= 1-\frac{1}{3}=\frac{2}{3}$$

(i) P (the problem is solved)

$$= P(\text{at least one of them will solve})$$
$$= P(A\cup B)$$
$$= 1-P(\overline{A\cup B})$$
$$= 1-P(\overline{A}\cap\overline{B})$$
$$= 1-P(\overline{A})P(\overline{B})$$
$$= 1-\frac{1}{2}\times\frac{2}{3}=\frac{2}{3}.$$

(ii) P (exactly one of them solved)

$$= P(A)P(\overline{B})+P(\overline{A})P(B)$$
$$=\frac{1}{2}\times\frac{2}{3}+\frac{1}{2}\times\frac{1}{3}=\frac{1}{2}\left(\frac{2}{3}+\frac{1}{3}\right)=\frac{1}{2}\times 1=\frac{1}{2}.$$

Ans.

SECTION – C

23. Using matrix method, solve the following system of equations :

$$\frac{2}{x}+\frac{3}{y}+\frac{10}{z}=4,\ \frac{4}{x}-\frac{6}{y}+\frac{5}{z}=1,$$
$$\frac{6}{x}+\frac{9}{y}-\frac{20}{z}=2;\ x, y, z\neq 0.$$ **[6]**

Solution : The given system of equations are

$$\frac{2}{x}+\frac{3}{y}+\frac{10}{z}=4,\ \frac{4}{x}-\frac{6}{y}+\frac{5}{z}=1,$$
$$\frac{6}{x}+\frac{9}{y}-\frac{20}{z}=2$$

Given equations can be written as AX = B

$$\begin{bmatrix}2 & 3 & 10\\4 & -6 & 5\\6 & 9 & -20\end{bmatrix}\begin{bmatrix}1/x\\1/y\\1/z\end{bmatrix}=\begin{bmatrix}4\\1\\2\end{bmatrix},$$

where $A=\begin{bmatrix}2 & 3 & 10\\4 & -6 & 5\\6 & 9 & -20\end{bmatrix}, X=\begin{bmatrix}1/x\\1/y\\1/z\end{bmatrix}$ and $B=\begin{bmatrix}4\\1\\2\end{bmatrix}$

$$|A| = [2+(120-45)-3(-80-30)+10(36+36)]$$
$$= [150+330+720]=1200\neq 0$$

$\therefore A^{-1}$ exists.

Cofactors of A,

$$A_{11}=(120-45)=75,\quad A_{13}=(36+36)=72,$$
$$A_{12}=-(-80-30)=110,$$
$$A_{21}=-(-60-90)=150,\quad A_{23}=-(18-18)=0,$$
$$A_{22}=(-40-60)=-100,$$
$$A_{31}=(15+60)=75,\quad A_{33}=(-12-12)=-24,$$
$$A_{32}=-(10-40)=30.$$

$$\therefore\quad \text{adj } A=\begin{bmatrix}75 & 110 & 72\\150 & -100 & 0\\75 & 30 & -24\end{bmatrix}^T$$
$$=\begin{bmatrix}75 & 150 & 75\\110 & -100 & 30\\72 & 0 & -24\end{bmatrix}$$

$$A^{-1}=\frac{1}{|A|}\times\text{adj } A$$

$$\Rightarrow\quad A^{-1}=\frac{1}{1200}\begin{bmatrix}75 & 150 & 75\\110 & -100 & 30\\72 & 0 & -24\end{bmatrix}$$

$$\therefore\quad X=A^{-1}B$$

$$\Rightarrow\quad \begin{bmatrix}1/x\\1/y\\1/z\end{bmatrix}=\frac{1}{1200}\begin{bmatrix}75 & 150 & 75\\110 & -100 & 30\\72 & 0 & -24\end{bmatrix}\begin{bmatrix}4\\1\\2\end{bmatrix}$$

$$=\frac{1}{1200}\begin{bmatrix}600\\400\\240\end{bmatrix}=\begin{bmatrix}1/2\\1/3\\1/5\end{bmatrix}$$

$$\frac{1}{x}=\frac{1}{2}\Rightarrow x=2$$
$$\frac{1}{y}=\frac{1}{3}\Rightarrow y=3$$

and $\frac{1}{z}=\frac{1}{5}\Rightarrow z=5.$ **Ans.**

OR

Using elementary transformations, find the inverse of the matrix $\begin{bmatrix}1 & 3 & -2\\-3 & 0 & -1\\2 & 1 & 0\end{bmatrix}$

Solution : Given

$$A=\begin{bmatrix}1 & 3 & -2\\-3 & 0 & -1\\2 & 1 & 0\end{bmatrix}$$

We have A = IA

$$\begin{bmatrix}1 & 3 & -2\\-3 & 0 & -1\\2 & 1 & 0\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}A$$

Applying $R_2\to R_2+3R_1$ and $R_3\to R_3-2R_1$, we get

$$\begin{bmatrix}1 & 3 & -2\\0 & 9 & -7\\0 & -5 & 4\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\3 & 1 & 0\\-2 & 0 & 1\end{bmatrix}A$$

Applying $R_2 \to \frac{R_2}{9}$, we get

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -\frac{7}{9} \\ 0 & -5 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{9} & 0 \\ -2 & 0 & 1 \end{bmatrix} A$$

Applying $R_3 \to R_3 + 5R_2$, we get

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -\frac{7}{9} \\ 0 & 0 & \frac{1}{9} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{9} & 0 \\ -\frac{1}{3} & \frac{5}{9} & 1 \end{bmatrix} A$$

Applying $R_3 \to 9 R_3$, we get

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -\frac{7}{9} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{9} & 0 \\ -3 & 5 & 9 \end{bmatrix} A$$

Applying $R_1 \to R_1 - 3R_2$, we get

$$\begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & -\frac{7}{9} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{9} & 0 \\ -3 & 5 & 9 \end{bmatrix} A$$

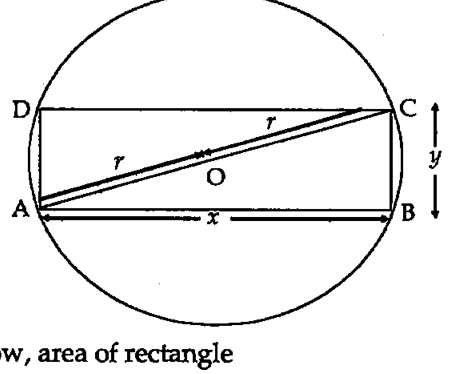
Applying $R_1 \to R_1 - \frac{1}{3} R_3$, $R_2 \to R_2 + \frac{7}{9} R_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -3 \\ -2 & 4 & 7 \\ -3 & 5 & 9 \end{bmatrix} A$$

$$\therefore \quad A^{-1} = \begin{bmatrix} 1 & -2 & -3 \\ -2 & 4 & 7 \\ -3 & 5 & 9 \end{bmatrix}$$ **Ans.**

24. Show that of all the rectangles inscribed in a given fixed circle, the square has the maximum area. [6]

Solution : Let length of rectangle be x and breadth of rectangle be y.



Now, area of rectangle

$$A = l \times b = xy \quad \text{...(i)}$$

Let r be radius of circle

In Δ ABC,

$$AC^2 = AB^2 + BC^2$$
$$(2r)^2 = x^2 + y^2$$
$$\Rightarrow \quad 4r^2 - x^2 = y^2 \quad \text{...(ii)}$$
$$A = xy$$
$$\Rightarrow \quad A^2 = x^2y^2 \quad \text{...(iii)}$$

From (ii) and (iii),

$$A^2 = x^2(4r^2 - x^2)$$
$$= 4r^2x^2 - x^4$$

Let $A^2 = f(x)$, $f(x) = 4r^2x^2 - x^4$

Differentiating w.r.t. x, we get

$$f'(x) = 8r^2x - 4x^3 \quad \text{...(iv)}$$

For maximum or minimum,

$$f'(x) = 0$$
$$\Rightarrow \quad 0 = 8r^2x - 4x^3$$
$$\Rightarrow \quad 4x^3 = 8r^2x$$
$$\Rightarrow \quad x^2 = 2r^2$$
$$\Rightarrow \quad x = \sqrt{2}r$$

Again differentiating equation (iv) w.r.t. x, we get

$$f''(x) = 8r^2 - 12x^2$$
$$f''(x)_{x=\sqrt{2}r} = 8r^2 - 12 \times 12r^2$$
$$= 8r^2 - 24r^2$$
$$= -16r^2 < 0.$$

$\therefore f(x)$ or A is maximum at $x = \sqrt{2}\, r$

Putting $r = \frac{x}{\sqrt{2}}$ in equation (ii), we get

$$4\frac{x^2}{2} - x^2 = y^2$$
$$\Rightarrow \quad 2x^2 - x^2 = y^2$$
$$\Rightarrow \quad x^2 = y^2$$
$$\Rightarrow \quad x = y$$

$\therefore$ Rectangle having maximum area is a square.

Hence Proved.

25. Using integration find the area of the triangular region whose sides have equations $y = 2x + 1$, $y = 3x + 1$ and $x = 4$. [6]

Solution : Given equations are

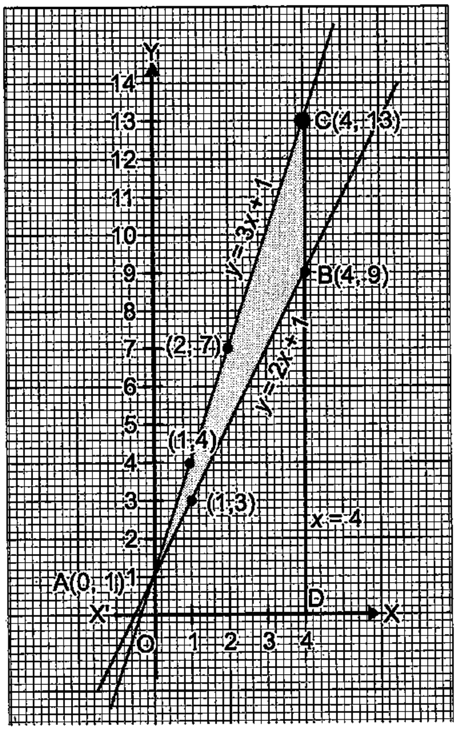
$$y = 2x + 1 \quad \text{...(i)}$$
$$y = 3x + 1 \quad \text{...(ii)}$$

Table for line (i)

x	0	1	2	4
y	1	3	5	9

Table for line (ii),

x	0	1	2	4
y	1	4	7	13



Area of triangular region ABC
= Area of the region OACDO – Area of the region OABDO

$= \int_0^4 [y \text{ line (ii)} - y \text{ line (i)}]dx$

$= \int_0^4 [(3x+1)-(2x+1)]dx$

$= \int_0^4 (3x+1-2x-1) = \int_0^4 x.dx$

$= \left[\frac{x^2}{2}\right]_0^4 = \left[\frac{(4)^2}{2}\right] = 8$ sq. units. **Ans.**

26. Evaluate : $\int_0^{\pi/2} 2\sin x \cos x \tan^{-1}(\sin x)\,dx.$ **[6]**

Solution : Let, $I = \int_0^{\pi/2} 2\sin x\cos x\tan^{-1}(\sin x)dx$

Putting $\sin x = t$

$\Rightarrow \quad \cos x\, dx = dt$

If $x = 0, t = 0$

and $x = \frac{\pi}{2}, t = 1$

$$I = 2\int_0^1 t \times \tan^{-1} t.dt$$

$$= 2\left[\frac{t^2}{2} \times \tan^{-1} t\right]_0^1 - 2\int_0^1 \frac{1}{1+t^2} \times \frac{t^2}{2} dt$$

$$= 2 \times \frac{1}{2} \times \tan^{-1}(1) - \int_0^1 \frac{1+t^2-1}{1+t^2} dt$$

$$= 1 \times \frac{\pi}{4} - \int_0^1 \left(\frac{1+t^2}{1+t^2} - \frac{1}{1+t^2}\right) dt$$

$$= \frac{\pi}{4} - \left[t - \tan^{-1} t\right]_0^1$$

$$= \frac{\pi}{4} - 1 + \tan^{-1}(1) = \frac{2\pi}{4} - 1 = \left(\frac{\pi}{2} - 1\right). \textbf{ Ans.}$$

OR

Evaluate : $\int_0^{\pi/2} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x} dx.$

Solution : Let, $I = \int_0^{\pi/2} \frac{x\sin x\cos x}{\sin^4 x + \cos^4 x} dx$ ...(i)

$$I = \int_0^{\pi/2} \frac{\left(\frac{\pi}{2}-x\right)\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)}{\sin^4\left(\frac{\pi}{2}-x\right)+\cos^4\left(\frac{\pi}{2}-x\right)} dx$$

$$\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$I = \int_0^{\pi/2} \frac{\left(\frac{\pi}{2}-x\right)\cos x.\sin x}{\cos^4 x + \sin^4 x} dx. \quad \text{...(ii)}$$

On adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{x\sin x\cos x + \left(\frac{\pi}{2}-x\right)\cos x.\sin x}{\sin^4 x + \cos^4 x} dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} \frac{\frac{\pi}{2}\sin x\cos x}{\sin^4 x + \cos^4 x} dx$$

$$= \frac{\pi}{2}.\frac{1}{2}\int_0^{\pi/2} \frac{2\sin x\cos x}{(\sin^2 x)^2 + (1-\sin^2 x)^2} dx$$

Putting $\sin^2 x = t$

$\Rightarrow \quad 2\sin x\cos x\, dx = dt$

when $x = 0$ then $t = 0$

when $x = \frac{\pi}{2}$ then $t = \sin^2\frac{\pi}{2} = (1)^2 = 1$

$$\therefore \quad 2I = \frac{\pi}{4}\int_0^1 \frac{dt}{t^2 + (1-t)^2}$$

$$\Rightarrow \quad 2I = \frac{\pi}{4}\int_0^1 \frac{dt}{t^2+1+t^2-2t}$$

$$\Rightarrow \quad 2I = \frac{\pi}{4}\int_0^1 \frac{dt}{2t^2-2t+1}$$

$$\Rightarrow \quad 2I = \frac{\pi}{4}\int_0^1 \frac{dt}{2\left(t^2-t+\frac{1}{2}\right)}$$

$$\Rightarrow \quad 2I = \frac{\pi}{8}\int_0^1 \frac{dt}{\left(t-\frac{1}{2}\right)^2-\frac{1}{4}+\frac{1}{2}}$$

$$\Rightarrow \quad 2I = \frac{\pi}{8}\int_0^1 \frac{dt}{\left(t-\frac{1}{2}\right)^2+\frac{1}{4}}$$

$$\Rightarrow \quad 2I = \frac{\pi}{8}\int_0^1 \frac{dt}{\left(t-\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}$$

$$\Rightarrow \quad 2I = \frac{\pi}{8}\cdot\frac{1}{1/2}\left[\tan^{-1}\left(\frac{t-\frac{1}{2}}{\frac{1}{2}}\right)\right]_0^1$$

$$\Rightarrow \quad 2I = \frac{\pi}{4}\left[\tan^{-1}\left(\frac{1/2}{1/2}\right)-\tan^{-1}\left(\frac{-1/2}{1/2}\right)\right]$$

$$\Rightarrow \quad 2I = \frac{\pi}{4}[\tan^{-1}(1)+\tan^{-1}(1)]$$

$$\Rightarrow \quad 2I = \frac{\pi}{4}\left[\frac{\pi}{4}+\frac{\pi}{4}\right]$$

$$\Rightarrow \quad 2I = \frac{\pi}{4}\left[\frac{\pi}{2}\right]$$

$$\Rightarrow \quad 2I = \frac{\pi^2}{8}$$

$$\therefore \quad I = \frac{\pi^2}{16}.$$

Ans.

27. Find the equation of the plane which contains the line of intersection of the planes $\vec{r}.(\hat{i}+2\hat{j}+3\hat{k})-4=0$, $\vec{r}.(2\hat{i}+\hat{j}-\hat{k})+5=0$ and which is perpendicular to the plane $\vec{r}.(5\hat{i}+3\hat{j}-6\hat{k})+8=0$. [6]

Solution : The given equations are

$$\vec{r}.(\hat{i}+2\hat{j}+3\hat{k})-4 = 0$$

$$\vec{r}.(2\hat{i}+\hat{j}-\hat{k})+5 = 0$$

The equation of plane containing them is

$$\vec{r}.[\hat{i}+2\hat{j}+3\hat{k}]-4+\lambda(\vec{r}.(2\hat{i}+\hat{j}-\hat{k})+5)=0 \quad \text{...(i)}$$

$$\Rightarrow \quad \vec{r}.[(1+2\lambda)\hat{i}+(2+\lambda)\hat{j}+(3-\lambda)\hat{k}] = 4-5\lambda$$

This is perpendicular to plane

$$\vec{r}.(5\hat{i}+3\hat{j}-6\hat{k})+8=0$$

$$\Rightarrow \quad (1+2\lambda)\times 5+(2+\lambda)\times 3+(3-\lambda)(-6) = 0$$

$$\Rightarrow \quad 5+10\lambda+6+3\lambda-18+6\lambda = 0$$

$$\Rightarrow \quad \lambda = \frac{7}{19}$$

Put $\lambda = \frac{7}{19}$ in equation (i), we get

The required equation of the plane is

$$\vec{r}.\left[\frac{33}{19}\hat{i}+\frac{45}{19}\hat{j}+\frac{50}{19}\hat{k}\right] = \frac{41}{19}$$

$$\Rightarrow \quad \vec{r}.(33\hat{i}+45\hat{j}+50\hat{k}) = 41.$$

Ans.

28. A factory makes tennis rackets and cricket bats. A tennis racket takes 1.5 hours of machine time and 3 hours of craftsman's time in its making while a cricket bat takes 3 hours of machine time and 1 hour of craftsman's time. In a day, the factory has the availability of not more than 42 hours of machine time and 24 hours of craftsman's time. If the profit on a racket and on a bat is ₹ 20 and ₹ 10 respectively, find the number of tennis rackets and circket bats that the factory must manufacture to earn the maximum profit. Make it as an L.P.P. and solve graphically. [6]

Solution : Let x be the number of tennis rackets and y that of cricket bats produced in one day in the factory.

Item	Number	Machine hours	Craftsman hours	Maximize (Profit)
Tennis Racket	x	1.5	3	₹ 20
Cricket Bats	y	3	1	₹10
Total		42	24	

Maximize $Z = 20x + 10y$

Subject to constraints :

$$1.5x + 3y \le 42$$

$$3x + y \le 24$$

$$x \ge 0, y \ge 0$$

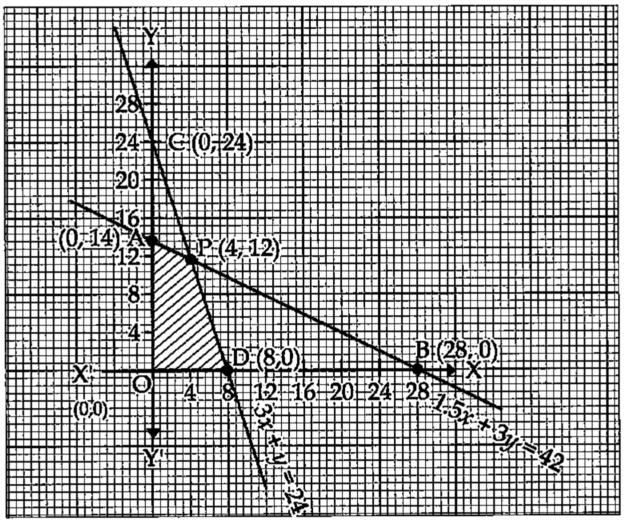
First we draw the lines AB and CD whose equations are

$$1.5x + 3y = 42 \quad \text{...(i)}$$

	A	B
x	0	28
y	14	0

and $\quad 3x + y = 24 \quad$ ...(ii)

	C	D
x	0	8
y	24	0



The feasible region is ODPAO which is shaded in the figure.

The vertices of the feasible region are O(0, 0), D(8, 0), P (4, 12) and A(0, 14).

P is the point of intersection of the lines

$$1.5x + 3y = 42 \text{ and } 3x + y = 24$$

Solving these equations, we get point P(4, 12).

The value of objective function Z = 20x + 10y at these vertices are as follows :

Corner Points	$Z = 20x + 10y$
O (0, 0)	0
D (8, 0)	$20 \times 8 + 0 = 160$
P (4, 12)	$20 \times 4 + 10 \times 12$ $= 200$ (maximum)
A (0, 14)	$0 + 10 \times 14 = 140$

For maximum profit ₹ 200, 4 tennis rackets and 12 cricket bats should be produced. **Ans.**

29. Suppose 5% of men and 0.25% of women have grey hair. A grey haired person is selected at random. What is the probability of this person being male ? Assume that there are equal number of males and females. [6]

Solution : Let E_1 and E_2 be the number of men and women respectively.

∴ Probability of men and women

$$P(E_1) = P(E_2) = \frac{1}{2} \text{ or } 0.5$$

Let A be event of selecting a grey person.

$$P(A/E_1) = 5\% = 0.05$$

$$P(A/E_2) = 0.25\% = 0.0025$$

∴ Probability of person being male (By Bayes' Theorem)

$$P(E_1/A) = \frac{P(E_1) \times P(A/E_1)}{P(E_1) \times P(A/E_1) + P(E_2) \times P(A/E_2)}$$

$$= \frac{0.5 \times 0.05}{0.5 \times 0.05 + 0.5 \times 0.0025}$$

$$= \frac{0.025}{0.025 + 0.00125}$$

$$= \frac{0.025}{0.02625} = 0.95.$$ **Ans.**

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Mathematics 2011 (Delhi) SET II

Time allowed : 3 hours **Maximum marks : 100**

Note : Except for the following questions, all the remaining questions have been asked in previous set.

SECTION — A

9. Write the value of $\tan^{-1}\left[\tan \frac{3\pi}{4}\right]$. [1]

Solution : We know that $\tan^{-1}(\tan x) = x$ if $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, which is the principal value branch of $\tan^{-1} x$.

Here, $\frac{3\pi}{4} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Now, $\tan^{-1}\left[\tan \frac{3\pi}{4}\right]$ can be written as

$$\tan^{-1}\left[\tan \frac{3\pi}{4}\right] = \tan^{-1}\left[\tan\left(\pi - \frac{\pi}{4}\right)\right]$$

$$= \tan^{-1}\left[-\tan\frac{\pi}{4}\right]$$

$$= \tan^{-1}\left[\tan\left(-\frac{\pi}{4}\right)\right] = \frac{-\pi}{4}$$

where $\frac{-\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\therefore \tan^{-1}\left(\tan\frac{3\pi}{4}\right) = \tan^{-1}\left[\tan\left(\frac{-\pi}{4}\right)\right] = \frac{-\pi}{4}.$$

Ans.

10. Write the value of $\int \frac{\sec^2 x}{\text{cosec}^2 x} dx$. **[1]**

Solution : Given $\int \frac{\sec^2 x}{\text{cosec}^2 x} dx = \int \frac{\sin^2 x}{\cos^2 x} dx$

$$= \int \tan^2 x\, dx = \int (\sec^2 x - 1) dx$$

$$(\because \sec^2 x - \tan^2 x = 1)$$

$$= \int \sec^2 x\, dx - \int 1.dx = \tan x - x + C.$$ **Ans.**

SECTION – B

15. Form the differential equation of the family of parabolas having vertex at the origin and axis along positive y-axis. **[4]**

Solution : The equation of the family of parabolas having vertex at the origin and axis along positive y-axis.

$$x^2 = 4ay \quad ...(i)$$

Differentiating w.r. t. x, we get

$$2x = 4a\frac{dy}{dx}$$

$$\Rightarrow \quad 2x.\frac{dx}{dy} = 4a \quad ...(ii)$$

Putting the value of $4a$ in equation (i), we get

$$x^2 = \left(2x.\frac{dx}{dy}\right).(y) \Rightarrow x\frac{dy}{dx} = 2y.$$ **Ans.**

16. Find a vector of magnitude 5 units and parallel to the resultant of the vectors $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$. **[4]**

Solution: We have $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$.

Let $\vec{c}$ be the resultant of $\vec{a}$ and $\vec{b}$, then

$$\vec{c} = \vec{a} + \vec{b}$$

$$= (2\hat{i} + 3\hat{j} - \hat{k}) + (\hat{i} - 2\hat{j} + \hat{k})$$

$$= 3\hat{i} + \hat{j} + 0\hat{k}$$

$$\therefore \quad |\vec{c}| = \sqrt{3^2 + 1^2 + 0} = \sqrt{9+1} = \sqrt{10}$$

Now, unit vector in the direction of $\vec{c}$ is

$$\therefore \quad \hat{c} = \frac{\vec{c}}{|\vec{c}|} = \frac{3\hat{i} + \hat{j}}{\sqrt{10}}$$

Hence, the vector of magnitude 5 units and parallel to the resultant of vector $\vec{a}$ and $\vec{b}$ is

$$\pm 5\hat{c} = \pm 5.\frac{(3\hat{i} + \hat{j})}{\sqrt{10}}.$$ **Ans.**

19. If the function $f(x)$ given by

$$f(x) = \begin{cases} 3ax + b, & \text{if } x > 1 \\ 11, & \text{if } x = 1 \\ 5ax - 2b, & \text{if } x < 1 \end{cases}$$ **is continuous at $x = 1$, find the values of a and b.** **[4]**

Solution : Since f is continuous at $x = 1$, therefore,

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1) \quad ...(i)$$

$$\text{L.H.L.} = \lim_{x \to 1^-} (5ax - 2b) = 5a - 2b$$

$$\text{R.H.L.} = \lim_{x \to 1^+} (3ax + b) = (3a + b)$$

$$\therefore \quad 5a - 2b = 3a + b = 11 \quad \text{[Using (i)]}$$

$$3a + b = 11 \quad ...(ii)$$

and $$5a - 2b = 11 \quad ...(iii)$$

Multiplying (ii) by 2 and adding it to (iii), we get

$$2(3a + b) + 5a - 2b = (2 \times 11) + 11$$

$$\Rightarrow \quad 6a + 2b + 5a - 2b = 22 + 11$$

$$\Rightarrow \quad 11a = 33$$

$$\Rightarrow \quad a = 3$$

Substituting $a = 3$ in (ii), we get

$$3 \times 3 + b = 11$$

$$\Rightarrow \quad b = 11 - 9 = 2$$

Thus, $a = 3$ and $b = 2$ will make $f(x)$ continuous at $x = 1$. **Ans.**

20. Using properties of determinants, prove the following :

$$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = xyz(x-y)(y-z)(z-x).$$ **[4]**

Solution : L.H.S. $= \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix}$

Taking common x, y, z from C_1, C_2 and C_3, we get

$$= xyz \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

Applying $C_1 \to C_1 - C_2$; $C_2 \to C_2 - C_3$, we get

$$\Delta = xyz \begin{vmatrix} 0 & 0 & 1 \\ x-y & y-z & z \\ x^2-y^2 & y^2-z^2 & z^2 \end{vmatrix}$$

Expanding along C_3, we get

$$= xyz \begin{vmatrix} x-y & y-z \\ x^2-y^2 & y^2-z^2 \end{vmatrix}$$

Taking $(x-y)$ common from C_1 and $y-z$ common from C_2

$$= xyz(x-y)(y-z) \begin{vmatrix} 1 & 1 \\ x+y & y+z \end{vmatrix}$$

$$= xyz\,(x-y)\,(y-z)\,(y+z-x-y)$$

$$= xyz\,(x-y)\,(y-z)\,(z-x) = \text{R.H.S.}$$

Hence Proved.

SECTION – C

23. Bag I contains 3 red and 4 black balls and Bag II contains 5 red and 6 black balls. One ball is drawn at random from one of the bags and is found to be red. Find the probability that it was drawn from Bag II. [6]

Solution :

Bag I	
R	B
3	4

Bag II	
R	B
5	6

Let E_1 be the event that Bag I is chosen and E_2 be the event that Bag II is chosen.
Let A be the event that the chosen ball is red

$\therefore \quad P(E_1) = \frac{1}{2} = P(E_2)$

$\therefore$ Probability of red ball from Bag I,

$$P(A/E_1) = \frac{3}{3+4} = \frac{3}{7};$$

and probability of red ball from Bag II,

$$P(A/E_2) = \frac{5}{5+6} = \frac{5}{11}$$

By Bayes' theorem,

$$P(E_2/A) = \frac{P(E_2).P(A/E_2)}{P(E_1).P(A/E_1)+P(E_2).P(A/E_2)}$$

$$= \frac{\frac{1}{2}\times\frac{5}{11}}{\frac{1}{2}\times\frac{3}{7}+\frac{1}{2}\times\frac{5}{11}} = \frac{\frac{5}{11}}{\frac{3}{7}+\frac{5}{11}}$$

$$= \frac{\frac{5}{11}}{\frac{33+35}{77}} = \frac{\frac{5}{11}}{\frac{68}{77}} = \frac{5\times 77}{68\times 11} = \frac{35}{68}. \qquad \textbf{Ans.}$$

29. Show that of all the rectangles with a given perimeter, the square has the largest area. [6]

Solution : Let x and y be the length and breadth of the rectangle whose perimeter is given $4a$ (say)

$\therefore \quad$ Area, $A = xy \quad$ ...(i)

$2x + 2y = 4a$ (Given)

$\Rightarrow \quad y = 2a - x \quad$ ...(ii)

Putting y in (i), we get

$A = x(2a - x)$

$\therefore \quad A = 2ax - x^2 \quad$ ...(iii)

Differentiating w.r. t. x, we get

$\therefore \quad \frac{dA}{dx} = 2a - 2x \quad$ ...(iv)

For maxima or minima,

$\frac{dA}{dx} = 0$

$\therefore \quad 2a - 2x = 0$

$\Rightarrow \quad 2x = 2a$

$\Rightarrow \quad x = a$

Again differentiating w.r. t. x, we get

$\frac{d^2A}{dx^2} = -2$

$\therefore \quad \left[\frac{d^2A}{dx^2}\right]_{\text{at } x=a} = -2 < 0$

$\therefore$ Area A is maximum at $x = a$

$\therefore \quad y = 2a - a$

$\Rightarrow \quad y = a$

Hence, It is proved that all the rectangles with a given perimeter, the square has the largest area.

Hence Proved.

Mathematics 2011 (Delhi) SET III

Time allowed : 3 hours **Maximum marks : 100**

Note : Except for the following questions, all the remaining questions have been asked in previous sets.

SECTION — A

1. Write the value of $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$. [1]

Solution : We know that $\cos^{-1}(\cos x) = x$ if $x \in [0, \pi]$, which is the principal value branch of $\cos^{-1} x$.

Here, $\frac{7\pi}{6} \notin [0, \pi]$.

Now, $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$ can be written as :

$$\cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \cos^{-1}\left[\cos\left(2\pi - \frac{7\pi}{6}\right)\right]$$

$[\because \cos(2\pi - x) = \cos x]$

$$= \cos^{-1}\left[\cos\frac{5\pi}{6}\right],$$

where $\frac{5\pi}{6} \in [0, \pi]$

$$\therefore \cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \cos^{-1}\left(\cos\frac{5\pi}{6}\right) = \frac{5\pi}{6}.$$ **Ans.**

2. Write the value of $\int \frac{2 - 3\sin x}{\cos^2 x} dx$. [1]

Solution : Let $I = \int \frac{2-3\sin x}{\cos^2 x} dx$.

$$= \int\left(\frac{2}{\cos^2 x} - \frac{3\sin x}{\cos^2 x}\right)dx$$

$$= \int 2\sec^2 x\, dx - 3\int \tan x \sec x\, dx$$

$$= 2\tan x - 3\sec x + C.$$ **Ans.**

SECTION — B

11. Using properties of determinants, prove the following :

$$\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x+4)(4-x)^2.$$ [4]

Solution : Taking L.H.S.

Let $\Delta = \begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$.

Applying $R_1 \to R_1 + R_2 + R_3$, we get

$$\Delta = \begin{vmatrix} 5x+4 & 5x+4 & 5x+4 \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$$

$$= (5x+4)\begin{vmatrix} 1 & 1 & 1 \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, we get

$$\Delta = (5x+4)\begin{vmatrix} 1 & 0 & 0 \\ 2x & -x+4 & 0 \\ 2x & 0 & -x+4 \end{vmatrix}$$

Taking $(4 - x)$ common from C_2 and C_3, we get

$$= (5x+4)(4-x)(4-x)\begin{vmatrix} 1 & 0 & 0 \\ 2x & 1 & 0 \\ 2x & 0 & 1 \end{vmatrix}$$

Expanding along C_3, we get

$$\Delta = (5x+4)(4-x)^2\begin{vmatrix} 1 & 0 \\ 2x & 1 \end{vmatrix}$$

$$= (5x + 4)(4 - x)^2 = \text{R.H.S.}$$

Hence Proved.

12. Find the value of a and b such that the following function $f(x)$ is a continuous function :

$$f(x) = \begin{cases} 5, & x \le 2 \\ ax + b, & 2 < x < 10 \\ 21, & x \ge 10 \end{cases}$$ [4]

Solution : If f is a continuous function, f is continuous at all real numbers.

In particular, $f(x)$ is continuous at $x = 2$ and $x = 10$.

Since f is continuous at $x = 2$, we obtain

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2)$$

$$\Rightarrow \lim_{x \to 2^-} (5) = \lim_{x \to 2^+} (ax + b) = 5$$

$$\Rightarrow 5 = 2a + b = 5$$

$$\Rightarrow 2a + b = 5 \quad \text{...(i)}$$

Since f is continuous at $x = 10$, we obtain

$$\lim_{x \to 10^-} f(x) = \lim_{x \to 10^+} f(x) = f(10)$$

$$\Rightarrow \lim_{x\to 10^-}(ax+b) = \lim_{x\to 10^+}(21) = 21$$

$$\Rightarrow 10a + b = 21 \quad ...(ii)$$

On subtracting equation (i) from equation (ii), we obtain

$$8a = 16$$

$$\Rightarrow a = 2$$

By putting $a = 2$ in equation (i), we obtain

$$2 \times 2 + b = 5$$

$$\Rightarrow 4 + b = 5$$

$$\Rightarrow b = 1$$

Therefore, the values of a and b for which $f(x)$ is a continuous function are 2 and 1 respectively.

Ans.

13. Solve the following differential equation :

$$(1 + y^2)(1 + \log x)\,dx + x\,dy = 0 \quad [4]$$

Solution : Given $(1 + y^2)(1 + \log x)\,dx + x\,dy = 0$

$\Rightarrow (1 + y^2)(1 + \log x)\,dx = -x\,dy$

$$\Rightarrow \frac{1}{x}(1+\log x)dx = \frac{-1}{(1+y^2)}dy$$

Integrating on both sides, we get

$$\int \frac{1}{x}(1+\log x)dx = \int -\frac{1}{(1+y^2)}dy \quad ...(i)$$

Let $\log x = t$

$$\Rightarrow \frac{1}{x}dx = dt$$

Now, from (i)

$$\int (1+t)dt = -\int \frac{1}{1+y^2}dy$$

$$\Rightarrow t + \frac{t^2}{2} = -\tan^{-1} y + C_1$$

$$\Rightarrow \log x + \frac{(\log x)^2}{2} = -\tan^{-1} y + C_1$$

$$\Rightarrow 2\log x + (\log x)^2 = -2\tan^{-1} y + 2C_1$$

$$\Rightarrow (\log x)^2 + 2\log x + 2\tan^{-1} y - 2C_1 = 0$$

$$\Rightarrow (\log x)^2 + 2\log x + 2\tan^{-1} y + C = 0, \text{ where } -2C_1 = C$$

Ans.

14. If two vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}| = 2$, $|\vec{b}| = 1$ and $\vec{a}.\vec{b} = 1$, then find the value of $(3\vec{a} - 5\vec{b}).(2\vec{a} + 7\vec{b})$. **[4]**

Solution : It is given that $|\vec{a}| = 2$, $|\vec{b}| = 1$ and $\vec{a}.\vec{b} = 1$

$$\therefore (3\vec{a}-5\vec{b}).(2\vec{a}+7\vec{b})$$

$$= 3\vec{a}.2\vec{a} + 3\vec{a}.7\vec{b} - 5\vec{b}.2\vec{a} - 5\vec{b}.7\vec{b}$$

$$= 6|\vec{a}|^2 + 21(\vec{a}.\vec{b}) - 10(\vec{b}.\vec{a}) - 35|\vec{b}|^2$$

$$= 6(2)^2 + 21(1) - 10(1) - 35(1)^2 \quad [\vec{b}.\vec{a} = \vec{a}.\vec{b}]$$

$$= 24 + 21 - 10 - 35 = 0.$$

Ans.

SECTION – C

23. A man is known to speak truth 3 out of 4 times. He throws a die and reports that it is a six. Find the probability that it is actually a six. **[6]**

Solution : Let E be the event that the man reports that six occurs in the throwing of a die and let S_1 be the event that six occurs and S_2 be the event that six does not occur,

$P(S_1)$ = Probability that six occurs $= \frac{1}{6}$

$P(S_2)$ = Probability that six does not occur $= \frac{5}{6}$

$P(E/S_1)$ = Probability that the man reports that six occurs when six has actually occurred on the die
= Probability that the man speaks the truth
$= \frac{3}{4}$

$P(E/S_2)$ = Probability that the man reports that six occurs when six has not actually occurred on the die
= Probability that the man does not speaks the truth
$= 1 - \frac{3}{4} = \frac{1}{4}$

Using Bayes' theorem,

$P(S_1/E)$ = Probability that the report of the man that six has occurred is actually a six

$$= \frac{P(S_1)P(E/S_1)}{P(S_1)P(E/S_1) + P(S_2)P(E/S_2)}$$

$$= \frac{\frac{1}{6}\times\frac{3}{4}}{\frac{1}{6}\times\frac{3}{4} + \frac{5}{6}\times\frac{1}{4}} = \frac{1}{8}\times\frac{24}{8} = \frac{3}{8}$$

Thus, the required probability is $\frac{3}{8}$.

Ans.

24. Show that of all the rectangles of given area, the square has the smallest perimeter. **[6]**

Solution : Let l and b respectively be the length and the breadth of the rectangle of given area A.

$$A = l \times b$$

$$\Rightarrow b = \frac{A}{l} \quad ...(i)$$

Perimeter of the rectangle, $P = 2(l + b)$

$$\therefore P = 2\left(l + \frac{A}{l}\right)$$

Differentiating w.r.t. l, we get

$$\frac{dP}{dl} = 2\left(1-\frac{A}{l^2}\right) \quad ...(ii)$$

Again differentiating, we get

$$\frac{d^2P}{dl^2} = 2\left(0+\frac{2A}{l^3}\right)$$

$$\Rightarrow \quad \frac{d^2P}{dl^2} = \frac{4A}{l^3} \quad ...(iii)$$

For maximum or minimum perimeter, $\frac{dP}{dl} = 0$

$$\Rightarrow \quad 2\left(1-\frac{A}{l^2}\right) = 0$$

$$\Rightarrow \quad 1-\frac{A}{l^2} = 0$$

$$\Rightarrow \quad \frac{A}{l^2} = 1$$

$$\Rightarrow \quad A = l^2$$

Substituting the value of A in equation (i), we get

$$b = \frac{A}{l} = \frac{l^2}{l} = l$$

$$\therefore \quad b = l = \sqrt{A}$$

From (iii),

$$\frac{d^2P}{dl^2} = \frac{4l^2}{l^3}$$

$$\frac{d^2P}{dl^2} = \frac{4}{l} = \frac{4}{\sqrt{A}}$$

The value of $\sqrt{A}$ cannot be negative,

$$\therefore \quad \frac{d^2P}{dl^2} > 0$$

Hence, of all the rectangles of given area, the square has the smallest perimeter.

Hence Proved.

Mathematics 2012 (Outside Delhi) Set I

Time allowed : 3 Hours **Maximum marks : 100**

SECTION — A

1. **The binary operation * : R × R → R is defined as $a * b = 2a + b$. Find (2 * 3) * 4.**** **[1]**

2. **Find the principal value of $\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$** **[1]**

Solution : $\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$ ...(i)

We know that the range of principal value of $\tan^{-1}\theta$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

$$\therefore \quad \tan^{-1}\sqrt{3} = \tan^{-1}\left(\tan\frac{\pi}{3}\right) = \frac{\pi}{3}$$

Now $\sec^{-1}(-2)$

We know that the range of principal value of $\sec^{-1}\theta$ is $[0, \pi] - \{\pi/2\}$

$$\Rightarrow \quad \sec^{-1}(-2) = \pi - \sec^{-1}(2) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

$[\because \sec^{-1}(-x) = \pi - \sec^{-1} x]$

$\therefore$ From equation (i)

$$\tan^{-1}\sqrt{3} - \sec^{-1}(-2) = \frac{\pi}{3} - \frac{2\pi}{3} = -\frac{\pi}{3}.$$ **Ans.**

3. **Find the value of $x + y$ from the following equation:**

$$2\begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$ **[1]**

Solution : Given that,

$$2\begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

$$\begin{bmatrix} 2x & 10 \\ 14 & 2y-6 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

$$\begin{bmatrix} 2x+3 & 10-4 \\ 14+1 & 2y-6+2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

$$\begin{bmatrix} 2x+3 & 6 \\ 15 & 2y-4 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

Equating values, we get x and y

$$2x + 3 = 7$$
$$\Rightarrow \quad 2x = 7 - 3$$
$$\Rightarrow \quad 2x = 4$$
$$\Rightarrow \quad x = 2$$

**Answer is not given due to the change in present syllabus

and $\quad 2y - 4 = 14$

$$\Rightarrow \quad 2y = 14 + 4$$
$$\Rightarrow \quad 2y = 18$$
$$\Rightarrow \quad y = 9$$
$$\Rightarrow \quad x = 2, y = 9$$
$$\therefore \quad x + y = 2 + 9 = 11.$$

Ans.

4. **If $A^T = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$, then find $A^T - B^T$.** **[1]**

Solution : $A^T = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

$$\Rightarrow \quad B^T = \begin{bmatrix} -1 & 1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix}$$

$$\Rightarrow \quad A^T - B^T = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3+1 & 4-1 \\ -1-2 & 2-2 \\ 0-1 & 1-3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 3 \\ -3 & 0 \\ -1 & -2 \end{bmatrix}.$$ **Ans.**

5. **Let A be a square matrix of order 3 × 3. Write the value of $|2A|$, where $|A| = 4$.** **[1]**

Solution : In a square matrix of order 3 × 3,

$$|KA| = K^3|A|$$
$$\therefore \quad |2A| = 2^3|A|$$
$$= 8 \times 4 = 32.$$ **Ans.**

6. **Evaluate : $\int_0^2 \sqrt{4 - x^2}\, dx$.** **[1]**

Solution : We know that,

$$\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

Therefore,

$$\int_0^2 \sqrt{2^2 - x^2}\,dx = \left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2$$

$$= \left[\frac{2}{2}\sqrt{4-4} + 2\sin^{-1}\frac{2}{2}\right] - [0 + 2\sin^{-1} 0]$$

$$= 2\sin^{-1}(1) - 2\sin^{-1}(0)$$

$$= 2\left(\frac{\pi}{2}\right) - 0$$

$$= \pi.$$

Ans.

7. Given $\int e^x(\tan x + 1)\sec x\, dx = e^x f(x) + c$. Write $f(x)$ satisfying the above. [1]

Solution : $\int e^x(\tan x + 1)\sec x\, dx = e^x f(x) + c$

Taking L.H.S.

Let $I = \int e^x(\tan x + 1)\sec x\, dx$

$= \int e^x(\sec x \tan x + \sec x)\, dx$

$= e^x \sec x + c$

$[\because \int e^x\{f(x) + f'(x)\}\,dx = e^x f(x) + c\ ]$

On comparing L.H.S. and R.H.S.

we get, $f(x) = \sec x$. **Ans.**

8. Write the value of $(\hat{i}\times\hat{j}).\hat{k} + \hat{i}.\hat{j}$ [1]

Solution : $(\hat{i}\times\hat{j}).\hat{k} + \hat{i}.\hat{j}$

$= \hat{k}.\hat{k} + 0 \quad \left[\because (\hat{i}\times\hat{j}) = \hat{k} \text{ and } \hat{k}.\hat{k} = 1\right]$

$= 1.$ **Ans.**

9. Find the scalar components of the vector $\vec{AB}$ with initial point A (2, 1) and terminal point B (– 5, 7). [1]

Solution : Position vector of A

$\vec{OA} = 2\hat{i} + \hat{j}$

Position vector of B

$\vec{OB} = -5\hat{i} + 7\hat{j}$

$\vec{AB} = (-5\hat{i} + 7\hat{j}) - (2\hat{i} + \hat{j})$

$= -7\hat{i} + 6\hat{j}$

Scalar components of vector $\vec{AB}$ are – 7 and 6.

Ans.

10. Find the distance of the plane $3x - 4y + 12z = 3$ from the origin. [1]

Solution : We know that the distance of the point (x_1, y_1, z_1) from the plane $ax + by + cz + d = 0$ is

$$= \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{|-3|}{\sqrt{3^2 + (-4)^2 + 12^2}}$$

$$= \frac{|-3|}{\sqrt{169}}$$

$$= \frac{3}{13}.$$

Ans.

SECTION — B

11. Prove the following :

$$\cos\left(\sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}\right) = \frac{6}{5\sqrt{13}}.$$ **[4]**

Solution : Taking L.H.S.

$$= \cos\left(\sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}\right)$$

$$= \cos\left(\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{2}{\sqrt{13}}\right)$$

$$\left(\because \cot^{-1}\frac{3}{2} = \sin^{-1}\frac{2}{\sqrt{13}}\right)$$

...(i)

We know that,

$$\sin^{-1}x + \sin^{-1}y = \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$$

Therefore, equation (i) becomes,

$$= \cos\left[\sin^{-1}\left(\frac{3}{5}\sqrt{1-\frac{4}{13}} + \frac{2}{\sqrt{13}}\sqrt{1-\frac{9}{25}}\right)\right]$$

$$= \cos\left[\sin^{-1}\left(\frac{3}{5}\sqrt{\frac{9}{13}} + \frac{2}{\sqrt{13}}\sqrt{\frac{16}{25}}\right)\right]$$

$$= \cos\left[\sin^{-1}\left(\frac{3}{5}.\frac{3}{\sqrt{13}} + \frac{2}{\sqrt{13}}.\frac{4}{5}\right)\right]$$

$$= \cos\left[\sin^{-1}\left(\frac{9}{5\sqrt{13}} + \frac{8}{5\sqrt{13}}\right)\right]$$

$$= \cos\left[\sin^{-1}\left(\frac{17}{5\sqrt{13}}\right)\right]$$

$$= \cos\left[\cos^{-1}\left(\frac{6}{5\sqrt{13}}\right)\right]$$

$$\left[\because \sin^{-1}\left(\frac{17}{5\sqrt{13}}\right) = \cos^{-1}\left(\frac{6}{5\sqrt{13}}\right)\right]$$

$$= \frac{6}{5\sqrt{13}} = \text{R.H.S.}$$ **Hence Proved.**

12. Using properties of determinants, show that

$$\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc$$ **[4]**

Solution : Taking L.H.S. $= \begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$

Applying $R_1 \to R_1 + R_2 + R_3$

$$= \begin{vmatrix} 2(b+c) & 2(a+c) & 2(a+b) \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$

$$= 2\begin{vmatrix} b+c & a+c & a+b \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$

Applying $R_1 \to R_1 - R_2$

$$= 2\begin{vmatrix} c & 0 & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$

On Expanding along R_1

$= 2[c\{(c+a)(a+b) - bc\} - 0 + a\{bc - c(c+a)\}]$

$= 2[c\{ca + bc + a^2 + ab - bc\} + a\{bc - c^2 - ca\}]$

$= 2[c^2a + ca^2 + abc + abc - c^2a - ca^2]$

$= 2[abc + abc]$

$= 4abc =$ R.H.S. **Hence Proved.**

13. Show that $f : N \to N$, given by

$$f(x) = \begin{cases} x+1, \text{ if } x \text{ is odd} \\ x-1, \text{ if } x \text{ is even} \end{cases}$$

is both one-one and onto. **[4]**

Solution : Let $x, y \in R$ such that $f(x) = f(y)$

$\therefore \quad f(x) = f(y)$

If x and y are odd, then

$f(x) = f(y)$

$\Rightarrow \quad x + 1 = y + 1$

$\Rightarrow \quad x = y$

If x and y are even, then

$f(x) = f(y)$

$\Rightarrow \quad x - 1 = y - 1$

$\Rightarrow \quad x = y$

If x is odd and y is even, then

$f(x) = x + 1$ is even and $f(y) = y + 1$ is odd.

$\therefore \quad x \neq y \Rightarrow f(x) \neq f(y)$

Similarly if x is even and y is odd, then

$x \neq y \Rightarrow f(x) \neq f(y)$

Hence, $f : N \to N$ is one-one

Also, $\quad f(1) = 1 + 1 = 2$

$f(1) = 2 \quad (\because 1 \text{ is odd})$

If x is odd number, then $\exists$ an even natural number, $x + 1 \in N$ such that,

$f(x + 1) = x + 1 - 1$

$= x$

If x is even number, then there exist a odd natural number $x - 1 \in N$ such that,

$f(x - 1) = x - 1 + 1$

$= x$

Hence for every $y \in N \; \exists \; x \in N$ such that $f(x) = y$, so f is onto.

Hence f is both one-one and onto.

Hence Proved.

OR

Consider the binary operations $* : R \times R \to R$ and $o : R \times R \to R$ defined as $a * b = |a - b|$ and $aob = a$ for all $a, b \in R$. Show that '*' is commutative but not associative, 'o' is associative but not commutative.**

14. If $x = \sqrt{a^{\sin^{-1} t}}, y = \sqrt{a^{\cos^{-1} t}}$, show that $\frac{dy}{dx} = -\frac{y}{x}$. **[4]**

Solution : Given, $x = \sqrt{a^{\sin^{-1} t}}$ and $y = \sqrt{a^{\cos^{-1} t}}$

Squaring both sides

$$x^2 = a^{\sin^{-1} t} \quad \text{...(i)}$$

$$y^2 = a^{\cos^{-1} t} \quad \text{...(ii)}$$

Differentiating both w.r.t. t

$$2x\frac{dx}{dt} = a^{\sin^{-1} t} \log a . \frac{1}{\sqrt{1-t^2}},$$

and $$2y\frac{dy}{dx} = a^{\cos^{-1} t} \log a . \frac{-1}{\sqrt{1-t^2}}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{a^{\sin^{-1} t} \log a}{2x\sqrt{1-t^2}},$$

$$\Rightarrow \quad \frac{dy}{dt} = -\frac{a^{\cos^{-1} t} \log a}{2y\sqrt{1-t^2}}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= -\frac{a^{\cos^{-1} t} \log a / 2y\sqrt{1-t^2}}{a^{\sin^{-1} t} \log a / 2x\sqrt{1-t^2}}$$

**Answer is not given due to the change in present syllabus

$$= \frac{-xa^{\cos^{-1}t}}{ya^{\sin^{-1}t}}$$

$$= \frac{-x}{y} \cdot \frac{y^2}{x^2} \quad \text{[Using (i) \& (ii)]}$$

$$= \frac{-y}{x} \quad \textbf{Hence Proved.}$$

OR

Differentiate $\tan^{-1}\left[\frac{\sqrt{1+x^2}-1}{x}\right]$ with respect to x.

Solution : Let $y = \tan^{-1}\left[\frac{\sqrt{1+x^2}-1}{x}\right]$

Put $x = \tan A \Rightarrow A = \tan^{-1} x$ ...(i)

$$y = \tan^{-1}\left[\frac{\sqrt{1+\tan^2 A}-1}{\tan A}\right]$$

$$y = \tan^{-1}\left[\frac{\sqrt{\sec^2 A}-1}{\tan A}\right]$$

$$= \tan^{-1}\left[\frac{\sec A - 1}{\tan A}\right]$$

$$= \tan^{-1}\left[\frac{1-\cos A}{\sin A}\right]$$

$$= \tan^{-1}\left[\frac{2\sin^2\left(\frac{A}{2}\right)}{2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)}\right]$$

$$= \tan^{-1}\left[\tan\left(\frac{A}{2}\right)\right]$$

$$\Rightarrow \quad y = \frac{A}{2}$$

Put the value of A from equation (i),

$$y = \frac{\tan^{-1} x}{2}$$

On differentiating w.r.t. x

$$\frac{dy}{dx} = \frac{1}{2(1+x^2)}. \quad \textbf{Ans.}$$

15. If $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$, $0 < t < \frac{\pi}{2}$, find $\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}$ and $\frac{d^2y}{dx^2}$. [4]

Solution : Given, $x = a(\cos t + t\sin t)$

On differentiating w.r.t. t, we get

$$\frac{dx}{dt} = a[-\sin t + t\cos t + \sin t]$$

$$\Rightarrow \quad \frac{dx}{dt} = a(t\cos t)$$

Again differentiating w.r.t. t, we get

$$\frac{d^2x}{dt^2} = a[\cos t - t\sin t] \quad \text{...(i)}$$

Given, $y = a(\sin t - t\cos t)$

On differentiating w.r.t. t, we get

$$\frac{dy}{dt} = a[\cos t + t\sin t - \cos t]$$

$$\Rightarrow \quad \frac{dy}{dt} = a(t\sin t)$$

Again differentiating w.r.t. t, we get

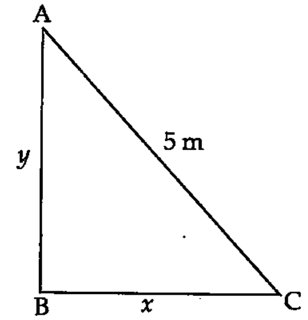
$$\frac{d^2y}{dt^2} = a[\sin t + t\cos t] \quad \text{...(ii)}$$

On dividing equation (ii) by equation (i), we get

$$\frac{d^2y}{dx^2} = \frac{t\cos t + \sin t}{\cos t - t\sin t}. \quad \textbf{Ans.}$$

16. A ladder 5 m long is leaning against a wall. The bottom of the ladder is pulled along the ground, away from the wall, at the rate of 2 cm/s. How fast is its height on the wall decreasing when the foot of the ladder is 4 m away from the wall ? [4]

Solution : Let AC = 5 m be the ladder and y be the height of the wall at which the ladder touches. Also, let the foot of the ladder be at C whose distance from the wall is x m



It is given that

$$\frac{dx}{dt} = 2 \text{ cm/sec} = \frac{2}{100} \text{ m/sec} \quad \text{...(i)}$$

As we know that Δ ABC is right angled triangle,

$$\therefore \quad x^2 + y^2 = 5^2 \quad \text{...(ii)}$$

On differentiating, we get

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\Rightarrow \quad 2x\frac{dx}{dt} = -2y\frac{dy}{dt}$$

$$\Rightarrow \quad x \times \frac{2}{100} = -y\frac{dy}{dt} \quad \text{[from (i)]}$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{-2x}{100y}$$

When $x = 4$ m from eq. (ii)

$$y^2 = 25 - x^2$$

$$\Rightarrow \quad y = \sqrt{25-16}$$

$$\Rightarrow \quad y = 3 \text{ m}$$

Thus, $$\frac{dy}{dt} = \frac{-2\times 4}{100\times 3} = \frac{-2}{75}$$

[Negative sign shows that height of ladder on the wall is decreasing at the rate of $\frac{2}{75}$ m/s] **Ans.**

17. Evaluate : $\int_{-1}^{2} |x^3 - x|\, dx$. **[4]**

Solution : Let $f(x) = x^3 - x$

$$= x(x-1)(x+1)$$

Sign of $f(x)$ for different value of x will be different

$$I = \int_{-1}^{0} |x^3 - x| dx + \int_{0}^{1} |x^3 - x| dx + \int_{1}^{2} |x^3 - x| dx$$

$$= \int_{-1}^{0} (x^3 - x) dx - \int_{0}^{1} (x^3 - x) dx + \int_{1}^{2} (x^3 - x) dx$$

$$\left[\begin{array}{l} \because \text{At } (-1, 0),\ x(x-1)(x+1) = (-ve)(-ve)(+ve) = +ve \\ \text{At } (0,1),\ x(x-1)(x+1) = (+ve)(-ve)(+ve) = -ve \\ \text{At } (1, 2),\ x(x-1)(x+1) = (+ve)(+ve)(+ve) = +ve \end{array}\right]$$

$$= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} - \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{0}^{1} + \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{1}^{2}$$

$$= 0 - \left[\frac{1}{4} - \frac{1}{2}\right] - \left[\left\{\frac{1}{4} - \frac{1}{2}\right\} - 0\right] + \left[\frac{16}{4} - \frac{4}{2}\right] - \left[\frac{1}{4} - \frac{1}{2}\right]$$

$$= \frac{11}{4}.$$ **Ans.**

OR

Evaluate : $\int_{0}^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx$.

Solution : Let $I = \int_{0}^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$ ...(i)

Use the following $\int_{0}^{a} f(x) dx = \int_{0}^{a} f(a-x) dx$

Therefore,

$$I = \int_{0}^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1 + \cos^2(\pi - x)} dx$$

$$\Rightarrow \quad I = \int_{0}^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x} dx \quad ...(ii)$$

Adding equation (i) and (ii), we get

$$2I = \int_{0}^{\pi} \frac{x \sin x}{1+\cos^2 x} dx + \int_{0}^{\pi} \frac{(\pi - x)\sin x}{1 + \cos^2 x} dx$$

$$= \int_{0}^{\pi} \left[\frac{x \sin x + \pi \sin x - x \sin x}{1 + \cos^2 x}\right] dx$$

$$\Rightarrow \quad 2I = \int_{0}^{\pi} \frac{\pi \sin x}{1 + \cos^2 x} dx$$

$$\Rightarrow \quad I = \frac{\pi}{2}\int_{0}^{\pi} \frac{\sin x}{1+\cos^2 x} dx \quad ...(iii)$$

Let $\cos x = t$

When $x = 0, t = \cos 0 = 1$

$x = \pi, t = \cos \pi = -1$

$$-\sin x = \frac{dt}{dx}$$

$$\sin x\, dx = -dt$$

Put the value of $\sin x\, dx$ in equation (iii), we get

$$I = \frac{\pi}{2}\int_{1}^{-1} \frac{-1}{1+t^2} dt$$

$$= \frac{-\pi}{2}\left[\tan^{-1} t\right]_{1}^{-1}$$

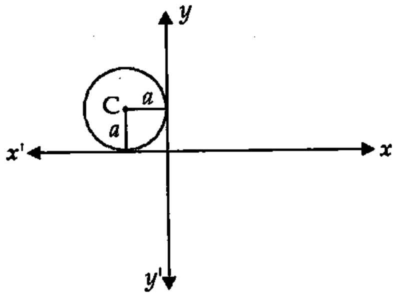
$$= \frac{-\pi}{2}\left[\tan^{-1}(-1) - \tan^{-1}(1)\right]$$

$$= \frac{-\pi}{2}\left[\frac{-\pi}{4} - \frac{\pi}{4}\right]$$

$$\Rightarrow \quad I = \frac{\pi^2}{4}.$$ **Ans.**

18. Form the differential equation of the family of circles in the second quadrant and touching the coordinate axes. **[4]**

Solution : The equation of circle in second quadrant which touches the coordinate axis is



Let a be the radius of circle

$\therefore$ centre $= (-a, a)$.

$$(x + a)^2 + (y - a)^2 = a^2 \quad ...(i)$$

On differentiating w.r.t. x, we get

$$2(x+a)+2(y-a)\frac{dy}{dx} = 0$$

$$\Rightarrow \quad x+a+y\frac{dy}{dx}-a\frac{dy}{dx} = 0$$

$$\Rightarrow \quad a\frac{dy}{dx}-a = x+y\frac{dy}{dx}$$

$$\Rightarrow \quad a\left(\frac{dy}{dx}-1\right) = x+y\frac{dy}{dx}$$

$$\Rightarrow \quad a = \frac{x+y\frac{dy}{dx}}{\frac{dy}{dx}-1}$$

$$\Rightarrow \quad a = \frac{x+yy_1}{y_1-1}, \text{ where } y_1 = \frac{dy}{dx}$$

Putting the value of a in eq. (i),

$$\left[x+\frac{x+yy_1}{y_1-1}\right]^2+\left[y-\frac{x+yy_1}{y_1-1}\right]^2=\left[\frac{x+yy_1}{y_1-1}\right]^2$$

$$\Rightarrow y_1^2(x+y)^2+(x+y)^2=(x+yy_1)^2$$

$$\Rightarrow (x+y)^2[(y_1)^2+1]=(x+yy_1)^2.$$ **Ans.**

OR

Find the particular solution of the differential equation $x(x^2-1)\frac{dy}{dx}=1; y=0$ when $x=2$.

Solution : Given, $x(x^2-1)\frac{dy}{dx}=1; y=0, x=2$

$$dy = \frac{dx}{x(x^2-1)}$$

Integrating both sides, we get

$$\int dy = \int\frac{dx}{x(x^2-1)}$$

$$y = \int\frac{dx}{x(x^2-1)} \qquad \text{...(i)}$$

Now $$\frac{1}{x(x^2-1)} = \frac{1}{x(x-1)(x+1)}$$

Solving by partial fractions, we get

$$\frac{1}{x(x-1)(x+1)} = \frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+1} \qquad \text{...(ii)}$$

$A(x^2-1) + Bx(x+1) + Cx(x-1) = 1$

Putting $x = 0$,

$\Rightarrow A(-1) + B(0) + C(0) = 1$

$\therefore \quad A = -1$

Putting $x = 1$

$\Rightarrow \quad A(0) + B(2) + C(0) = 1$

$\therefore \quad B = \frac{1}{2}$

Putting $x = -1$

$A(0) + B(0) + C(-1)(-2) = 1$

$\therefore \quad C = \frac{1}{2}$

On substituting the values of A, B, C in equation (ii), we get

$$\frac{1}{x(x-1)(x+1)} = \frac{-1}{x}+\frac{1}{2(x-1)}+\frac{1}{2(x+1)}$$

From eq. (i)

$$y = \int\frac{1}{x(x^2-1)}dx$$

$$= -\int\frac{dx}{x}+\frac{1}{2}\int\frac{dx}{x-1}+\frac{1}{2}\int\frac{dx}{x+1}$$

$$= -\log x+\frac{1}{2}\log(x-1)+\frac{1}{2}\log(x+1)+c$$

$$= \log\left(\frac{1}{x}\right)+\frac{1}{2}\log(x^2-1)+c$$

$$= \log\left[\frac{\sqrt{x^2-1}}{x}\right]+c$$

Now, putting $y = 0$, $x = 2$, we get

$$0 = \log\frac{\sqrt{3}}{2}+c$$

$$\Rightarrow \quad c = -\log\frac{\sqrt{3}}{2}$$

$\therefore$ The particular solution is

$$y = \log\left[\frac{\sqrt{x^2-1}}{x}\right]+\left(-\log\frac{\sqrt{3}}{2}\right)$$

$$= \frac{2}{2}\log\left[\frac{\sqrt{x^2-1}}{x}\right]+\frac{2}{2}\log\left[\frac{2}{\sqrt{3}}\right]$$

$$= \frac{1}{2}\log\left[\frac{x^2-1}{x^2}\right]+\frac{1}{2}\log\left[\frac{4}{3}\right]$$

$$= \frac{1}{2}\log\frac{4(x^2-1)}{3x^2}.$$ **Ans.**

19. Solve the following differential equation :
$(1 + x^2)\, dy + 2xy\, dx = \cot x\, dx; x \neq 0.$ **[4]**

Solution : $(1 + x^2)\, dy + 2xy\, dx = \cot x\, dx$

Dividing with $(1 + x^2)\, dx$ on both sides, we get

$$\frac{dy}{dx}+\left(\frac{2x}{1+x^2}\right)y = \frac{\cot x}{1+x^2}$$

On comparing this equation with

$$\frac{dy}{dx}+Py = Q,$$

Where $P = \frac{2x}{1+x^2}, Q = \frac{\cot x}{1+x^2}$

Now, $\quad$ I.F. $= e^{\int P dx} = e^{\int \frac{2x}{1+x^2}dx}$

$= e^{\log(1+x^2)} = 1 + x^2$

Solution is

$$y\,(\text{I.F.}) = \int Q.(\text{I.F.})dx + C$$

$$y\,(1+x^2) = \int \frac{\cot x}{(1+x^2)}.(1+x^2)dx + C$$

$$\Rightarrow \quad y\,(1+x^2) = \int \cot x + C$$

$$\Rightarrow \quad y\,(1+x^2) = \log|\sin x| + C.$$

Ans.

20. Let $\vec{a} = \hat{i} + 4\hat{j} + 2\hat{k}$, $\vec{b} = 3\hat{i} - 2\hat{j} + 7\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$. Find a vector $\vec{p}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$ and $\vec{p}.\vec{c} = 18$. **[4]**

Solution :

Given that, $\vec{a} = \hat{i} + 4\hat{j} + 2\hat{k}$, $\vec{b} = 3\hat{i} - 2\hat{j} + 7\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$

Since $\vec{p}$ is perpendicular to both $\vec{a}$ and $\vec{b}$ therefore $\vec{p}$ is parallel to $(\vec{a} \times \vec{b})$

So, $\quad \vec{p} = \lambda(\vec{a} \times \vec{b})$

$$= \lambda \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 4 & 2 \\ 3 & -2 & 7 \end{vmatrix}$$

$$= \lambda[\hat{i}(28+4) - \hat{j}(7-6) + \hat{k}(-2-12)]$$

$$= \lambda(32\hat{i} - \hat{j} - 14\hat{k})$$

Now, $\quad \vec{p}.\vec{c} = 18$

$$\lambda(32\hat{i} - \hat{j} - 14\hat{k}).(2\hat{i} - \hat{j} + 4\hat{k}) = 18$$

$$\Rightarrow \quad \lambda\,(32 \times 2 + 1 - 14 \times 4) = 18$$

$$\Rightarrow \quad 9\lambda = 18$$

$$\therefore \quad \lambda = 2$$

$$\therefore \quad \vec{p} = 2(32\hat{i} - \hat{j} - 14\hat{k})$$

$$\Rightarrow \quad \vec{p} = 64\hat{i} - 2\hat{j} - 28\hat{k}.$$

Ans.

21. Find the coordinates of the point where the line through the points A = (3, 4, 1) and B = (5, 1, 6) crosses the XY-plane. **[4]**

Solution : Given, A = (3, 4, 1), B = (5, 1, 6)

The equation of the line passing through above points is

$$\frac{x-3}{5-3} = \frac{y-4}{1-4} = \frac{z-1}{6-1}$$

$$\Rightarrow \quad \frac{x-3}{2} = \frac{y-4}{-3} = \frac{z-1}{5}$$

This line crosses the XY-plane

$$\therefore \quad z = 0$$

$$\frac{x-3}{2} = \frac{y-4}{-3} = \frac{0-1}{5}$$

Taking 1st and 3rd terms, we get

$$\frac{x-3}{2} = \frac{-1}{5}$$

$$\Rightarrow \quad 5x - 15 = -2$$

$$\Rightarrow \quad 5x = 13$$

$$\Rightarrow \quad x = \frac{13}{5}$$

Taking 2nd and 3rd terms, we get

$$\frac{y-4}{-3} = \frac{-1}{5}$$

$$\Rightarrow \quad 5y - 20 = 3$$

$$\Rightarrow \quad 5y = 23$$

$$\Rightarrow \quad y = \frac{23}{5}$$

So, coordinates of the point where that line through the points A and B crosses the XY-plane is $\left(\frac{13}{5}, \frac{23}{5}, 0\right)$. **Ans.**

22. Two cards are drawn simultaneously (without replacement) from a well-shuffled pack of 52 cards. Find the mean and variance of the number of red cards. **[4]**

Solution : Let P (A) = Probability of getting one red card.

Number of red cards = 26

Let X be the random variable which can take values 0, 1, 2 where X is the number of red cards selected

$$P(X=0) = \frac{^{26}C_2}{^{52}C_2} = \frac{26 \times 25}{52 \times 51} = \frac{25}{102}$$

$$P(X=1) = \frac{^{26}C_1 \times {}^{26}C_1}{^{52}C_2} = \frac{26 \times 26 \times 2}{52 \times 51} = \frac{52}{102}$$

$$P(X=2) = \frac{^{26}C_2}{^{52}C_2} = \frac{26 \times 25}{52 \times 51} = \frac{25}{102}$$

Probability distribution of random variable X is

X	0	1	2
P(X)	$\frac{25}{102}$	$\frac{52}{102}$	$\frac{25}{102}$

$$\text{Mean} = \Sigma X P(X) = \frac{52}{102} + \frac{50}{102} = 1$$

$$\text{Variance of } x = \Sigma X^2 P(X) - (\Sigma X P(X))^2$$

$$\Rightarrow \quad \Sigma X^2 P(X) = \frac{52}{102} + \frac{4 \times 25}{102} = \frac{152}{102}$$

$$\Rightarrow \quad (\Sigma X P(X))^2 = 1^2 = 1$$

$$\therefore \text{Variance of } x = \frac{152}{102} - 1 = \frac{50}{102}$$

$$= \frac{25}{51}$$

Ans.

SECTION — C

23. Using matrices, solve the following system of equations:
$2x + 3y + 3z = 5, x - 2y + z = -4, 3x - y - 2z = 3.$ **[6]**

Solution : Given, $2x + 3y + 3z = 5,$
$x - 2y + z = -4,$
$3x - y - 2z = 3$

Here, $A = \begin{bmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$

$$|A| = 2(4+1) - 3(-2-3) + 3(-1+6)$$
$$= 10 + 15 + 15 = 40 \neq 0$$

$\therefore A^{-1}$ exists.

Co-factors of matrix A is

$A_{11} = 5, \quad A_{12} = 5, \quad A_{13} = 5$
$A_{21} = 3, \quad A_{22} = -13, \quad A_{23} = 11$
$A_{31} = 9, \quad A_{32} = 1, \quad A_{33} = -7$

$$\text{adj } A = \begin{bmatrix} 5 & 5 & 5 \\ 3 & -13 & 11 \\ 9 & 1 & -7 \end{bmatrix}^T = \begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{\text{adj } A}{|A|}$$

$$= \frac{1}{40}\begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$$

$$\therefore \quad X = A^{-1} B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}\begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$$

$$= \frac{1}{40}\begin{bmatrix} 25 - 12 + 27 \\ 25 + 52 + 3 \\ 25 - 44 - 21 \end{bmatrix}$$

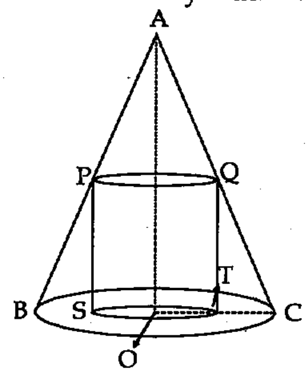
$$= \frac{1}{40}\begin{bmatrix} 40 \\ 80 \\ -40 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$\therefore \quad x = 1, y = 2$ and $z = -1.$ **Ans.**

24. Prove that the radius of the right circular cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone. **[6]**

Solution : Let r = radius of cylinder
R = radius of cone
h = height of cylinder
H = height of cone

Curved surface area of cylinder $= 2\pi r h$



In Δ QTC and Δ AOC,

$$\frac{AO}{QT} = \frac{OC}{TC}$$

$$\frac{H}{h} = \frac{R}{R - r}$$

$$\Rightarrow \quad h = \frac{H(R - r)}{R}$$

$$S = 2\pi r h$$

$$\Rightarrow \quad S = 2\pi r H \frac{(R - r)}{R}$$

$$\Rightarrow \quad S = 2\pi H \frac{(Rr - r^2)}{R}$$

Differentiating w.r. t. r, we get

$$\frac{dS}{dr} = \frac{2\pi H}{R}(R - 2r) \quad \text{...(i)}$$

Again differentiating w.r. t. r, we get

$$\frac{d^2S}{dr^2} = \frac{-4\pi H}{R} \quad \text{...(ii)}$$

For maxima and minima,

$$\frac{dS}{dr} = 0$$

From (i),

$$\frac{2\pi H}{R}(R - 2r) = 0$$

$$\Rightarrow \quad R = 2r$$

$$\Rightarrow \quad r = \frac{R}{2}$$

$$\frac{d^2S}{dr^2} = \frac{-4\pi H}{R} < 0$$

Hence, S is maximum when $r = \frac{R}{2}$

Hence Proved.

OR

An open box with a square base is to be made out of a given quantity of cardboard of area c^2 square units. Show that the maximum volume of the box is $\frac{c^3}{6\sqrt{3}}$ cubic units.

Solution : Let $l = x, b = x, h = y$

$\therefore$ Area of open box = Area of cardboard

$$x^2 + 4xy = c^2 \Rightarrow y = \frac{c^2 - x^2}{4x} \quad ...(i)$$

Let V = volume of the box

$$V = x^2y$$

$$V = x^2\left(\frac{c^2-x^2}{4x}\right)$$

$$V = \frac{xc^2}{4} - \frac{x^3}{4}$$

Differentiating w.r. t. x, we get

$$\frac{dV}{dx} = \frac{c^2}{4} - \frac{3x^2}{4} \quad ...(ii)$$

Again differentiating w.r. t. x, we get

$$\frac{d^2V}{dx^2} = \frac{-3x}{2}$$

For maxima or minima, $\frac{dV}{dx} = 0$.

$$\frac{c^2}{4} - \frac{3x^2}{4} = 0$$

From (ii),

$$\frac{c^2}{4} = \frac{3x^2}{4}$$

$$\Rightarrow \quad x = \frac{c}{\sqrt{3}}$$

Put the value of x in equation (i), we get

$$y = \frac{c^2 - \frac{c^2}{3}}{4.\frac{c}{\sqrt{3}}}$$

$$\Rightarrow \quad y = \frac{2c^2}{3}.\frac{\sqrt{3}}{4c} = \frac{c}{2\sqrt{3}}$$

Also $\quad \frac{d^2V}{dx^2} = \frac{-3x}{2} < 0$

$\therefore$ Volume is maximum.

$\therefore$ Maximum volume $V = x^2y$

$$= \frac{c^2}{3}.\frac{c}{2\sqrt{3}} = \frac{c^3}{6\sqrt{3}} \quad \textbf{Ans.}$$

25. Evaluate : $\int \frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx.$ **[6]**

Solution : Let, $I = \int \frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx \quad ...(i)$

Let $\quad t = \sin^{-1}x \Rightarrow x = \sin t$

$$dt = \frac{1}{\sqrt{1-x^2}}dx$$

Put the value of dt in equation (i), we get

$$I = \int t\sin t\,dt$$

On integrating by parts method, we get

$$I = t\int \sin t\,dt - \int\left(\frac{d(t)}{dt}\int \sin t\,dt\right)dt$$

$$\Rightarrow \quad I = -t\cos t + \int \cos t\,dt$$

$$\Rightarrow \quad I = -t\cos t + \sin t + c$$

$$\Rightarrow \quad I = -t\sqrt{1-\sin^2 t} + \sin t + c$$

$$= -\sin^{-1}x\sqrt{1-x^2} + x + c$$

$$= x - \sin^{-1}x.\sqrt{1-x^2} + c. \quad \textbf{Ans.}$$

OR

Evaluate : $\int \frac{x^2+1}{(x-1)^2(x+3)}dx$

Solution : Given, $\int \frac{x^2+1}{(x-1)^2(x+3)}dx$

By using partial fractions

$$\frac{x^2+1}{(x-1)^2(x+3)}dx = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+3)}$$

$$x^2 + 1 = A(x-1)(x+3) + B(x+3) + C(x-1)^2$$

Put $x = 1$ in above equation, we get

$$1^2 + 1 = B(1+3)$$

$$\Rightarrow \quad B = \frac{2}{4} = \frac{1}{2}$$

Put $x = -3$ in above equation, we get

$$9 + 1 = C(-3-1)^2$$

$$\Rightarrow \quad C = \frac{10}{16} = \frac{5}{8}$$

Put $x = 0$ in above equation, we get

$$1 = A(-3) + B(3) + C$$

$$\Rightarrow \quad 3A = -1 + \frac{3}{2} + \frac{5}{8} = \frac{-8+12+5}{8} = \frac{9}{8}$$

$$\Rightarrow \quad A = \frac{3}{8}$$

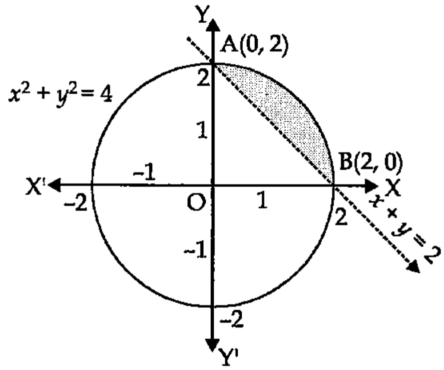
$$\int \frac{x^2+1}{(x-1)^2(x+1)}dx = \frac{3}{8}\int \frac{dx}{x-1} + \frac{1}{2}\int \frac{dx}{(x-1)^2} + \frac{5}{8}\int \frac{dx}{x+3}$$

$= \frac{3}{8}\log|x-1| - \frac{1}{2(x-1)} + \frac{5}{8}\log|x+3| + c.$ **Ans.**

26. Find the area of the region $\{(x, y) : x^2 + y^2 \le 4, x + y \ge 2\}$. [6]

Solution : We have two equations

$x^2 + y^2 \le 4$ and $x + y \ge 2$



Area under shaded region

$= \text{Area of OBCAO} - \text{Area of } \Delta\text{BOA}$

$= \text{Area under circle} - \text{Area under line}$

$$= \int_0^2 (\sqrt{4-x^2})\,dx - \int_0^2 (2-x)\,dx$$

$$= \left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2 - \left[2x - \frac{x^2}{2}\right]_0^2$$

$$= \left[\frac{2}{2}\sqrt{4-4} + 2\sin^{-1}\frac{2}{2}\right] - [0 + 2\sin^{-1} 0] - [2\times 2 - 2] + 0$$

$= 2\sin^{-1}(1) - 2.0 - 2$

$= 2\left(\frac{\pi}{2}\right) - 2$

$= (\pi - 2)$ sq. units. **Ans.**

27. If the lines $\frac{x-1}{-3} = \frac{y-2}{-2k} = \frac{z-3}{2}$ and $\frac{x-1}{k} = \frac{y-2}{1} = \frac{z-3}{5}$ are perpendicular, find the value of k and hence find the equation of plane containing these lines. [6]

Solution : The equation of the given lines are

$$\frac{x-1}{-3} = \frac{y-2}{-2k} = \frac{z-3}{2} \text{ and } \frac{x-1}{k} = \frac{y-2}{1} = \frac{z-3}{5}$$

These lines are perpendicular, therefore

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

$$-3k + (-2k \times 1) + 2 \times 5 = 0$$

$\Rightarrow \quad -5k = -10$

$\Rightarrow \quad k = 2$

$\therefore$ The equation of lines becomes

$$\frac{x-1}{-3} = \frac{y-2}{-4} = \frac{z-3}{2}, \quad \frac{x-1}{2} = \frac{y-2}{1} = \frac{z-3}{5}$$

Equation of plane containing 2 lines is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0,$$

Where $x_1 = 1, \quad y_1 = 2, \quad z_1 = 3$

$l_1 = -3, m_1 = -4, n_1 = 2$

$l_2 = 2, \quad m_2 = 1, \quad n_2 = 5$

$$\begin{vmatrix} x-1 & y-2 & z-3 \\ -3 & -4 & 2 \\ 2 & 1 & 5 \end{vmatrix} = 0$$

$(x-1)(-20-2) - (y-2)(-15-4) + (z-3)(-3+8) = 0$

$\Rightarrow \quad -22x + 22 + 19y - 38 + 5z - 15 = 0$

$\Rightarrow \quad 22x - 19y - 5z + 31 = 0.$ **Ans.**

28. Suppose a girl throws a die. If she gets a 5 or 6, she tosses a coin 3 times and notes the number of heads. If she gets 1, 2, 3 or 4 she tosses a coin once and notes whether a head or tail is obtained. If she obtained exactly one head, what is the probability that she threw 1, 2, 3 or 4 with the die? [6]

Solution : The outcome of an experiment can be represented as

Die is thrown { Getting 5 or 6 → coin is tossed 3 times; Getting 1, 2, 3 or 4 → coin is tossed once }

If she gets 1, 2, 3 or 4, sample space will be

(1H), (2H), (3H), (4H), (1T), (2T), (3T), (4T)

If she gets 5 or 6, sample space will be

(5 HHH), (5 HTT), (5 HHT), (5 HTH), (5 THT), (5 TTH), (5 THH), (5 TTT), (6 HHH), (6 HTT), (6 HHT), (6 HTH), (6 THT), (6 TTH), (6 THH), (6 TTT)

Let

A = Getting 1, 2, 3 or 4 on die

B = Getting exactly 1 Head

A = (1H), (2H), (3H), (4H), (1T), (2T), (3T), (4T)

B = (1H), (2H), (3H), (4H), (5HTT), (5THT), (5TTH), (6HTT), (6TTH), (6THT)

$A \cap B$ = (1H), (2H), (3H), (4H)

$P(1H) = \frac{1}{6} \times \frac{1}{2} = \frac{1}{12} \qquad P(2H) = \frac{1}{6} \times \frac{1}{2} = \frac{1}{12}$

$P(1T) = \frac{1}{6} \times \frac{1}{2} = \frac{1}{12} \qquad P(2T) = \frac{1}{6} \times \frac{1}{2} = \frac{1}{12}$

$P(3T) = \frac{1}{6} \times \frac{1}{2} = \frac{1}{12}$ $\quad$ $P(4T) = \frac{1}{6} \times \frac{1}{2} = \frac{1}{12}$

$P(5TTH) = \frac{1}{6} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{48}$ $\quad$ $P(5THT) = \frac{1}{48}$

$P(5HTT) = \frac{1}{48}$, $P(6THT) = \frac{1}{48}$, $P(6TTH) = \frac{1}{48}$,

$P(6HTT) = \frac{1}{48}$

$$P(A \cap B) = \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} = \frac{4}{12} = \frac{1}{3}$$

$P(B) = P(1H) + P(2H) + P(3H) + P(4H) + P(5THT) + P(5HTT) + P(5TTH) + P(6HTT) + P(6THT) + P(6TTH)$

$$= \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{48} + \frac{1}{48} + \frac{1}{48} + \frac{1}{48} + \frac{1}{48} + \frac{1}{48}$$

$$= \frac{4}{12} + \frac{6}{48} = \frac{22}{48} = \frac{11}{24}$$

∴ Required probability is

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{3}}{\frac{11}{24}} = \frac{8}{11}.$$ **Ans.**

29. A dietician wishes to mix two types of foods in such a way that the vitamin contents of the mixture contains at least 8 units of vitamin A and 10 units of vitamin C. Food I contains 2 units/kg of vitamin A and 1 unit/kg of vitamin C while food II contains 1 unit/kg of vitamin A and 2 units/kg of vitamin 1 unit/kg of vitamin C. It costs ₹ 5 per kg to purchase food I and ₹ 7 per kg to purchase Food II. Determine the minimum cost of such a mixture. Formulate the above as a LPP and solve it graphically. [6]

Solution :

Vitamins	Food 1	Food 2	Requirement
Vitamin A	2	1	8
Vitamin C	1	2	10
Cost (in ₹)	5	7	

Let the amount of food I = x kg
Let the amount of food II = y kg
If Z denotes the total cost
To minimize the cost we have to minimize Z.
Subject to the constraints,

$$2x + y \geq 8$$
$$x + 2y \geq 10$$
$$x \geq 0$$
$$y \geq 0$$
$$\text{Minimize } Z = 5x + 7y$$

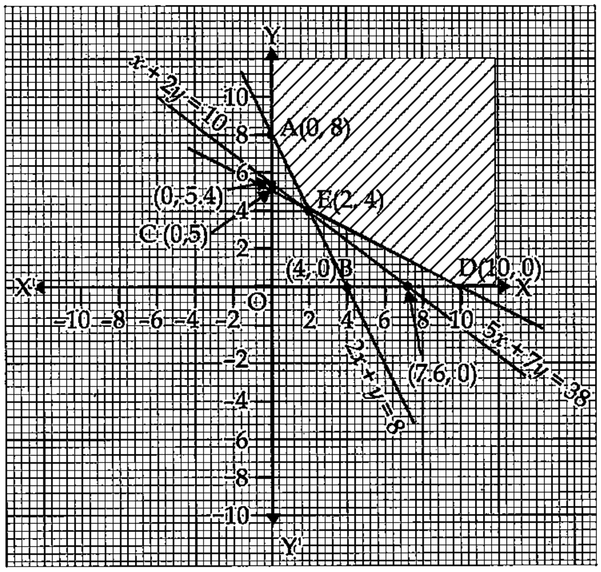
First we draw the lines AB and CD whose equations are

$2x + y = 8$ ...(i)

	A	B
x	0	4
y	8	0

$x + 2y = 10$ ...(ii)

	C	D
x	0	10
y	5	0



The feasible region is shaded in the figure. The lines are intersecting at the point E (2, 4).

∴ The vertices of the feasible region are A (0, 8), E (2, 4) and D (10, 0).

Corner points	$Z = 5x + 7y$
At A(0, 8)	$Z = 5(0) + 7(8) = 56$
At E (2, 4)	$Z = 5(2) + 7(4) = 38 \leftarrow$ minimum
At D (10, 0)	$Z = 5(10) + 7(0) = 50$

Since the feasible region is unbounded 38 may or may not be minimum value of total cost for this we draw graph of inequality.

$$5x + 7y < 38$$

x	0	$38/5 = 7.6$
y	$38/7 = 5.4$	0

Clearly graph of L has no common point with the feasible region.

∴ The minimum value of Z is 38 at the point E (2, 4). Hence, the amount of food I is 2 kg and amount of food II is 4 kg should be included in the mixture for minimum cost of ₹ 38. **Ans.**

Mathematics 2012 (Outside Delhi) SET II

Time allowed : 3 hours **Maximum marks : 100**

Note : Except for the following questions, all the remaining questions have been asked in previous set.

SECTION — A

10. Write the value of $(\hat{k}\times\hat{j}).\hat{i}+\hat{j}.\hat{k}$ **[1]**

Solution : $(\hat{k}\times\hat{j}).\hat{i}+\hat{j}.\hat{k}$

$= -(\hat{j}\times\hat{k}).\hat{i}+\hat{j}.\hat{k}$ $\quad (\because \hat{k}\times\hat{j}=-\hat{j}\times\hat{k})$

$= -\hat{i}.\hat{i}+\hat{j}.\hat{k}$ $\quad (\because \hat{i}.\hat{i}=1, \hat{j}.\hat{k}=0)$

$= -1+0$

$= -1.$ **Ans.**

SECTION — B

19. Prove the following :

$$\cos^{-1}\left(\frac{4}{5}\right)+\cos^{-1}\left(\frac{12}{13}\right)=\cos^{-1}\left(\frac{33}{65}\right)$$ **[4]**

Solution : L.H.S. $= \cos^{-1}\frac{4}{5}+\cos^{-1}\frac{12}{13}$

$$= \cos^{-1}\left(\frac{4}{5}\times\frac{12}{13}-\sqrt{1-\left(\frac{4}{5}\right)^2}.\sqrt{1-\left(\frac{12}{13}\right)^2}\right)$$

$[\because \cos^{-1}x+\cos^{-1}y=\cos^{-1}(xy-\sqrt{1-x^2}\sqrt{1-y^2})]$

$$= \cos^{-1}\left(\frac{48}{65}-\frac{3}{5}\times\frac{5}{13}\right)=\cos^{-1}\left(\frac{48}{65}-\frac{15}{65}\right)$$

$$= \cos^{-1}\left(\frac{33}{65}\right)=\text{R.H.S.}$$ **Hence Proved.**

20. If $y = (\tan^{-1} x)^2$, show that

$$(x^2+1)^2\frac{d^2y}{dx^2}+2x(x^2+1)\frac{dy}{dx}=2.$$ **[4]**

Solution : Given, $y = (\tan^{-1} x)^2$

Differentiating w.r. t. x, we get

$$\frac{dy}{dx} = 2\tan^{-1}x.\frac{d}{dx}(\tan^{-1}x)$$

$$= 2\tan^{-1}x.\frac{1}{1+x^2}$$

$$\Rightarrow \quad (x^2+1)\frac{dy}{dx} = 2\tan^{-1}x$$

Again differentiating w.r. t. x, we get

$$(x^2+1).\frac{d^2y}{dx^2}+2x.\frac{dy}{dx}=2.\frac{1}{x^2+1}$$

$$\Rightarrow (x^2+1)^2\frac{d^2y}{dx^2}+2x(x^2+1)\frac{dy}{dx}=2.$$

Hence Proved.

21. Find the particular solution of the differential equation $\frac{dy}{dx}+y\cot x=4x\operatorname{cosec} x, (x\neq 0)$, given that $y=0$ when $x=\frac{\pi}{2}$. **[4]**

Solution : Given;

$$\frac{dy}{dx}+y\cot x=4x\operatorname{cosec}x, (x\neq 0) \quad ...(i)$$

On comparing equation (i) with $\frac{dy}{dx}+\text{P}y=\text{Q}$

Here P = $\cot x$ and Q = $4x\operatorname{cosec}x$

$$\therefore \quad \text{I.F.} = e^{\int \text{P}dx} = e^{\int \cot x dx}$$

$$= e^{\log \sin x} = \sin x$$

$\therefore$ The solution is

$$y(\text{I.F.}) = \int \text{Q. (I.F.)}\,dx+\text{C}$$

$$y.\sin x = \int 4x\operatorname{cosec}x.\sin x\,dx+\text{C}$$

$$= \int 4x\,dx+\text{C}$$

$$\therefore \quad y.\sin x = 2x^2+\text{C} \quad ...(ii)$$

Putting $y = 0$, when $x=\frac{\pi}{2}$

$$\therefore \quad 0.\sin\frac{\pi}{2} = 2.\left(\frac{\pi}{2}\right)^2+\text{C}$$

$$\Rightarrow \quad \text{C} = -\frac{\pi^2}{2}$$

Substituting the value of C in equation (ii), we get

$$y.\sin x = 2x^2-\frac{\pi^2}{2}.$$

This is the required particular solution of the given differential equation. **Ans.**

22. Find the coordinates of the point where the line through the point (3, –4, –5) and (2, –3, 1) crosses the plane $2x + y + z = 7$. **[4]**

Solution : Equation of the line passes through the points (3, – 4, – 5) and (2, – 3, 1) is

$$\frac{x-3}{2-3} = \frac{y+4}{-3+4}=\frac{z+5}{1+5}$$

$$\Rightarrow \quad \frac{x-3}{-1} = \frac{y+4}{1}=\frac{z+5}{6}=k \text{ (say)} \quad ...(i)$$

$\therefore$ Any point on line is $(-k+3, k-4, 6k-5)$.

Line crosses the plane $2x+y+z=7$ ...(ii)

Point lies on it

$\Rightarrow \quad 2(-k+3)+k-4+6k-5=7$

$\Rightarrow \quad 5k-3=7$

$\Rightarrow \quad k=2$

$\therefore$ The point is $(-2+3, 2-4, 12-5)$

$= (1, -2, 7)$. **Ans.**

SECTION — C

28. Using matrices, solve the following system of equations :

$x+y-z=3;\ 2x+3y+z=10;\ 3x-y-7z=1$ **[6]**

Solution : The given system of equations are

$$x+y-z = 3;$$
$$2x+3y+z = 10;$$
$$3x-y-7z = 1.$$

These equations can be written in matrix form

$$AX = B \quad ...(i)$$

$$\begin{bmatrix}1 & 1 & -1\\ 2 & 3 & 1\\ 3 & -1 & -7\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}3\\ 10\\ 1\end{bmatrix}$$

Where $A = \begin{bmatrix}1 & 1 & -1\\ 2 & 3 & 1\\ 3 & -1 & -7\end{bmatrix}, X = \begin{bmatrix}x\\ y\\ z\end{bmatrix}$ and $B = \begin{bmatrix}3\\ 10\\ 1\end{bmatrix}$

Now $\quad |A| = \begin{vmatrix}1 & 1 & -1\\ 2 & 3 & 1\\ 3 & -1 & -7\end{vmatrix}$

$= (-21+1)-(-14-3)-1(-2-9)$

$= -20+17+11 = -20+28 = 8 \neq 0$

$\Rightarrow A^{-1}$ exists.

For adj A,

$A_{11} = -21+1 = -20 \qquad A_{21} = -(-7-1) = 8,$

$A_{12} = -(-14-3) = 17, \qquad A_{22} = -7+3 = -4,$

$A_{13} = -2-9 = -11, \qquad A_{23} = -(-1-3) = 4,$

$A_{31} = 1+3 = 4$

$A_{32} = -(1+2) = -3$

$A_{33} = 3-2 = 1$

$$\text{adj } A = \begin{bmatrix}-20 & 17 & -11\\ 8 & -4 & 4\\ 4 & -3 & 1\end{bmatrix}^T$$

$$= \begin{bmatrix}-20 & 8 & 4\\ 17 & -4 & -3\\ -11 & 4 & 1\end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}.\text{adj } A = \frac{1}{8}\begin{bmatrix}-20 & 8 & 4\\ 17 & -4 & -3\\ -11 & 4 & 1\end{bmatrix}$$

From (i), $X = A^{-1}B$

$$\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \frac{1}{8}\begin{bmatrix}-20 & 8 & 4\\ 17 & -4 & -3\\ -11 & 4 & 1\end{bmatrix}\begin{bmatrix}3\\ 10\\ 1\end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix}x\\ y\\ z\end{bmatrix} = \frac{1}{8}\begin{bmatrix}24\\ 8\\ 8\end{bmatrix} = \begin{bmatrix}3\\ 1\\ 1\end{bmatrix}$$

$\Rightarrow \quad x = 3, y = 1, z = 1.$ **Ans.**

29. Find the length and the foot of the perpendicular from the point P (7, 14, 5) to the plane $2x+4y-z = 2$. Also find the image of point P in the plane. **[6]**

Solution : The given plane is

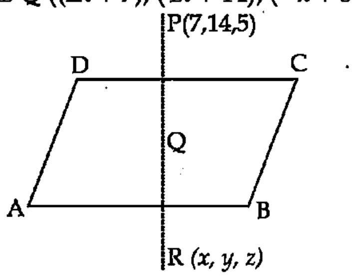
$$2x+4y-z = 2 \quad ...(i)$$

The d.r.s. of the normal to (i) are 2, 4, – 1.

$\therefore$ Equation of a line perpendicular to (i) passing through P(7, 14, 5) is

$$\frac{x-7}{2} = \frac{y-14}{4} = \frac{z-5}{-1} = k \text{ (say)} \quad ...(ii)$$

$\therefore$ Point is Q $((2k+7), (4k+14), (-k+5))$



Suppose it lies on the plane (i),

$\therefore 2(2k+7)+4(4k+14)-(-k+5) = 2$

$\Rightarrow \quad 21k+65 = 2 \Rightarrow 21k = -63$

$\Rightarrow \quad k = -3.$

$\therefore$ Q $(2\times(-3)+7, 4\times(-3)+14, 3+5)$

$= (1, 2, 8)$

This is the foot of perpendicular of the line (ii) on the plane (i).

$\Rightarrow \quad PQ = \sqrt{(7-1)^2+(14-2)^2+(5-8)^2}$

$= \sqrt{36+144+9} = \sqrt{189}$

which is the length of the perpendicular from P on (i)

Again let R(x, y, z) be the image of P in the plane (i). Then Q is the mid–point of PR.

∴ The coordinates of Q are given by

$$\left(\frac{x+7}{2}, \frac{y+14}{2}, \frac{z+5}{2}\right)$$

$$\therefore \quad \frac{x+7}{2} = 1, \frac{y+14}{2} = 2, \frac{z+5}{2} = 8$$

$$\Rightarrow \quad x = -5, y = -10, z = 11$$

$\Rightarrow R(-5, -10, 11)$

This is the image of P in the plane (i). **Ans.**

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Mathematics 2012 (Outside Delhi) SET III

Time allowed : 3 hours **Maximum marks : 100**

> **Note : Except for the following questions, all the remaining questions have been asked in previous sets.**

SECTION–A

9. Write the value of $(\hat{k} \times \hat{i}).\hat{j} + \hat{i}.\hat{k}$ **[1]**

Solution : $(\hat{k} \times \hat{i}).\hat{j} + \hat{i}.\hat{k} = \hat{j}.\hat{j} + 0$

$$\left[\because \hat{k} \times \hat{i} = \hat{j} \text{ and } \hat{i}.\hat{k} = 0\right]$$

$= 1 + 0$ $[\because \hat{j}.\hat{j} = 1]$

$= 1.$ **Ans.**

10. Find the value of $x + y$ from the following equations :

$$2\begin{pmatrix} 1 & 3 \\ 0 & x \end{pmatrix} + \begin{pmatrix} y & 0 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 6 \\ 1 & 8 \end{pmatrix}$$ **[1]**

Solution : Given,

$$2\begin{pmatrix} 1 & 3 \\ 0 & x \end{pmatrix} + \begin{pmatrix} y & 0 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 6 \\ 1 & 8 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2 & 6 \\ 0 & 2x \end{pmatrix} + \begin{pmatrix} y & 0 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 6 \\ 1 & 8 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2+y & 6+0 \\ 1 & 2x+2 \end{pmatrix} = \begin{pmatrix} 5 & 6 \\ 1 & 8 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2+y & 6 \\ 1 & 2x+2 \end{pmatrix} = \begin{pmatrix} 5 & 6 \\ 1 & 8 \end{pmatrix}$$

$$\therefore \quad 2 + y = 5 \Rightarrow y = 3$$

and $2x + 2 = 8 \Rightarrow 2x = 6 \Rightarrow x = 3$

$$\therefore \quad x + y = 3 + 3 = 6.$$ **Ans.**

SECTION–B

19. If $x = a\left(\cos t + \log \tan \frac{t}{2}\right), y = a \sin t$, find $\frac{d^2y}{dt^2}$ and $\frac{d^2y}{dx^2}$ **[4]**

Solution : Given, $x = a\left(\cos t + \log \tan \frac{t}{2}\right)$ and

$$y = a \sin t$$

Now, $x = a\left(\cos t + \log \tan \frac{t}{2}\right)$

Differentiating both sides w.r.t. t, we get

$$\frac{dx}{dt} = a\left(-\sin t + \frac{1}{\tan \frac{t}{2}} \times \sec^2 \frac{t}{2} \times \frac{1}{2}\right)$$

[Applying chain rule of differentiation]

$$= a\left(-\sin t + \frac{\cos \frac{t}{2}}{2 \sin \frac{t}{2} \cos^2 \frac{t}{2}}\right)$$

$$= a\left(-\sin t + \frac{1}{2 \sin \frac{t}{2} \cos \frac{t}{2}}\right)$$

$$= a\left(-\sin t + \frac{1}{\sin t}\right)$$

$$\left[\because \sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2}\right]$$

$$= a\left(\frac{1 - \sin^2 t}{\sin t}\right)$$

$$\therefore \quad \frac{dx}{dt} = \frac{a \cos^2 t}{\sin t} \quad ...(i)$$

$[\because 1 - \sin^2 t = \cos^2 t]$

Similarly $y = a \sin t$

Differentiating both sides w.r.t. t, we get

$$\frac{dy}{dt} = a \cos t \quad ...(ii)$$

Again, differentiating both sides w.r.t. t, we get

$$\frac{d^2y}{dt^2} = -a \sin t$$

From (i) and (ii), we get

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a\cos t}{\frac{a\cos^2 t}{\sin t}} = \frac{\sin t}{\cos t}$$

$$\therefore \quad \frac{dy}{dx} = \tan t$$

Again differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\tan t)$$

$$= \sec^2 t \frac{dt}{dx}$$

$$= \sec^2 t \times \frac{\sin t}{a\cos^2 t}$$

$$= \frac{1}{\cos^2 t} \times \frac{\sin t}{a\cos^2 t} \quad \text{[Using (i)]}$$

$$= \frac{\sin t}{a\cos^4 t}$$ **Ans.**

20. Find the coordinates of the point where the line through the points (3, – 4, – 5) and (2, – 3, 1) crosses the plane 3*x* + 2*y* + *z* + 14 = 0. [4]

Solution : The equation of the straight line passing through the points (3, – 4, – 5) and (2, – 3, 1) is

$$\frac{x-3}{2-3} = \frac{y-(-4)}{-3-(-4)} = \frac{z-(-5)}{1-(-5)}$$

$$\left[\because \frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}\right]$$

$$\Rightarrow \quad \frac{x-3}{-1} = \frac{y+4}{1} = \frac{z+5}{6} = \lambda \text{ (say)}$$

$$\Rightarrow \quad x = -\lambda + 3, y = \lambda - 4, z = 6\lambda - 5$$

So, the coordinates of a general point on this line are $(-\lambda + 3, \lambda - 4, 6\lambda - 5)$.

The line intersects the given plane $3x + 2y + z + 14 = 0$

$\therefore$ point $(-\lambda + 3, \lambda - 4, 6\lambda - 5)$ lies on the plane

$$3x + 2y + z + 14 = 0$$

$$\Rightarrow 3(-\lambda + 3) + 2(\lambda - 4) + 6\lambda - 5 + 14 = 0$$

$$\Rightarrow \quad -3\lambda + 9 + 2\lambda - 8 + 6\lambda + 9 = 0$$

$$\Rightarrow \quad 5\lambda + 10 = 0$$

$$\Rightarrow \quad \lambda = -2$$

Putting, $\lambda = -2$, we have

$x = -\lambda + 3 = -(-2) + 3 = 5$

$y = \lambda - 4 = -2 - 4 = -6$

$z = 6\lambda - 5 = 6 \times (-2) - 5 = -12 - 5 = -17$

Thus, the point of intersection of the line and the given plane is (5, – 6, – 17). **Ans.**

21. Find the particular solution of the differential equation $x\frac{dy}{dx} - y + x\sin\left(\frac{y}{x}\right) = 0$, given that when $x = 2, y = \pi$. [4]

Solution : Given differential equation is :

$$x\frac{dy}{dx} - y + x\sin\left(\frac{y}{x}\right) = 0$$

$$\Rightarrow \quad x\frac{dy}{dx} = y - x\sin\left(\frac{y}{x}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y - x\sin\left(\frac{y}{x}\right)}{x} \quad \text{...(i)}$$

Put $y = vx$

Differentiating w.r.t. x, we get

$$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting the values of y and $\frac{dy}{dx}$ in equation (i), we get

$$v + x\frac{dv}{dx} = \frac{vx - x\sin v}{x}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = v - \sin v$$

$$\Rightarrow \quad -\frac{dv}{\sin v} = \frac{dx}{x}$$

$$\Rightarrow \quad \text{cosec}\, v\, dv = -\frac{dx}{x}$$

Integrating both sides, we get

$$\log|\text{cosec}\, v - \cot v| = -\log x + \log C = \log\frac{C}{x}$$

$$\Rightarrow \quad \text{cosec}\left(\frac{y}{x}\right) - \cot\left(\frac{y}{x}\right) = \frac{C}{x}$$

$$\Rightarrow \quad \frac{1}{\sin\left(\frac{y}{x}\right)} - \frac{\cos\left(\frac{y}{x}\right)}{\sin\left(\frac{y}{x}\right)} = \frac{C}{x}$$

$$\Rightarrow \quad x\left[1 - \cos\left(\frac{y}{x}\right)\right] = C\sin\left(\frac{y}{x}\right) \quad \text{...(ii)}$$

It is given that when $x = 2, y = \pi$

$$2\left[1 - \cos\left(\frac{\pi}{2}\right)\right] = C\sin\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \quad 2[1 - 0] = C \times 1$$

$$\Rightarrow \quad C = 2$$

$$\therefore \quad x\left[1 - \cos\left(\frac{y}{x}\right)\right] = 2\sin\left(\frac{y}{x}\right)$$

This is the required particular solution of the given differential equation. **Ans.**

22. Prove the following :

$$\cos^{-1}\left(\frac{12}{13}\right)+\sin^{-1}\left(\frac{3}{5}\right)=\sin^{-1}\left(\frac{56}{65}\right)$$ **[4]**

Solution : Taking L.H.S.

$$= \cos^{-1}\left(\frac{12}{13}\right)+\sin^{-1}\left(\frac{3}{5}\right)$$

$$=\sin^{-1}\left(\sqrt{1-\left(\frac{12}{13}\right)^2}\right)+\sin^{-1}\left(\frac{3}{5}\right)$$

$$\left(\because \cos^{-1}x = \sin^{-1}\sqrt{1-x^2}\right)$$

$$=\sin^{-1}\left(\sqrt{\frac{25}{169}}\right)+\sin^{-1}\left(\frac{3}{5}\right)$$

$$=\sin^{-1}\left(\frac{5}{13}\right)+\sin^{-1}\left(\frac{3}{5}\right)$$

$$=\sin^{-1}\left(\frac{5}{13}\sqrt{1-\left(\frac{3}{5}\right)^2}+\frac{3}{5}\sqrt{1-\left(\frac{5}{13}\right)^2}\right)$$

$$\left(\because \sin^{-1}x+\sin^{-1}y=\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)\right)$$

$$=\sin^{-1}\left(\frac{5}{13}\sqrt{\frac{16}{25}}+\frac{3}{5}\sqrt{\frac{144}{169}}\right)$$

$$=\sin^{-1}\left(\frac{5}{13}\cdot\frac{4}{5}+\frac{3}{5}\cdot\frac{12}{13}\right)$$

$$=\sin^{-1}\left(\frac{20}{65}+\frac{36}{65}\right)$$

$$=\sin^{-1}\left(\frac{56}{65}\right)=\text{R.H.S.}$$

Hence Proved.

SECTION–C

28. Find the coordinates of the foot of perpendicular and the length of the perpendicular drawn from the point P (5, 4, 2) to the line $\vec{r}=-\hat{i}+3\hat{j}+\hat{k}+\lambda(2\hat{i}+3\hat{j}-\hat{k})$. Also find the image of P in this line. **[6]**

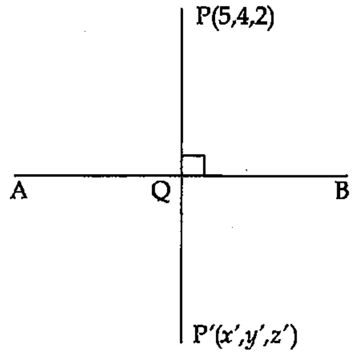
Solution : The given point is (5, 4, 2) and the given line is $\vec{r}=-\hat{i}+3\hat{j}+\hat{k}+\lambda(2\hat{i}+3\hat{j}-\hat{k})$.

This line passes through the point (– 1, 3, 1) and is parallel to the vector $(2\hat{i}+3\hat{j}-\hat{k})$. Cartesian equation of line is

$$\frac{x+1}{2}=\frac{y-3}{3}=\frac{z-1}{-1}=\lambda \text{ (say)}$$

$\therefore x = 2\lambda - 1, y = 3\lambda + 3, z = -\lambda + 1$

These are the coordinates of any general point on the line.



Let Q be the foot of the perpendicular on the line. Then, for some value of λ, the coordinates of Q are $(2\lambda-1, 3\lambda+3, -\lambda+1)$.

Direction ratios of PQ are $2\lambda - 1 - 5, 3\lambda + 3 - 4, -\lambda + 1 - 2$ *i.e.*, $2\lambda - 6, 3\lambda - 1, -\lambda - 1$.

PQ is perpendicular to given line, we have

$\therefore \quad 2(2\lambda-6)+3(3\lambda-1)-1(-\lambda-1)=0$

$\Rightarrow \quad 4\lambda-12+9\lambda-3+\lambda+1=0$

$\Rightarrow \quad 14\lambda-14=0$

$\Rightarrow \quad \lambda=1$

$\therefore$ Coordinates of Q $= (2\times1-1, 3\times1+3, -1+1)$
$= (1, 6, 0)$

Hence, the coordinates of the foot of perpendicular are (1, 6, 0).

Using distance formula, we have

$$PQ=|\vec{PQ}|=\sqrt{(1-5)^2+(6-4)^2+(0-2)^2}$$

$$=\sqrt{16+4+4}$$

$$=\sqrt{24}$$

$$=2\sqrt{6} \text{ units}$$

Let $P'(x', y', z')$ be the image of point P (5, 4, 2) in the given line.

Then, Q is mid-point of PP′

$$\therefore \quad \frac{x'+5}{2}=1\Rightarrow x'+5=2\Rightarrow x'=-3$$

$$\frac{y'+4}{2}=6\Rightarrow y'+4=12\Rightarrow y'=8$$

$$\frac{z'+2}{2}=0\Rightarrow z'+2=0\Rightarrow z'=-2$$

Hence, the image of P in the given line is (–3, 8, –2). **Ans.**

29. Using matrices, solve the following system of equations:
$3x + 4y + 7z = 4; 2x - y + 3z = -3; x + 2y - 3z = 8$ **[6]**

Solution : The given system of equation is

$3x + 4y + 7z = 4;$

$2x - y + 3z = -3;$

$$x + 2y - 3z = 8$$

The above system of equations can be represented as $AX = B$

$$\begin{bmatrix} 3 & 4 & 7 \\ 2 & -1 & 3 \\ 1 & 2 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ 8 \end{bmatrix}$$

Here, $A = \begin{bmatrix} 3 & 4 & 7 \\ 2 & -1 & 3 \\ 1 & 2 & -3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ -3 \\ 8 \end{bmatrix}$

$$|A| = \begin{vmatrix} 3 & 4 & 7 \\ 2 & -1 & 3 \\ 1 & 2 & -3 \end{vmatrix}$$

$$= 3(3-6) - 4(-6-3) + 7(4+1)$$
$$= 3 \times (-3) - 4 \times (-9) + 7 \times 5$$
$$= -9 + 36 + 35$$
$$= 62 \neq 0$$

$\therefore A^{-1}$ exists.

So, the given system of equations has a unique solution given by $X = A^{-1}B$

Let A_{ij} be the cofactors of elements a_{ij} in A.

$A_{11} = (3-6) = -3$, $A_{12} = -(-6-3) = 9$,

$A_{13} = (4+1) = 5$

$A_{21} = -(-12-14) = 26$, $A_{22} = (-9-7) = -16$,

$A_{23} = -(6-4) = -2$

$A_{31} = (12+7) = 19$, $A_{32} = -(9-14) = 5$,

$A_{33} = (-3-8) = -11$

$$\text{Adj } A = \begin{bmatrix} -3 & 9 & 5 \\ 26 & -16 & -2 \\ 19 & 5 & -11 \end{bmatrix}^T = \begin{bmatrix} -3 & 26 & 19 \\ 9 & -16 & 5 \\ 5 & -2 & -11 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{Adj } A$$

$$\Rightarrow A^{-1} = \frac{1}{62}\begin{bmatrix} -3 & 26 & 19 \\ 9 & -16 & 5 \\ 5 & -2 & -11 \end{bmatrix}$$

Now, $X = A^{-1}B$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{62}\begin{bmatrix} -3 & 26 & 19 \\ 9 & -16 & 5 \\ 5 & -2 & -11 \end{bmatrix}\begin{bmatrix} 4 \\ -3 \\ 8 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{62}\begin{bmatrix} -12-78+152 \\ 36+48+40 \\ 20+6-88 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{62}\begin{bmatrix} 62 \\ 124 \\ -62 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Hence, $x = 1, y = 2, z = -1$ is the required solution.

Ans.

Mathematics 2012 (Delhi) — SET I

Time allowed : 3 hours **Maximum marks : 100**

SECTION–A

1. If a line has direction ratios 2, – 1, – 2, then what are its direction cosines ? [1]

Solution : Given direction ratios are 2, – 1, – 2

i.e., $a = 2, b = -1, c = -2$.

Direction cosines are :

$$l = \frac{a}{\sqrt{a^2+b^2+c^2}} = \frac{2}{\sqrt{4+1+4}} = \frac{2}{\sqrt{9}} = \frac{2}{3}$$

$$m = \frac{b}{\sqrt{a^2+b^2+c^2}} = \frac{-1}{\sqrt{4+1+4}} = \frac{-1}{\sqrt{9}} = \frac{-1}{3}$$

$$n = \frac{c}{\sqrt{a^2+b^2+c^2}} = \frac{-2}{\sqrt{4+1+4}} = \frac{-2}{3}$$

$\therefore$ Direction cosines are $\frac{2}{3}, \frac{-1}{3}, \frac{-2}{3}$. **Ans.**

2. Find 'λ' when the projection of $\vec{a} = \lambda\hat{i} + \hat{j} + 4\hat{k}$ on $\vec{b} = 2\hat{i} + 6\hat{j} + 3\hat{k}$ is 4 units. [1]

Solution : Given vectors are, $\vec{a} = \lambda\hat{i} + \hat{j} + 4\hat{k}$, $\vec{b} = 2\hat{i} + 6\hat{j} + 3\hat{k}$

The projection of $\vec{a}$ on $\vec{b}$

$$= \frac{\vec{a}.\vec{b}}{|\vec{b}|}$$

$$\vec{a}.\vec{b} = (\lambda\hat{i} + \hat{j} + 4\hat{k}).(2\hat{i} + 6\hat{j} + 3\hat{k})$$
$$= 2\lambda + 6 + 12$$

$$= 2\lambda + 18$$

$$|\vec{b}| = \sqrt{2^2+6^2+3^2}$$

$$= \sqrt{49} = 7$$

$$\therefore \quad \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = 4$$

$$\Rightarrow \quad \frac{2\lambda+18}{7} = 4$$

$$\Rightarrow \quad 2\lambda + 18 = 28$$

$$\Rightarrow \quad 2\lambda = 10$$

$$\Rightarrow \quad \lambda = 5$$

Ans.

3. Find the sum of the vectors $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$, $\vec{b} = -2\hat{i} + 4\hat{j} + 5\hat{k}$ and $\vec{c} = \hat{i} - 6\hat{j} - 7\hat{k}$. **[1]**

Solution : Given vectors are $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$, $\vec{b} = -2\hat{i} + 4\hat{j} + 5\hat{k}$ and $\vec{c} = \hat{i} - 6\hat{j} - 7\hat{k}$.

$$\vec{a} + \vec{b} + \vec{c} = \hat{i} - 2\hat{j} + \hat{k} - 2\hat{i} + 4\hat{j} + 5\hat{k} + \hat{i} - 6\hat{j} - 7\hat{k}$$

$$= 0\hat{i} - 4\hat{j} - \hat{k}.$$

Ans.

4. Evaluate : $\int_2^3 \frac{1}{x} dx$. **[1]**

Solution : Given, $\int_2^3 \frac{1}{x} dx$

$$= [\log x]_2^3 \qquad \left[\because \int \frac{1}{x} dx = \log x\right]$$

$$= \log 3 - \log 2$$

$$= \log\left(\frac{3}{2}\right)$$

Ans.

5. Evaluate : $\int (1-x)\sqrt{x}\, dx$. **[1]**

Solution : Let, $t = \sqrt{x}$

$$\Rightarrow \quad t^2 = x$$

Differentiating on both sides w.r.t. 'x', we get

$$2t\,dt = dx$$

$$= \int 2(1-t^2)t^2 dt$$

$$= 2\left[\int t^2 dt - \int t^4 dt\right]$$

$$= 2\left[\frac{t^3}{3} - \frac{t^5}{5}\right] + C$$

Put, $t = \sqrt{x}$

$$= 2\frac{x^{3/2}}{3} - 2\frac{x^{5/2}}{5} + C.$$

Ans.

6. If $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$, write the minor of the element a_{23}. **[1]**

Solution : Let, $A = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$

Minor of the element

$$a_{23} = \begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix}$$

$$= 10 - 3$$

$$= 7.$$

Ans.

7. If $\begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix}\begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} -4 & 6 \\ -9 & x \end{pmatrix}$, write the value of x. **[1]**

Solution : $\begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix}\begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} -4 & 6 \\ -9 & x \end{pmatrix}$

$$\begin{pmatrix} 2-6 & -6+12 \\ 5-14 & -15+28 \end{pmatrix} = \begin{pmatrix} -4 & 6 \\ -9 & x \end{pmatrix}$$

$$\begin{pmatrix} -4 & 6 \\ -9 & 13 \end{pmatrix} = \begin{pmatrix} -4 & 6 \\ -9 & x \end{pmatrix}$$

Comparing both sides, we get $x = 13$. **Ans.**

8. Simplify :

$$\cos\theta\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta\begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$$ **[1]**

Solution :

$$\cos\theta\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta\begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ -\cos\theta\sin\theta & \cos^2\theta \end{bmatrix} + \begin{bmatrix} \sin^2\theta & -\cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

Ans.

9. Write the principal value of

$$\cos^{-1}\left(\frac{1}{2}\right) - 2\sin^{-1}\left(-\frac{1}{2}\right).$$ **[1]**

Solution : $\cos^{-1}\left(\frac{1}{2}\right) - 2\sin^{-1}\left(-\frac{1}{2}\right)$

$$= \cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right)$$

$[\because \sin^{-1}(-x) = -\sin^{-1} x]$

$$= \frac{\pi}{3} + 2\frac{\pi}{6}$$

$$= \frac{\pi}{3} + \frac{\pi}{3}$$

$$= \frac{2\pi}{3}.$$

Ans.

10. Let * be a 'binary' operation of N given by $a * b$ = LCM (a, b) for all $a, b \in$ N. Find 5 * 7.** **[1]**

SECTION–B

11. If $(\cos x)^y = (\cos y)^x$, find $\frac{dy}{dx}$. **[4]**

Solution : Given, $(\cos x)^y = (\cos y)^x$

Taking log on both sides, we get

$$\log (\cos x)^y = \log (\cos y)^x$$

$$\Rightarrow \quad y \log \cos x = x \log \cos y$$

On differentiating w.r. t. x, we get

$$y.\frac{1}{\cos x}(-\sin x) + \log \cos x.\frac{dy}{dx} = x\frac{1}{\cos y}(-\sin y)\frac{dy}{dx} + \log \cos y$$

$$\Rightarrow -y\tan x + \log \cos x\frac{dy}{dx} = -x\tan y\frac{dy}{dx} + \log \cos y$$

$$\Rightarrow (\log \cos x + x\tan y)\frac{dy}{dx} = \log \cos y + y\tan x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\log \cos y + y\tan x}{\log \cos x + x\tan y}.$$

Ans.

OR

If $\sin y = x \sin (a+y)$, prove that $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$.

Solution : Given, $\sin y = x \sin (a+y)$

$$\Rightarrow \quad x = \frac{\sin y}{\sin(a+y)}$$

Differentiating both sides w.r.t. y

$$\frac{dx}{dy} = \frac{\sin (a+y)\cos y - \sin y \cos (a+y)}{\sin^2(a+y)}$$

$$= \frac{\sin (a+y-y)}{\sin^2(a+y)}$$

$[\because \sin (A-B) = \sin A \cos B - \cos A \sin B]$

$$= \frac{\sin a}{\sin^2(a+y)}$$

Taking reciprocal

$$\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}.$$

Hence Proved.

12. How many times must a man toss a fair coin, so that the probability of having at least one head is more than 80%? **[4]**

Solution : Let us consider,

x = Number of times a man should toss a coin.

P(H) = Probability of getting atleast one head.

If $x = 1$, sample space will be H, T

$$\therefore \quad P(H) = \frac{1}{2} = 50\%$$

If $x = 2$, sample space will be HH, HT, TH, TT

Therefore, $P(H) = \frac{3}{4} = 75\%$

If $x = 3$, sample space will be HHH, HHT, HTH, HTT, TTT, TTH, THH, THT.

$$\therefore \quad P(H) = \frac{7}{8} = 87.5\% > 80\%$$

Hence, a coin should be tossed 3 times in order to have the probability of getting atleast one head is more than 80%. **Ans.**

13. Find the vector and cartesian equations of the line passing through the point (1, 2, – 4) and perpendicular to the two lines $\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7}$ and $\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$. **[4]**

Solution : Let, the direction ratios of required line be p, q, r and given that, line is perpendicular to two given lines,

Therefore, we get

$$3p - 16q + 7r = 0$$
$$3p + 8q - 5r = 0$$

On solving, we get

$$\frac{p}{80-56} = \frac{p}{-(-15-21)} = \frac{r}{24+48}$$

$$\Rightarrow \quad \frac{p}{24} = \frac{q}{36} = \frac{r}{72}$$

$$\text{or} \quad \frac{p}{2} = \frac{q}{3} = \frac{r}{6}.$$

The required line passing through (1, 2, – 4) has direction ratios proportional to 2, 3, 6.

So cartesian equation of line is

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6}$$

Equation of line passing through (1, 2, – 4) has

**Answer is not given due to the change in present syllabus

direction ratios, proportional to 2, 3, 6.
i.e. in vector form, this line passes through point having position vector,

$$\vec{a} = \hat{i}+2\hat{j}-4\hat{k}$$

This is parallel to vector

$$\vec{b} = 2\hat{i}+3\hat{j}+6\hat{k}$$

Therefore, vector form of line will be

$$\vec{r} = \vec{a}+\lambda\vec{b}$$

$$\Rightarrow \quad = (\hat{i}+2\hat{j}-4\hat{k})+\lambda(2\hat{i}+3\hat{j}+6\hat{k}).$$

Ans.

14. If $\vec{a}, \vec{b}, \vec{c}$ are three vectors such that $|\vec{a}|=5, |\vec{b}|$ = 12 and $|\vec{c}|=13$ and $\vec{a}+\vec{b}+\vec{c}=\vec{0}$ find the value of $\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}$. **[4]**

Solution : Given that, $|\vec{a}|=5, |\vec{b}|=12, |\vec{c}|=13$ and $\vec{a}+\vec{b}+\vec{c}=0$

We know that,

$$|\vec{a}+\vec{b}+\vec{c}|^2$$

$$= |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a})$$

$$\Rightarrow 0 = 25+144+169+2(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a})$$

$$\Rightarrow (\vec{a}.\vec{b}+\vec{b}\,\vec{c}+\vec{c}.\vec{a}) = \frac{-338}{2}$$

$$= -169.$$

Ans.

15. Solve the following differential equation :

$2x^2\frac{dy}{dx}-2xy+y^2=0.$ **[4]**

Solution : The given differential equation is :

$$2x^2\frac{dy}{dx}-2xy+y^2=0$$

$$\Rightarrow \quad 2x^2\frac{dy}{dx} = 2xy-y^2$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2xy-y^2}{2x^2} \quad ...(i)$$

Put, $\quad y = vx$

Differentiate w.r. t. 'x' on both sides, we get

$$\frac{dy}{dx} = v+\frac{xdv}{dx}$$

From (i)

$$\Rightarrow \quad v+\frac{xdv}{dx} = \frac{2x^2v-v^2x^2}{2x^2}$$

$$\Rightarrow \quad v+\frac{xdv}{dx} = \frac{2v-v^2}{2}$$

$$\Rightarrow \quad \frac{xdv}{dx} = \frac{2v-v^2}{2}-v$$

$$\Rightarrow \quad \frac{xdv}{dx} = -\frac{v^2}{2}$$

$$\frac{-1}{v^2}dv = \frac{1}{2x}dx$$

Integrating both sides, we get

$$-\int\frac{1}{v^2}dv = \int\frac{1}{2x}dx$$

$$\frac{1}{v} = \frac{1}{2}\log|x|+C$$

$$\frac{1}{\frac{y}{x}} = \frac{1}{2}\log|x|+C$$

$$\frac{x}{y} = \frac{1}{2}\log x+C$$

$$2x = y\log x+2yC.$$

Ans.

16. Find the particular solution of the following differential equation;

$\frac{dy}{dx}=1+x^2+y^2+x^2y^2$, given that $y = 1$ when $x=0.$ **[4]**

Solution : Given, $\frac{dy}{dx} = 1+x^2+y^2+x^2y^2$, $y=1, x=0.$

$$\frac{dy}{dx} = (1+x^2)(1+y^2)$$

$$\Rightarrow \quad \frac{dy}{1+y^2} = (1+x^2)\,dx$$

Integrating both sides

$$\int\frac{dy}{1+y^2} = \int(1+x^2)dx$$

$$\Rightarrow \quad \tan^{-1}y = x+\frac{x^3}{3}+C$$

and given, $y=1, x=0$

$$\tan^{-1}(1) = 0+C$$

$$\Rightarrow \quad \tan^{-1}(1) = C$$

$$\Rightarrow \quad C = \frac{\pi}{4}$$

$$\therefore \quad \tan^{-1}y = x+\frac{x^3}{3}+\frac{\pi}{4}.$$

Ans.

17. Evaluate : $\int\sin x\sin 2x\sin 3x\,dx.$ **[4]**

Solution : $\int\sin x\sin 2x\sin 3x\,dx$

Multiply and divide by 2,

$$= \frac{1}{2}\int(2\sin x\sin 2x)\sin 3x\,dx$$

$= \frac{1}{2}\int(\cos x - \cos 3x)\sin 3x\,dx$

$[\because 2\sin A\sin B = \cos(A-B) - \cos(A+B)]$

$= \frac{1}{2}\int(\sin 3x\cos x - \cos 3x\sin 3x)\,dx$

$= \frac{1}{2}\int(\sin 3x\cos x - \cos 3x\sin 3x)\,dx$

$= \frac{1}{4}\int(\sin 4x + \sin 2x - \sin 6x)\,dx$

$[\ 2\sin A\cos B = \sin(A+B) + \sin(A-B)]$

$= \frac{1}{4}\left[-\frac{\cos 4x}{4} - \frac{\cos 2x}{2} + \frac{\cos 6x}{6}\right] + C.$ **Ans.**

OR

Evaluate : $\int \frac{2}{(1-x)(1+x^2)}dx.$

Solution : By method of partial fractions :

$$\frac{2}{(1-x)(1+x^2)} = \frac{A}{1-x} + \frac{Bx+C}{x^2+1}$$

Now, $2 = A(x^2+1) + (Bx+C)(1-x)$

Putting $x = 1$, we get

$2 = 2A + (B+C)0$

$\Rightarrow 2 = 2A$

$\Rightarrow A = 1$

Putting $x = 0$, we get

$2 = A + C,$

$\Rightarrow 2 = 1 + C$

$\Rightarrow C = 1$

Putting $x = -1$, we get

$2 = 2A + 2(-B+C)$

$\Rightarrow 2 = 2 + 2(-B+1)$

$\Rightarrow -B + 1 = 0$

$\Rightarrow B = 1.$

$$\int\frac{2}{(1-x)(1+x^2)}dx = \int\frac{A}{1-x}dx + \int\frac{Bx+C}{x^2+1}dx$$

$$= \int\frac{1}{1-x}dx + \int\frac{x+1}{x^2+1}dx$$

$$= \int\frac{1}{1-x}dx + \frac{1}{2}\int\frac{2x}{1+x^2}dx + \int\frac{1}{1+x^2}dx$$

$= -\log|x-1| + \frac{1}{2}\log(1+x^2) + \tan^{-1}x + C.$ **Ans.**

18. Find the point on the curve $y = x^3 - 11x + 5$ at which the equation of tangent is $y = x - 11$. [4]

Solution : Given curve is $y = x^3 - 11x + 5$ ...(i)

Slope of tangent to curve

$\Rightarrow \frac{dy}{dx} = 3x^2 - 11$

and tangent is $y = x - 11$ ...(ii)

$\Rightarrow x - y - 11 = 0$

Slope of tangent,

$\frac{dy}{dx} = 1$

Equating slopes,

$3x^2 - 11 = 1$

$\Rightarrow 3x^2 = 12$

$\Rightarrow x^2 = 4$

$\Rightarrow x = \pm 2.$

put, $x = 2$, in (i),

$y = (2)^3 - 11\times 2 + 5$

$= 8 - 22 + 5$

$= -9$

put $x = -2$ in (i),

$y = (-2)^3 - 11(-2) + 5$

$\Rightarrow y = -8 + 22 + 5$

$\Rightarrow y = 19$

The points on the curve are, (2, – 9), (– 2, 19)

Now, put these values in (ii)

$-9 = 2 - 11$

So, (2, – 9) is satisfying the tangent equation.

But (–2, 19) does not satisfy tangent equation.

Hence (2, – 9) is the required point on curve.

Ans.

OR

Using differentials, find the approximate value of $\sqrt{49.5}$.

Solution : Let $y = \sqrt{x} = \sqrt{49}$ ...(i)

$y + \Delta y = \sqrt{x+\Delta x} = \sqrt{49.5}$ ...(ii)

(ii) – (i) gives

$\Delta y = \sqrt{49.5} - \sqrt{49}$

$= \sqrt{49.5} - 7$

$\therefore \sqrt{49.5} = \Delta y + 7$ ...(iii)

$\Delta y \approx dy = \frac{dy}{dx}.\Delta x = \frac{1}{2\sqrt{x}} \times 0.5$

$\left[\because y = \sqrt{x} \Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x}}\right]$

$= \frac{1}{2\times 7} \times 0.5 = 0.036$

Put value in eq. (iii)

$\sqrt{49.5} = 0.036 + 7$

$= 7.036.$ **Ans.**

19. If $y = (\tan^{-1}x)^2$, show that

$$(x^2+1)^2\frac{d^2y}{dx^2} + 2x(x^2+1)\frac{dy}{dx} = 2.$$ **[4]**

Solution : Given, $y = (\tan^{-1} x)^2$

Differentiating w.r.t. 'x' on both sides

$$\frac{dy}{dx} = 2\tan^{-1} x . \frac{1}{x^2+1}$$

$$\Rightarrow \quad (x^2+1)\frac{dy}{dx} = 2\tan^{-1} x$$

$$\Rightarrow \quad \frac{dy}{dx}x^2 + \frac{dy}{dx} = 2\tan^{-1} x$$

Again differentiating w.r.t. 'x' on both sides

$$\frac{dy}{dx}\times 2x + x^2\frac{d^2y}{dx^2} + \frac{d^2y}{dx^2} = \frac{2}{1+x^2}$$

$$\Rightarrow \quad (1+x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = \frac{2}{1+x^2}$$

$$\Rightarrow (1+x^2)^2\frac{d^2y}{dx^2} + 2x(1+x^2)\frac{dy}{dx} = 2.$$ **Hence Proved.**

20. Using properties of determinants, prove that

$$\begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix} = 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}.$$ **[4]**

Solution : Given,

$$\begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix} = 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$$

$$\text{L.H.S.} = \begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$

$$= 2\begin{vmatrix} a+b+c & p+q+r & x+y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$= 2\begin{vmatrix} a+b+c & p+q+r & x+y+z \\ -b & -q & -y \\ -c & -r & -z \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$

$$= 2\begin{vmatrix} a & p & x \\ -b & -q & -y \\ -c & -r & -z \end{vmatrix}$$

Taking -1 common from R_2 and R_3

$$= 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} = \text{R.H.S.}$$ **Hence Proved.**

21. Prove that : $\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right) = \frac{\pi}{4} - \frac{x}{2}, x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ **[4]**

Solution :

$$\text{L.H.S.} = \tan^{-1}\left[\frac{\cos x}{1+\sin x}\right]$$

$$= \tan^{-1}\left[\frac{\cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right) + \sin^2\left(\frac{x}{2}\right) + 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}\right]$$

$$\left[\because \begin{array}{l} \cos 2\theta = \cos^2\theta - \sin^2\theta \\ \cos^2\theta + \sin^2\theta = 1 \\ \text{and } \sin 2\theta = 2\sin\theta\cos\theta \end{array}\right]$$

$$= \tan^{-1}\left(\frac{\left\{\cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right)\right\}\left\{\cos\left(\frac{x}{2}\right) + \sin\left(\frac{x}{2}\right)\right\}}{\left\{\cos\left(\frac{x}{2}\right) + \sin\left(\frac{x}{2}\right)\right\}^2}\right)$$

$[\because a^2 - b^2 = (a-b)(a+b)]$

$$= \tan^{-1}\left[\frac{\left\{\cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right)\right\}}{\cos\left(\frac{x}{2}\right) + \sin\left(\frac{x}{2}\right)}\right]$$

Dividing Numerator & Denominator with $\cos\left(\frac{x}{2}\right)$.

$$= \tan^{-1}\left[\frac{1-\tan\left(\frac{x}{2}\right)}{1+\tan\left(\frac{x}{2}\right)}\right]$$

$$= \tan^{-1}\left(\frac{\tan\frac{\pi}{4} - \tan\left(\frac{x}{2}\right)}{\tan\frac{\pi}{4}\cdot\tan\left(\frac{x}{2}\right) + 1}\right)$$

$$= \tan^{-1}\left(\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\right)$$

$$= \frac{\pi}{4} - \frac{x}{2} = \text{R.H.S.}$$ **Hence Proved.**

OR

Prove that $\sin^{-1}\left(\frac{8}{17}\right) + \sin^{-1}\left(\frac{3}{5}\right) = \cos^{-1}\left(\frac{36}{85}\right)$.

Solution :

$$\text{L.H.S.} = \sin^{-1}\left(\frac{8}{17}\right) + \sin^{-1}\left(\frac{3}{5}\right)$$

and we know that

$$\sin^{-1} x + \sin^{-1} y = \sin^{-1}\left(x\sqrt{1-y^2} + (\sqrt{1-x^2})y\right)$$

$$= \sin^{-1}\left(\frac{8}{17}\right) + \sin^{-1}\left(\frac{3}{5}\right)$$

$$= \sin^{-1}\left\{\frac{8}{17}\sqrt{1-\left(\frac{3}{5}\right)^2} + \frac{3}{5}\sqrt{1-\left(\frac{8}{17}\right)^2}\right\}$$

$$= \sin^{-1}\left\{\frac{8}{17}\sqrt{\frac{16}{25}} + \frac{3}{5}\sqrt{\frac{225}{289}}\right\}$$

$$= \sin^{-1}\left\{\frac{8}{17}\times\frac{4}{5} + \frac{3}{5}\times\frac{15}{17}\right\}$$

$$= \sin^{-1}\left(\frac{77}{85}\right) = \cos^{-1}\left(\sqrt{1-\left(\frac{77}{85}\right)^2}\right)$$

$[\because \sin^{-1}x = \cos^{-1}\sqrt{1-x^2}]$

$$= \cos^{-1}\left(\sqrt{\frac{1296}{(85)^2}}\right)$$

$$= \cos^{-1}\left(\frac{36}{85}\right) = \text{R.H.S.}$$

Hence Proved.

22. Let A = R – {3} and B = R – {1}. Consider the function f : A → B defined by $f(x) = \left(\frac{x-2}{x-3}\right)$. Show that f is one-one and onto and hence find f^{-1}. **[4]**

Solution : Let $x, y \in R$ such that

$$f(x) = f(y)$$

$$\Rightarrow \quad \frac{x-2}{x-3} = \frac{y-2}{y-3}$$

$$\Rightarrow xy - 3x - 2y + 6 = xy - 2x - 3y + 6$$

$$\Rightarrow \quad x = y$$

∴ Function is one-one.

Let, $y = f(x)$

$$y = \frac{x-2}{x-3}$$

$$\Rightarrow \quad x - 2 = xy - 3y$$

$$\Rightarrow \quad x = \frac{3y-2}{y-1}$$

Now, consider,

$$f\left(\frac{3y-2}{y-1}\right) = \frac{\frac{3y-2}{y-1} - 2}{\frac{3y-2}{y-1} - 3} = \frac{3y-2-2y+2}{3y-2-3y+3} = y$$

Since $y \neq 1$ and $\frac{3y-2}{y-1} \neq 3$

$\therefore x \in A$

$\therefore$ for value $y \in B$, there exists $x = \frac{3y-2}{y-1}$

Such that $f(x) = y$

$\Rightarrow f : A \to B$ is onto. **Hence Proved.**

Now, $f(x) = y \Rightarrow x = f^{-1}(y)$ ...(i)

$$\frac{x-2}{x-3} = y$$

$$\Rightarrow \quad x - 2 = xy - 3y$$

$$\Rightarrow \quad x = \frac{3y-2}{y-1}$$

From (i),

$$f^{-1}(y) = \frac{3y-2}{y-1}$$

Thus, $f : A \to B$ is defined as for all $x \in A$

$$f^{-1}(x) = \frac{3x-2}{x-1}.$$

Ans.

SECTION–C

23. Find the equation of the plane determined by the points A (3, –1, 2), B (5, 2, 4) and C (–1, –1, 6) and hence find the distance between the plane and the point P (6, 5, 9). **[6]**

Solution : Any plane passes through A is,

$$a(x-3) + b(y+1) + c(z-2) = 0 \quad \text{...(i)}$$

Plane (i) passes through B and C

$$a(5-3) + b(2+1) + c(4-2) = 0$$

$$\Rightarrow \quad 2a + 3b + 2c = 0$$

$$a(-1-3) + b(-1+1) + c(6-2) = 0$$

$$\Rightarrow \quad -4a + 4c = 0$$

On solving equations, we get

$$\frac{a}{12} = \frac{b}{-16} = \frac{c}{12}$$

Let, $\frac{a}{3} = \frac{b}{-4} = \frac{c}{3} = k$ (say)

$$a = 3k, b = -4k, c = 3k$$

Putting the value of a, b and c in equation (i), we get

$$3k(x-3) - 4k(y+1) + 3k(z-2) = 0$$

$$\Rightarrow 3(x-3) - 4(y+1) + 3(z-2) = 0$$

$$\Rightarrow \quad 3x - 4y + 3z = 19$$

Thus the required equation of the plane is $3x - 4y + 3z = 19$

The distance of point (6, 5, 9) from the equation of plane $3x - 4y + 3z - 19 = 0$ is,

$$\frac{|3\times6 - 4\times5 + 3\times9 - 19|}{\sqrt{3^2 + (-4)^2 + 3^2}} = \frac{|18 - 20 + 27 - 19|}{\sqrt{9+16+9}}$$

$$= \frac{6}{\sqrt{34}}. \qquad \textbf{Ans.}$$

24. Of the students in a college, it is known that 60% reside in hostel and 40% are day scholars (not residing in hostel). Previous year results report that 30% of all students who reside in hostel attain 'A' grade and 20% of dayscholars attain 'A' grade in their annual examination. At the end of the year, one student is chosen at random from the college and he has an 'A' grade, what is the probability that the student is a hostler ? [6]

Solution : Let the events be defined as

E_1 = Students reside in hostel
E_2 = Selected student is a day scholar
A = Getting "A" grade

and $P(E_1) = 0.60$
$P(E_2) = 0.40$
$P(A/E_1) = 0.30$
$P(A/E_2) = 0.20$

We know that by Bayes' theorem

$$P(E_1/A) = \frac{P(E_1).P(A/E_1)}{P(E_1).P(A/E_1)+P(E_2).P(A/E_2)}$$

$$= \frac{0.60\times0.30}{0.60\times0.30+0.40\times0.20}$$

$$= \frac{0.18}{0.26} = \frac{9}{13} = 0.69 \qquad \textbf{Ans.}$$

25. A manufacturer produces nuts and bolts. It takes 1 hour of work on machine A and 3 hours on machine B to produce a package of nuts. It takes 3 hours on machine A and 1 hour on machine B to produce a package of bolts. He earns a profit of ₹ 17.50 per package on nuts and ₹ 7 per package of bolts. How many packages of each should be produced each day so as to maximize his profits if he operate his machines for at the most 12 hours a day? Form the above as a linear programming problem and solve it graphically. [6]

Solution :

Machine	Nuts (x)	Bolts (y)	Maximum hrs.
Machine A	1	3	12
Machine B	3	1	12
Cost (in ₹)	17.50	7	

Let Packages of Nuts = x
Packages of Bolt = y

If Z denotes the total cost.
To maximise the cost, we have to maximize Z.
Maximize $Z = 17.50\,x + 7y$
Subject to the constraints

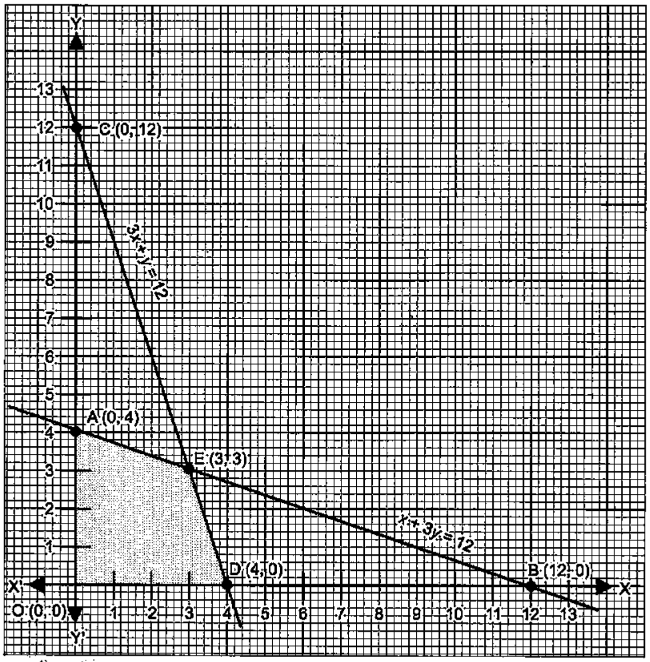
$$x + 3y \le 12$$
$$3x + y \le 12$$
$$x \ge 0, y \ge 0$$

First we draw the lines AB and CD whose equations are

$x + 3y = 12$ ...(i) $\qquad 3x + y = 12$ ...(ii)

	A	B
x	0	12
y	4	0

	C	D
x	0	4
y	12	0



The feasible region OAEDO is shaded in the figure.

The lines are intersecting the point E(3, 3).
∴ The vertices of the feasible region are O(0, 0), A (0, 4), E (3, 3) and D(4, 0).

Points	$Z = 17.50x + 7y$
At O(0, 0)	$Z = 17.50(0) + 7(0) = 0$
At A(0, 4)	$Z = 17.50(0) + 7(4) = 28$
At E(3, 3)	$Z = 17.50(3) + 7(3) = 73.50$
At D(4, 0)	$Z = 17.50(4) + 7(0) = 70$

∴ The maximum value of Z is 73.50 at the point E(3, 3).
Hence the 3 packages of nuts and 3 packages of bolt should be produced each day to get the maximum profit of ₹ 73.50. **Ans.**

26. Prove that : $\int_0^{\pi/4} (\sqrt{\tan x} + \sqrt{\cot x})\,dx = \sqrt{2}.\frac{\pi}{2}$. **[6]**

Solution : Taking L.H.S.

$$= \int_0^{\pi/4} (\sqrt{\tan x} + \sqrt{\cot x})\,dx$$

$$= \int_0^{\pi/4}\left(\sqrt{\frac{\sin x}{\cos x}}+\sqrt{\frac{\cos x}{\sin x}}\right)dx$$

$$= \int_0^{\pi/4}\left(\frac{\sin x+\cos x}{\sqrt{\sin x\cos x}}\right)dx$$

$$= \sqrt{2}\int_0^{\pi/4}\left(\frac{\sin x+\cos x}{\sqrt{2\sin x\cos x}}\right)dx$$

$$= \sqrt{2}\int_0^{\pi/4}\left(\frac{\sin x+\cos x}{\sqrt{1-(\sin x-\cos x)^2}}\right)dx$$

Let, $t = \sin x - \cos x \Rightarrow dt = (\sin x + \cos x)\, dx$

At $x = 0, t = \sin 0 - \cos 0 = -1$

At $x = \frac{\pi}{4}, t = \sin\frac{\pi}{4} - \cos\frac{\pi}{4} = 0$

$$= \sqrt{2}\int_{-1}^{0}\frac{dt}{\sqrt{1-t^2}}$$

$$= \sqrt{2}\left|\sin^{-1} t\right|_{-1}^{0}$$

$$= \sqrt{2}[\sin^{-1}(0)-\sin^{-1}(-1)]$$

$$= \sqrt{2}\sin^{-1}(1)$$

$= \sqrt{2}.\frac{\pi}{2} = \text{R.H.S.}$ **Hence Proved.**

OR

Evaluate : $\int_1^3 (2x^2+5x)\,dx$ **as a limit of a sum.**

Solution : Given, $\int_1^3 (2x^2+5x)\,dx$

We know that

$$\int_a^b f(x)dx = \lim_{h\to 0} h[f(a)+f(a+h)$$
$$+f(a+2h)+...+f\{a+(n-1)h\}],$$

where $h = \frac{b-a}{n}$

Here, $f(x) = 2x^2 + 5x$

$$h = \frac{3-1}{n} = \frac{2}{n}$$

$$\int_1^3 (2x^2+5x)dx$$

$$= \lim_{h\to 0} h[f(1)+f(1+h)+f(1+2h)+..+f\{1+(n-1)h\}]$$

$$= \lim_{h\to 0} h[\{2(1)^2+5(1)\}+\{2(1+h)^2+5(1+h)\}+\{2(1+2h)^2$$
$$+5(1+2h)\}+...+\{2(1+(n-1)h)^2+5(1+(n-1)h)\}]$$

$$= \lim_{h\to 0} h[2\{1^2+(1+h)^2+(1+2h)^2+....+(1+(n-1)h)^2\}$$
$$+5\{1+(1+h)+(1+2h)+...+(1+(n-1)h)\}]$$

$$= \lim_{h\to 0} h[2\{n+2h(1+2+3+...+(n-1))+h^2(1^2+2^2$$
$$......+(n-1)^2)\}+5\{n+h(1+2+.....+(n-1))\}]$$

$$= \lim_{h\to 0} h\left[2\left\{n+2h.\frac{n(n-1)}{2}+h^2.\frac{n(n-1)(2n-1)}{6}\right\}\right.$$
$$\left.+\left\{5n+5h\frac{(n(n-1))}{2}\right\}\right]$$

$$= \lim_{h\to 0} h\left[\left\{2n+2h.n(n-1)+h^2\frac{n(n-1)(2n-1)}{3}\right\}\right.$$
$$\left.+\left\{5n+5h\frac{(n(n-1))}{2}\right\}\right]$$

Put $h = \frac{2}{n}$

$$= \lim_{n\to\infty}\frac{2}{n}\left[\left\{2n+2.\frac{2}{n}n(n-1)+\left(\frac{2}{n}\right)^2.\frac{n(n-1)(2n-1)}{3}\right\}\right.$$
$$\left.+\left\{5n+5.\frac{2}{n}.\frac{n(n-1)}{2}\right\}\right]$$

$$= \lim_{n\to\infty}\frac{2}{n}\left[\left\{2n+4(n-1)+\frac{4(n-1)(2n-1)}{3n}\right\}\right.$$
$$\left.+\left\{5n+5(n-1)\right\}\right]$$

$$= \lim_{n\to\infty}\frac{2}{n}\left[\left\{6n-4+\frac{4(2n^2-3n+1)}{3n}\right\}+(10n-5)\right]$$

$$= \lim_{n\to\infty}\frac{2}{n}\left[\left\{\frac{18n^2-12n+8n^2-12n+4}{3n}\right\}+(10n-5)\right]$$

$$= \lim_{n\to\infty}\frac{2}{n}\left[\left(\frac{26n^2-24n+4}{3n}\right)+(10n-5)\right]$$

$$= \lim_{n\to\infty}\frac{2}{n}\left[\frac{26n^2-24n+4+30n^2-15n}{3n}\right]$$

$$= \lim_{n\to\infty}\frac{2}{n}\left[\frac{56n^2-39n+4}{3n}\right]$$

$$= \lim_{n\to\infty} \frac{2}{3}\left[\frac{56n^2 - 39n + 4}{n^2}\right]$$

$$= \lim_{n\to\infty} \frac{2}{3}\left[56 - \frac{39}{n} + \frac{4}{n^2}\right]$$

$$= \frac{2}{3}(56)$$

$$= \frac{112}{3}.$$ **Ans.**

27. Using the method of integration, find the area of the region bounded by the lines $3x - 2y + 1 = 0$, $2x + 3y - 21 = 0$ and $x - 5y + 9 = 0$. [6]

Solution : Given equations are,

$$3x - 2y + 1 = 0$$
$$2x + 3y - 21 = 0$$
$$x - 5y + 9 = 0$$

Taking, $3x - 2y + 1 = 0$

$\Rightarrow$ $$y = \frac{3x+1}{2} \quad \text{...(i)}$$

x	1	3	0
y	2	5	0.5

$$2x + 3y - 21 = 0$$

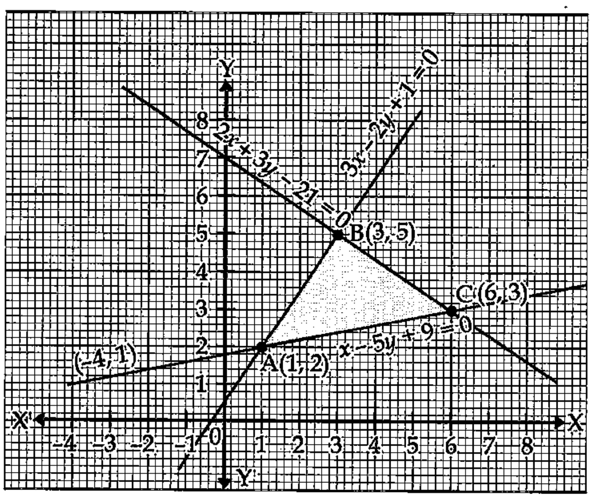
$\Rightarrow$ $$y = \frac{21-2x}{3} \quad \text{...(ii)}$$

x	3	6	0
y	5	3	7

$$x - 5y + 9 = 0$$

$\Rightarrow$ $$y = \frac{x+9}{5} \quad \text{...(iii)}$$

x	1	6	–4
y	2	3	1



The required area of shaded bounded region ABCA

= Area under line AB + Area under line BC – Area under line AC

$$= \int_1^3 \left(\frac{3x+1}{2}\right)dx + \int_3^6 \left(\frac{21-2x}{3}\right)dx - \int_1^6 \left(\frac{x+9}{5}\right)dx$$

$$= \frac{1}{2}\left[\frac{3x^2}{2} + x\right]_1^3 + \frac{1}{3}\left[21x - \frac{2x^2}{2}\right]_3^6 - \frac{1}{5}\left[\frac{x^2}{2} + 9x\right]_1^6$$

$$= \frac{1}{2}\left[\frac{3}{2}(9-1) + 3 - 1\right] + \frac{1}{3}[21(6-3) - (6^2 - 3^2)] - \frac{1}{5}\left[\frac{1}{2}(6^2 - 1^2) + 9(6-1)\right]$$

$$= \frac{1}{2}[12 + 3 - 1] + \frac{1}{3}[21(3) - (36 - 9)] - \frac{1}{5}\left[\frac{1}{2}(35) + 9(5)\right]$$

$$= \frac{1}{2}(14) + \frac{1}{3}(63 - 27) - \frac{1}{5}\left[\frac{35 + 90}{2}\right]$$

$$= 7 + \frac{36}{3} - \frac{1}{5}\left(\frac{125}{2}\right)$$

$$= 19 - \frac{25}{2} = \frac{13}{2} \text{ sq. units.}$$ **Ans.**

28. Show that the height of a closed right circular cylinder of given surface and maximum volume, is equal to the diameter of its base. [6]

Solution : Let the surface area of cylinder is 'S' and volume is 'V'.

$$S = 2\pi rh + 2\pi r^2$$

$\Rightarrow$ $$h = \frac{S - 2\pi r^2}{2\pi r} \quad \text{...(i)}$$

and $$V = \pi r^2 h$$

$\Rightarrow$ $$V = \pi r^2\left[\frac{S - 2\pi r^2}{2\pi r}\right]$$

$\Rightarrow$ $$V = r\left[\frac{S - 2\pi r^2}{2}\right]$$

$\Rightarrow$ $$V = \left[\frac{Sr - 2\pi r^3}{2}\right]$$

Differentiating both sides w.r. t. r, we get

$$\frac{dV}{dr} = \left[\frac{S - 6\pi r^2}{2}\right]$$

For maximum or minimum,

$$\frac{dV}{dr} = 0$$

$\Rightarrow$ $$\frac{S - 6\pi r^2}{2} = 0$$

$\Rightarrow$ $$S = 6\pi r^2$$

Substitute the value of S in equation (i),

$$h = \frac{6\pi r^2 - 2\pi r^2}{2\pi r}$$

$$\Rightarrow \quad h = \frac{4\pi r^2}{2\pi r}$$

$$\Rightarrow \quad h = 2r$$

Again differentiating,

$$\frac{d^2V}{dr^2} = -6\pi r < 0$$

$\therefore$ Volume is maximum when $h = 2r$.

Hence Proved.

29. Using matrices, solve the following system of linear equations :

$$x - y + 2z = 7$$
$$3x + 4y - 5z = -5$$
$$2x - y + 3z = 12 \qquad [6]$$

Solution : Given equations are

$$x - y + 2z = 7$$
$$3x + 4y - 5z = -5$$
$$2x - y + 3z = 12$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix}$$

$$AX = B \qquad ...(i)$$

$$\Rightarrow \quad X = A^{-1}B$$

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{bmatrix}$$

$$|A| = 1(12-5) + 1(9+10) + 2(-3-8)$$
$$= 7 + 19 - 22$$
$$= 4 \neq 0$$

$\therefore$ A^{-1} exists.

Cofactors of matrix A are

$A_{11} = 7$ $\quad A_{12} = -19$ $\quad A_{13} = -11$

$A_{21} = 1$ $\quad A_{22} = -1$ $\quad A_{23} = -1$

$A_{31} = -3$ $\quad A_{32} = 11$ $\quad A_{33} = 7$

$$\text{adj } A = \begin{bmatrix} 7 & -19 & -11 \\ 1 & -1 & -1 \\ -3 & 11 & 7 \end{bmatrix}^T$$

$$\Rightarrow \quad \text{adj } A = \begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj } A$$

$$= \frac{1}{4}\begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}$$

$$X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}\begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 49-5-36 \\ -133+5+132 \\ -77+5+84 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 8 \\ 4 \\ 12 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$\therefore$ Required values are, $x = 2, y = 1, z = 3$.

Ans.

OR

Using elementary operations, find the inverse of the following matrix :

$$\begin{bmatrix} -1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

Solution : Let $A = \begin{bmatrix} -1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

We know that,

$$A = I.A$$

$$\begin{bmatrix} -1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}A$$

Applying $R_1 \to 2R_1 + R_3$, we get

$$\begin{bmatrix} 1 & 3 & 5 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}A$$

Applying $R_2 \to R_2 - R_1$, we get

$$\begin{bmatrix} 1 & 3 & 5 \\ 0 & -1 & -2 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ -2 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}A$$

Applying $R_1 \to R_1 + 3R_2$ and $R_3 \to R_3 - 3R_1$, we get

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & -2 \\ 0 & -8 & -14 \end{bmatrix} = \begin{bmatrix} -4 & 3 & -2 \\ -2 & 1 & -1 \\ -6 & 0 & -2 \end{bmatrix}A$$

Applying $R_2 \to -R_2$, we get

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & -8 & -14 \end{bmatrix} = \begin{bmatrix} -4 & 3 & -2 \\ 2 & -1 & 1 \\ -6 & 0 & -2 \end{bmatrix} A$$

Applying $R_3 \to R_3 + 8R_2$, we get

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -4 & 3 & -2 \\ 2 & -1 & 1 \\ 10 & -8 & 6 \end{bmatrix} A$$

Applying $R_1 \to R_1 + \frac{1}{2} R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 1 \\ 10 & -8 & 6 \end{bmatrix} A$$

Applying $R_2 \to R_2 - R_3$ and $R_3 \to ½ R_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ -8 & 7 & -5 \\ 5 & -4 & 3 \end{bmatrix} A$$

$$I = \begin{bmatrix} 1 & -1 & 1 \\ -8 & 7 & -5 \\ 5 & -4 & 3 \end{bmatrix} A$$

$$\therefore \quad A^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ -8 & 7 & -5 \\ 5 & -4 & 3 \end{bmatrix}.$$ **Ans.**

Mathematics 2012 (Delhi) SET II

Time allowed : 3 hours **Maximum marks : 100**

Note : Except for the following questions, all the remaining questions have been asked in previous set.

SECTION — A

9. Find the sum of the following vectors :
$\vec{a} = \hat{i} - 2\hat{j}, \vec{b} = 2\hat{i} - 3\hat{j}, \vec{c} = 2\hat{i} + 3\hat{k}.$ **[1]**

Solution : The given vectors are

$$\vec{a} = \hat{i} - 2\hat{j}, \vec{b} = 2\hat{i} - 3\hat{j}, \vec{c} = 2\hat{i} + 3\hat{k}$$

$$\therefore \vec{a} + \vec{b} + \vec{c} = (\hat{i} - 2\hat{j}) + (2\hat{i} - 3\hat{j}) + (2\hat{i} + 3\hat{k})$$

$$= 5\hat{i} - 5\hat{j} + 3\hat{k}.$$ **Ans.**

10. If $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$, write the cofactor of the element a_{32}. **[1]**

Solution : Given, $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$ and $a_{32} = 2$

Minor of a_{32}

$$= \begin{vmatrix} 5 & 8 \\ 2 & 1 \end{vmatrix} = 5 - 16 = -11$$

$\therefore$ Cofactor of $a_{32} = (-1)^{3+2}(-11) = 11.$ **Ans.**

SECTION — B

19. Using properties of determinants, prove the following :

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c).$$ **[4]**

Solution : L.H.S.

$$= \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$$

Applying $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$, we get

$$= \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ a^3-b^3 & b^3-c^3 & c^3 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ (a-b)(a^2+ab+b^2) & (b-c)(b^2+bc+c^2) & c^3 \end{vmatrix}$$

Taking $(a-b)$ and $(b-c)$ common from C_1 and C_2 respectively, we get

$$= (a-b)(b-c)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ (a^2+ab+b^2) & (b^2+bc+c^2) & c^3 \end{vmatrix}$$

Expanding along R_1, we get,

$(a-b).(b-c)\ [(b^2+bc+c^2)-(a^2+ab+b^2)]$

$= (a-b).(b-c)\ [(c^2-a^2)+(bc-ab)]$

$= (a-b).(b-c)\ [(c-a)(c+a)+b(c-a)]$

$= (a-b)(b-c)(c-a)(c+a+b)$

$= (a-b)(b-c)(c-a)(a+b+c) =$ R.H.S.

Hence Proved.

20. If $y = 3\cos(\log x) + 4\sin(\log x)$, show that

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0 \qquad [4]$$

Solution : Given, $y = 3\cos(\log x) + 4\sin(\log x)$ ...(i)

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = -3\sin(\log x).\frac{d}{dx}(\log x)+4\cos(\log x).\frac{d}{dx}(\log x)$$

$$= -3\sin(\log x).\frac{1}{x}+4\cos(\log x).\frac{1}{x}$$

$$\Rightarrow \quad x.\frac{dy}{dx} = -3\sin(\log x)+4\cos(\log x)$$

Again differentiating w.r.t. x, we get

$$x.\frac{d^2y}{dx^2}+1\frac{dy}{dx} = -3\cos(\log x).\frac{1}{x}-4\sin(\log x).\frac{1}{x}$$

$$\Rightarrow x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx} = -[3\cos(\log x)+4\sin(\log x)]$$

$$= -y \qquad \text{[using (i)]}$$

$$\Rightarrow x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0.$$ **Hence Proved.**

21. Find the equation of the line passing through the point $(-1, 3, -2)$ and perpendicular to the lines

$$\frac{x}{1}=\frac{y}{2}=\frac{z}{3} \text{ and } \frac{x+2}{-3}=\frac{y-1}{2}=\frac{z+1}{5} \qquad [4]$$

Solution : Let a, b, c be the direction ratios of the line which is perpendicular to the lines

$\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ and $\frac{x+2}{-3}=\frac{y-1}{2}=\frac{z+1}{5}$, then

$$a+2b+3c = 0 \qquad \text{...(i)}$$

and $$-3a+2b+5c = 0 \qquad \text{...(ii)}$$

Solving equation (i) and (ii), we get

$$\frac{a}{10-6} = \frac{b}{-9-5}=\frac{c}{2+6}$$

$$\Rightarrow \quad \frac{a}{4} = \frac{b}{-14}=\frac{c}{8}$$

$$\Rightarrow \quad \frac{a}{2} = \frac{b}{-7}=\frac{c}{4}$$

$\therefore$ Equation of the required line passing through $(-1, 3, -2)$ having d.r.s. $2, -7, 4$ is

$$\frac{x+1}{2} = \frac{y-3}{-7}=\frac{z+2}{4}. \qquad \textbf{Ans.}$$

22. Find the particular solution of the following differential equation;

$$(x+1)\frac{dy}{dx}=2e^{-y}-1; \quad y=0 \text{ when } x=0. \qquad [4]$$

Solution : The given differential equation is

$$(x+1)\frac{dy}{dx} = 2e^{-y}-1 \qquad \text{...(i)}$$

Separate the given differential equation, we get

$$\frac{dy}{2e^{-y}-1} = \frac{dx}{x+1}$$

$$\Rightarrow \quad \frac{e^y}{2-e^y}dy = \frac{dx}{x+1}$$

On integrating, we get

$$\int\frac{e^y}{2-e^y}dy = \int\frac{dx}{x+1}+C$$

$$\Rightarrow \quad -\log(2-e^y) = \log(x+1)+C \qquad \text{...(ii)}$$

Putting $y = 0$, when $x = 0$ in (ii), we get

$$-\log(2-1) = \log(0+1)+C$$

$$\Rightarrow \quad 0 = 0+C \Rightarrow C=0$$

$\therefore$ Equation (ii) becomes

$$-\log(2-e^y) = \log(x+1)$$

$$\Rightarrow \log(x+1)+\log(2-e^y) = 0$$

$$\Rightarrow \quad \log(x+1)(2-e^y) = 0$$

$$\therefore \quad (x+1)(2-e^y) = e^0 = 1. \qquad \textbf{Ans.}$$

SECTION — C

28. A girl throws a die. If she gets a 5 or 6, she tosses a coin three times and notes the number of heads. If she gets 1, 2, 3 or 4, she tosses a coin two times and notes the number of heads obtained. If she obtained exactly two heads, what is the probability that she threw 1, 2, 3 or 4 with the die ? **[6]**

Solution : Let A_1 be even that the girl gets 5 or 6 and hence tosses a coin 3-times.

A_2 be the even that girl gets 1, 2, 3 or 4 and hence tosses a coin 2-times.

and A be even that the girl gets exactly two heads.

Now $P(A_1) = P(5 \text{ or } 6) = P(5) + P(6) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$

$$P(A_2) = P(1, 2, 3 \text{ or } 4)$$
$$= P(1) + P(2) + P(3) + P(4)$$
$$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{2}{3}$$

and $P(A/A_1) = 3/8$

Here $n = 3$; sample space = {HHH, HTH, HHT, THH, HTT, THT, TTH, TTT}

and $P(A/A_2) = \frac{1}{4}$

Here $n = 2$;

Sample space = {HH, HT, TH, TT}

$\therefore$ By Bayes' theorem

$$P(A_2/A) = \frac{P(A_2).P(A/A_2)}{P(A_1).P(A/A_1) + P(A_2).P(A/A_2)}$$

$$= \frac{\frac{2}{3} \times \frac{1}{4}}{\frac{1}{3} \times \frac{3}{8} + \frac{2}{3} \times \frac{1}{4}}$$

$$= \frac{\frac{1}{6}}{\frac{1}{8} + \frac{1}{6}} = \frac{4}{7}.$$ **Ans.**

29. Using the method of integration, find the area of the region bounded by the following lines $3x - y - 3 = 0$, $2x + y - 12 = 0$, $x - 2y - 1 = 0$. **[6]**

Solution : The given lines are

$$3x - y - 3 = 0 \Rightarrow y = 3x - 3 \quad \text{...(i)}$$
$$2x + y - 12 = 0 \Rightarrow y = 12 - 2x \quad \text{...(ii)}$$
$$x - 2y - 1 = 0 \Rightarrow y = \frac{x-1}{2} \quad \text{...(iii)}$$

Table for the line (i),

x	1	3
y	0	6

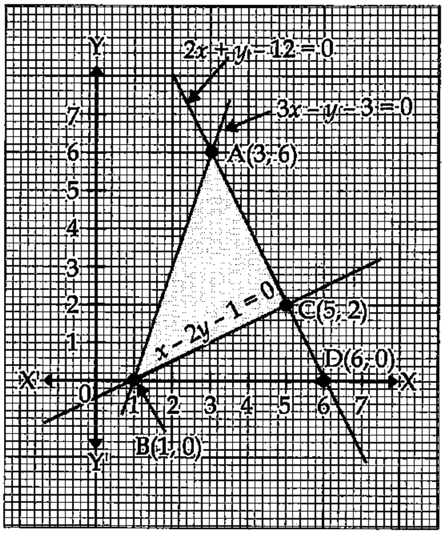
Table for the line (ii),

x	6	3	5
y	0	6	2

Table for the line (iii),

x	1	5
y	0	2

Now we draw these lines (i), (ii) and (iii).



These lines intersect at A(3, 6), B(1, 0) and C(5, 2)

$\therefore$ Area of ΔABC = Area under the line AB + Area under the line AC − Area under the line BC

$$= \int_1^3 [\text{line (i)}]dx + \int_3^5 [\text{line (ii)}]dx - \int_1^5 [\text{line (iii)}]dx$$

$$= \int_1^3 (3x-3)dx + \int_3^5 (12-2x)\,dx - \int_1^5 \left(\frac{x-1}{2}\right)dx$$

$$= \left[\frac{3}{2}x^2 - 3x\right]_1^3 + [12x - x^2]_3^5 - \frac{1}{2}\left[\frac{x^2}{2} - x\right]_1^5$$

$$= \left\{\frac{27}{2} - 9 - \left(\frac{3}{2} - 3\right)\right\} + \{60 - 25 - (36 - 9)\}$$
$$- \frac{1}{2}\left\{\frac{25}{2} - 5 - \left(\frac{1}{2} - 1\right)\right\}$$

$$= \frac{9}{2} + \frac{3}{2} + 35 - 27 - \frac{1}{2}\left(\frac{25 - 10 + 1}{2}\right)$$

$= 6 + 8 - 4 = 10$ sq. units. **Ans.**

Mathematics 2012 (Delhi)

SET III

Time allowed : 3 hours **Maximum marks : 100**

Note : Except for the following questions, all the remaining questions have been asked in previous sets.

SECTION — A

9. Find the sum of the following vectors :

$\vec{a}=\hat{i}-3\hat{k}, \vec{b}=2\hat{j}-\hat{k}, \vec{c}=2\hat{i}-3\hat{j}+2\hat{k}.$ **[1]**

Solution : The given vectors are

$\vec{a}=\hat{i}-3\hat{k}, \vec{b}=2\hat{j}-\hat{k}, \vec{c}=2\hat{i}-3\hat{j}+2\hat{k}$

$$\therefore \vec{a}+\vec{b}+\vec{c}=(\hat{i}-3\hat{k})+(2\hat{j}-\hat{k})+(2\hat{i}-3\hat{j}+2\hat{k})$$

$$=3\hat{i}-\hat{j}-2\hat{k}.$$ **Ans.**

10. If $\Delta=\begin{vmatrix}1 & 2 & 3\\ 2 & 0 & 1\\ 5 & 3 & 8\end{vmatrix}$, **write the minor of the element a_{22}.** **[1]**

Solution : Given, $\Delta=\begin{vmatrix}1 & 2 & 3\\ 2 & 0 & 1\\ 5 & 3 & 8\end{vmatrix}$

$$\therefore \quad \text{Minor of } a_{22}=\begin{vmatrix}1 & 3\\ 5 & 8\end{vmatrix}$$

$$=8-15=-7.$$ **Ans.**

SECTION–B

19. Using properties of determinants, prove the following :

$$\begin{vmatrix}1+a & 1 & 1\\ 1 & 1+b & 1\\ 1 & 1 & 1+c\end{vmatrix}=ab+bc+ca+abc.$$ **[4]**

Solution : L.H.S. = $\begin{vmatrix}1+a & 1 & 1\\ 1 & 1+b & 1\\ 1 & 1 & 1+c\end{vmatrix}$

Applying $C_1 \to C_1-C_3$ and $C_2 \to C_2-C_3$, we get

$$=\begin{vmatrix}a & 0 & 1\\ 0 & b & 1\\ -c & -c & 1+c\end{vmatrix}$$

Applying $C_1 \to C_1-a.C_3$, we get

$$=\begin{vmatrix}0 & 0 & 1\\ -a & b & 1\\ -c-a-ac & -c & 1+c\end{vmatrix}$$

Expanding along R_1, we get

$ac-b(-c-a-ac)$

$=ac+bc+ba+bac$

$=ab+bc+ca+abc=$ R.H.S. **Hence Proved.**

20. If $y=\sin^{-1}x$, show that $(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}=0$ **[4]**

Solution : Given, $y=\sin^{-1}x$,

Differentiating w.r. to x, we get

$$\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad \sqrt{1-x^2}.\frac{dy}{dx}=1$$

Again differentiating w.r. t. x, we get

$$\sqrt{1-x^2}.\frac{d^2y}{dx^2}+\frac{1}{2}.\frac{1}{\sqrt{1-x^2}}.(-2x)\frac{dy}{dx}=0$$

$$\Rightarrow \quad \sqrt{1-x^2}.\frac{d^2y}{dx^2}-\frac{x}{\sqrt{1-x^2}}.\frac{dy}{dx}=0$$

Multiplying by $\sqrt{1-x^2}$ on both sides, we get

$$(1-x^2)\frac{d^2y}{dx^2}-x.\frac{dy}{dx}=0.$$

Hence Proved.

21. Find the particular solution of the following differential equation :

$xy\frac{dy}{dx}=(x+2)(y+2);\ y=-1$ **when** $x=1.$ **[4]**

Solution : The given differential equation is

$$xy\frac{dy}{dx}=(x+2)(y+2) \quad \text{...(i)}$$

Separate the given differential equation, we get

$$\frac{y}{y+2}dy=\frac{x+2}{x}dx$$

On integrating, we get

$$\int \frac{y}{y+2}dy = \int \frac{x+2}{x}dx + C$$

$$\Rightarrow \int \left[1-\frac{2}{y+2}\right]dy = \int \left[1+\frac{2}{x}\right]dx + C$$

$$\Rightarrow \quad y - 2\log(y+2) = x + 2\log x + C \qquad ...(ii)$$

Putting $y = -1$, when $x = 1$ in equation (ii), we get

$$-1 - 2\log 1 = 1 + 2\log 1 + C$$

$$\Rightarrow -1 - 0 = 1 + 0 + C \Rightarrow C = -2$$

$\therefore$ From equation (ii),

$$y - 2\log(y+2) = x + 2\log x - 2$$

$$\Rightarrow \quad y - x + 2 = 2[\log x + \log(y+2)]$$

$$\Rightarrow \quad y - x + 2 = 2\log\{x(y+2)\}$$

This is the required particular solution of the given differential equation. **Ans.**

22. Find the equation of a line passing through the point P(2, – 1, 3) and perpendicular to the lines

$$\vec{r} = (\hat{i}+\hat{j}-\hat{k}) + \lambda(2\hat{i}-2\hat{j}+\hat{k}) \text{ and}$$

$$\vec{r} = (2\hat{i}-\hat{j}-3\hat{k}) + \mu(\hat{i}+2\hat{j}+2\hat{k}). \qquad [4]$$

Solution : The given lines are

$$\vec{r} = (\hat{i}+\hat{j}-\hat{k}) + \lambda(2\hat{i}-2\hat{j}+\hat{k})$$

$$\vec{r} = (2\hat{i}-\hat{j}-3\hat{k}) + \mu(\hat{i}+2\hat{j}+2\hat{k})$$

The required line is $\perp$ to both these lines which are parallel to the vectors

$\vec{b_1} = 2\hat{i}-2\hat{j}+\hat{k}$ and $\vec{b_2} = \hat{i}+2\hat{j}+2\hat{k}$, respectively

$\Rightarrow$ The required line is parallel to the vector

$$\vec{b} = \vec{b_1} \times \vec{b_2}$$

Now, $\vec{b} = \vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -2 & 1 \\ 1 & 2 & 2 \end{vmatrix} = -6\hat{i} - 3\hat{j} + 6\hat{k}$

Now, to find the line passing through (2, – 1, 3) which is parallel to vector $\vec{b}$.

$$\therefore \quad \vec{r} = (2\hat{i}-\hat{j}+3\hat{k}) + t(-6\hat{i}-3\hat{j}+6\hat{k})$$

$$= (2\hat{i}-\hat{j}+3\hat{k}) - 3t(2\hat{i}+\hat{j}-2\hat{k})$$

$$= (2\hat{i}-\hat{j}+3\hat{k}) + \lambda(2\hat{i}+\hat{j}-2\hat{k})$$

where $\lambda = -3t$. **Ans.**

SECTION–C

28. Bag I contains 3 red and 4 black balls and Bag II contains 4 red and 5 black balls. Two balls are transferred at random from Bag I to Bag II and then a ball is drawn from Bag II. The ball so drawn is found to be red in colour. Find the probability that the transferred balls were both black. **[6]**

Solution : Let,

E_1 = Both transferred balls from bag I to bag II are red.

E_2 = Both transferred balls from bag I to bag II are black.

E_3 = Out of transferred balls one is black and other is red.

A = Drawing a red ball from bag II

$$P(E_1) = \frac{^3C_2}{^7C_2} = \frac{3!}{2! \times 1!} \times \frac{2! \times 5!}{7!} = \frac{1}{7}$$

$$P(E_2) = \frac{^4C_2}{^7C_2} = \frac{4!}{2! \times 1!} \times \frac{2! \times 5!}{7!} = \frac{2}{7}$$

$$P(E_3) = \frac{^3C_1 \times {^4C_1}}{^7C_2} = \frac{3\times 4}{7!} \times \frac{2! \times 5!}{7!} = \frac{4}{7}$$

$$P(A/E_1) = \frac{6}{11}, \quad P(A/E_2) = \frac{4}{11}, \quad P(A/E_3) = \frac{5}{11}$$

Required probability

$P(E_2/A)$ =

$$\frac{P(E_2).P(A/E_2)}{P(E_1).P(A/E_1) + P(E_2).P(A/E_2) + P(E_3).P(A/E_3)}$$

$$= \frac{\frac{2}{7} \times \frac{4}{11}}{\frac{1}{7} \times \frac{6}{11} + \frac{2}{7} \times \frac{4}{11} + \frac{4}{7} \times \frac{5}{11}}$$

$$= \frac{\frac{8}{77}}{\frac{6}{77} + \frac{7}{77} + \frac{20}{77}}$$

$$= \frac{4}{17}$$ **Ans.**

29. Using the method of integration, find the area of the region bounded by the following lines $5x-2y-10=0, x+y-9=0, 2x-5y-4=0$. [6]

Solution : The given lines are

$$5x-2y-10=0 \Rightarrow y=\frac{5x-10}{2} \quad \text{...(i)}$$
$$x+y-9=0 \Rightarrow y=9-x \quad \text{...(ii)}$$
$$2x-5y-4=0 \Rightarrow y=\frac{2x-4}{5} \quad \text{...(iii)}$$

Table for the line (i),

x	2	4
y	0	5

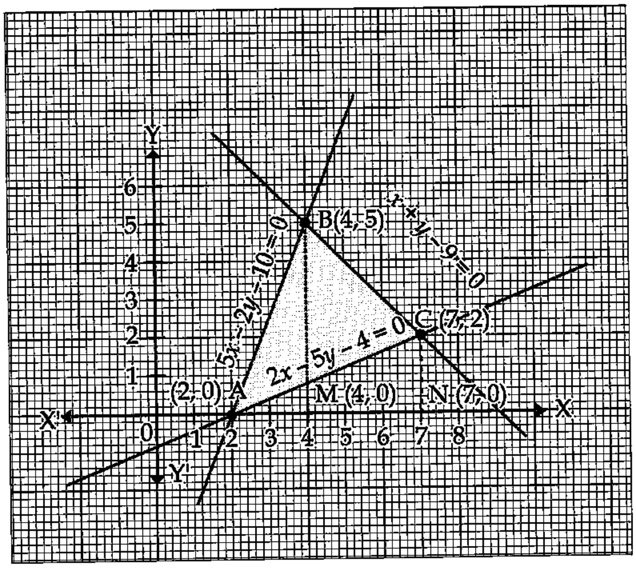
Table for the line (ii),

x	7	4
y	2	5

Table for the line (iii),

x	2	7
y	0	2

Now we draw these lines (i), (ii) and (iii),



These lines intersect at A(2, 0), B(4, 5) and C(7, 2).

$\therefore$ Area of Δ ABC = Area AMB + Area BMNC − Area ANC

$$=\int_2^4[\text{line (i)}]dx+\int_4^7[\text{line (ii)}]dx-\int_2^7[\text{line (iii)}]dx$$

$$=\int_2^4\frac{5x-10}{2}dx+\int_4^7(9-x)dx-\int_2^7\frac{2x-4}{5}dx$$

$$=\frac{1}{2}\left[\frac{5x^2}{2}-10x\right]_2^4+\left[9x-\frac{x^2}{2}\right]_4^7-\frac{1}{5}\left[x^2-4x\right]_2^7$$

$$=\frac{1}{2}[40-40-(10-20)]$$

$$+\left[63-\frac{49}{2}-(36-8)\right]-\frac{1}{5}[49-28-(4-8)]$$

$$=5+35-\frac{49}{2}-5=\frac{21}{2}\text{sq. units.}$$ **Ans.**

Mathematics 2013 (Outside Delhi) SET I

Time allowed : 3 hours **Maximum marks : 100**

SECTION — A

1. Write the principal value of $\tan^{-1}(\sqrt{3}) - \cot^{-1}(-\sqrt{3})$ [1]

Solution : Given,

$\tan^{-1}(\sqrt{3}) - \cot^{-1}(-\sqrt{3})$

$$= \tan^{-1}\sqrt{3} - \left(\pi - \cot^{-1}\sqrt{3}\right)$$

$$[\because \cot^{-1}(-x) = \pi - \cot^{-1}x]$$

$$= \tan^{-1}\sqrt{3} + \cot^{-1}\sqrt{3} - \pi$$

$$= \frac{\pi}{2} - \pi = -\frac{\pi}{2}$$ **Ans.**

2. Write the value of $\tan^{-1}\left[2\sin\left(2\cos^{-1}\frac{\sqrt{3}}{2}\right)\right]$ [1]

Solution : $\tan^{-1}\left[2\sin\left(2\cos^{-1}\frac{\sqrt{3}}{2}\right)\right]$

$$= \tan^{-1}\left[2\sin\left\{2\cos^{-1}\left(\cos\frac{\pi}{6}\right)\right\}\right]$$

$$= \tan^{-1}\left[2\sin\left\{2\left(\frac{\pi}{6}\right)\right\}\right]$$

$$= \tan^{-1}\left[2\sin\left(\frac{\pi}{3}\right)\right]$$

$$= \tan^{-1}\left[2\left(\frac{\sqrt{3}}{2}\right)\right]$$

$$= \tan^{-1}(\sqrt{3})$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{3}\right)\right]$$

$$= \frac{\pi}{3}.$$ **Ans.**

3. For what value of x, is the matrix $A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ x & -3 & 0 \end{bmatrix}$ a skew symmetric matrix ? [1]

Solution : We see that,

$$a_{31} = x$$

Given that the matrix 'A' is skew symmetric

$\therefore$ $$a_{ij} = -a_{ji}$$

$\Rightarrow$ $$a_{31} = -a_{13}$$

$\therefore$ $$x = -(-2) = 2.$$ **Ans.**

4. If matrix $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $A^2 = KA$, then write the value of K. [1]

Solution : Given, $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

and $$A^2 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times1+(-1)\times(-1) & 1\times(-1)+(-1)\times1 \\ (-1)\times1+1\times(-1) & (-1)\times(-1)+1\times1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1 & -1-1 \\ -1-1 & 1+1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

$\Rightarrow$ $$A^2 = 2\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 2A$$

Also,

$$A^2 = KA$$

$\therefore$ $$K = 2.$$ **Ans.**

5. Write the differential equation representing the family of curves $y = mx$, where m is an arbitrary constant. [1]

Solution : We have,

$$y = mx$$

On differentiating, we get

$$\frac{dy}{dx} = m \quad ...(i)$$

and $$m = \frac{y}{x} \quad ...(ii) \text{ (Given)}$$

From (i) and (ii), we get

$$\frac{dy}{dx} = \frac{y}{x}$$

The differential equation representing the family of curves $y = mx$ is

$$xdy - ydx = 0.$$ **Ans.**

6. If A_{ij} is the cofactor of the element a_{ij} of the determinant $\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$ then write the value of a_{32}. A_{32}. [1]

Solution : Let $A = \begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$

$$A_{32} = -\begin{vmatrix} 2 & 5 \\ 6 & 4 \end{vmatrix} = -(8-30) = 22$$

$$a_{32} = 5$$

Thus, $a_{32}.A_{32} = 5 \times 22 = 110.$ **Ans.**

7. P and Q are two points with position vectors $3\vec{a} - 2\vec{b}$ and $\vec{a} + \vec{b}$ respectively. Write the position vector of a point R which divides the line segment PQ in the ratio 2 : 1 externally. [1]

Solution : Position vector of point

$$R = \frac{2(\vec{a}+\vec{b})-1(3\vec{a}-2\vec{b})}{2-1}$$

$$= 2\vec{a} + 2\vec{b} - 3\vec{a} + 2\vec{b}$$

$$= -\vec{a} + 4\vec{b}.$$ **Ans.**

8. Find $|\vec{x}|$ if for a unit vector $\vec{a}$, $(\vec{x}-\vec{a}).(\vec{x}+\vec{a}) = 15$. [1]

Solution : We have,

$(\vec{x}-\vec{a}).(\vec{x}+\vec{a}) = 15$

$\Rightarrow |\vec{x}|^2 - |\vec{a}|^2 = 15$

$\Rightarrow \quad |\vec{x}|^2 = 15 + |\vec{a}|^2 \quad (\because |\vec{a}| = 1)$

$\Rightarrow \quad |\vec{x}| = \sqrt{15 + 1^2}$

$\therefore \quad |\vec{x}| = 4.$ **Ans.**

9. Find the length of the perpendicular drawn from the origin to the plane $2x - 3y + 6z + 21 = 0$. [1]

Solution : Length of perpendicular

$$= \left|\frac{2(0)-3(0)+6(0)+21}{\sqrt{2^2+(-3)^2+6^2}}\right| = \frac{21}{\sqrt{49}}$$

$$= \frac{21}{7}$$

$$= 3.$$ **Ans.**

10. The money to be spent for the welfare of the employees of a firm is proportional to the rate of change of its total revenue (marginal revenue). If the total revenue (in rupee) received from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$, find the marginal revenue, when $x = 5$ and write which value does the question indicates. [1]

Solution : Total revenue, $R(x) = 3x^2 + 36x + 5$

Marginal revenue,

$$\frac{dR(x)}{dx} = 6x + 36$$

At $x = 5$,

$$\frac{dR(x)}{dx} = 6(5) + 36 = 66$$

Thus, marginal revenue = 66.

Money for welfare of employees is a nice step, there should be a growth in raising funds for the welfare of the employees. **Ans.**

SECTION — B

11. Consider $f : R^+ \to [4, \infty)$ given by $f(x) = x^2 + 4$. Show that f is invertible with the inverse f^{-1} of f given by $f^{-1}(y) = \sqrt{y-4}$ where R^+ is the set of all non-negative real numbers. [4]

Solution : Given, $f(x) = x^2 + 4$

Let $f(x_1) = f(x_2)$

$\Rightarrow \quad x_1^2 + 4 = x_2^2 + 4$

$\Rightarrow \quad x_1 = x_2$

Thus, $f(x)$ is one-one.

Since, $x^2 + 4$ is a real number. Thus, for every y in the co-domain of f, there exists a number x in R^+ such that

$$f(x) = y = x^2 + 4$$

Thus, we can say that $f(x)$ is onto.

Now, $f(x)$ is one-one and onto. Hence, $f(x)$ is invertible.

Let $\quad f(x) = y \Rightarrow x^2 + 4 = y$

$\Rightarrow \quad x^2 = y - 4$

i.e. $\quad x = \sqrt{y-4}$

Also, $\quad x = f^{-1}(y)$

$f^{-1}(y) = \sqrt{y-4}.$ **Hence Proved.**

12. Show that : $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) = \frac{4-\sqrt{7}}{3}$. [4]

Solution : Let $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) = x$

$$\Rightarrow \quad \frac{1}{2}\sin^{-1}\frac{3}{4} = \tan^{-1}x$$

$$\sin^{-1}\frac{3}{4} = 2\tan^{-1}x$$

$$\Rightarrow \quad \sin^{-1}\frac{3}{4} = \sin^{-1}\frac{2x}{1+x^2}$$

$$\left[\because \sin^{-1}\left(\frac{2x}{1+x^2}\right) = 2\tan^{-1}x\right]$$

Thus,

$$\frac{2x}{1+x^2} = \frac{3}{4}$$

$$\Rightarrow \quad 8x = 3+3x^2$$

$$\Rightarrow \quad 3x^2-8x+3 = 0$$

On comparing with

$$ax^2+bx+c = 0, \text{ we get}$$

$$a = 3, b=-8, c=3$$

$$\therefore \quad x = \frac{8\pm\sqrt{64-36}}{6}$$

$$\left[\because x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}\right]$$

$$\Rightarrow \quad x = \frac{4\pm\sqrt{7}}{3}$$

But $\sin 2\theta = \frac{3}{4}$

$$\Rightarrow \quad 0<2\theta<\frac{\pi}{2}$$

$$\Rightarrow \quad 0<\theta<\frac{\pi}{4}$$

Accordingly,

$$0<\tan\theta<\tan\frac{\pi}{4}$$

$$0<\tan\theta<1$$

or $\quad 0<x<1$

Thus $\quad x = \frac{4+\sqrt{7}}{3}$ is rejected

$$\Rightarrow \tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) = \frac{4-\sqrt{7}}{3} = \text{R.H.S.}$$

Hence Proved.

OR

Solve the following equation :

$$\cos(\tan^{-1}x) = \sin\left(\cot^{-1}\frac{3}{4}\right)$$

Solution : Given,

$$\cos(\tan^{-1}x) = \sin\left(\cot^{-1}\frac{3}{4}\right)$$

$$\Rightarrow \quad \cos\left(\tan^{-1}\frac{x}{1}\right) = \sin\left(\tan^{-1}\frac{4}{3}\right)$$

$$\Rightarrow \quad \cos\left(\tan^{-1}\frac{x}{1}\right) = \cos\left(\frac{\pi}{2}-\left(\tan^{-1}\frac{4}{3}\right)\right)$$

On comparing

$$\tan^{-1}\frac{x}{1} = \frac{\pi}{2}-\tan^{-1}\frac{4}{3}$$

$$\Rightarrow \tan^{-1}x+\tan^{-1}\frac{4}{3} = \frac{\pi}{2}$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{x+\frac{4}{3}}{1-\frac{4}{3}x}\right) = \frac{\pi}{2}$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{\frac{3x+4}{3}}{\frac{3-4x}{3}}\right) = \frac{\pi}{2}$$

$$\Rightarrow \quad \frac{3x+4}{3-4x} = \tan\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \quad \frac{3x+4}{3-4x} = \infty$$

$$\Rightarrow \quad \frac{3-4x}{3x+4} = 0$$

$$\Rightarrow \quad 3-4x = 0$$

$$\therefore \quad x = \frac{3}{4}$$

Ans.

13. Using properties of determinants prove the following :

$$\begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix} = 9y^2(x+y).$$ **[4]**

Solution : L.H.S.

$$= \begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$$

Applying $R_1 \to R_1+R_2+R_3$, we get

$$= \begin{vmatrix} 3x+3y & 3x+3y & 3x+3y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$$

Taking $(3x+3y)$ common from R_1, we get

$$= (3x+3y)\begin{vmatrix} 1 & 1 & 1 \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$$

Applying $C_2 \to C_2-C_1$ and $C_3 \to C_3-C_1$,

$$= 3(x+y)\begin{vmatrix} 1 & 0 & 0 \\ x+2y & -2y & -y \\ x+y & y & -y \end{vmatrix}$$

Taking $-y$ and y common from C_3 and C_2 respectively,

$$= -3y^2(x+y)\begin{vmatrix} 1 & 0 & 0 \\ x+2y & -2 & 1 \\ x+y & 1 & 1 \end{vmatrix}$$

Expanding along R_1, we get

$$= -3y^2(x+y)\,[1.(-2-1)]$$
$$= 9y^2(x+y) = \text{R.H.S.}$$

Hence Proved

14. **If $y^x = e^{y-x}$, prove that $\dfrac{dy}{dx} = \dfrac{(1+\log y)^2}{\log y}$.** **[4]**

Solution : Given,

$$y^x = e^{y-x}$$

Taking log on both sides,

$$x \log y = (y-x)\log e$$
$$\Rightarrow \quad x\log y = y - x \qquad (\because \log e = 1)$$
$$\Rightarrow \quad y = x\log y + x \qquad \text{...(i)}$$

On differentiating, we get

$$\frac{dy}{dx} = x.\frac{1}{y}.\frac{dy}{dx} + \log y + 1$$
$$\Rightarrow \quad \frac{dy}{dx} = \frac{x}{y}.\frac{dy}{dx} + \log y + 1$$
$$\Rightarrow \quad \left(1-\frac{x}{y}\right)\frac{dy}{dx} = 1 + \log y$$
$$\Rightarrow \quad \left(\frac{y-x}{y}\right)\frac{dy}{dx} = 1 + \log y$$
$$\Rightarrow \quad \frac{dy}{dx} = \frac{y(1+\log y)}{y-x}$$

Put the value of y from equation (i),

$$\frac{dy}{dx} = \frac{(x\log y + x)\,(1+\log y)}{x\log y + x - x}$$
$$\frac{dy}{dx} = \frac{(x\log y + x)(1+\log y)}{x\log y}$$
$$\therefore \quad \frac{dy}{dx} = \frac{(1+\log y)^2}{\log y}.$$

Hence Proved.

15. **Differentiate the following with respect to x :**

$$\sin^{-1}\left(\frac{2^{x+1}.3^x}{1+(36)^x}\right). \qquad \textbf{[4]}$$

Solution :

Let, $$y = \sin^{-1}\left[\frac{2^{x+1}.3^x}{1+(36)^x}\right]$$
$$= \sin^{-1}\left[\frac{2.2^x.3^x}{1+(6\times6)^x}\right]$$
$$= \sin^{-1}\left[\frac{2(6^x)}{1+(6^x)^2}\right]$$
$$\therefore \quad y = 2\tan^{-1}(6^x)$$
$$\left[\because \sin^{-1}\left(\frac{2x}{1+x^2}\right) = 2\tan^{-1}x\right]$$

On differentiating, we get

$$\frac{dy}{dx} = \frac{2}{1+(6^x)^2} \times [(6^x)\log 6]$$
$$\frac{dy}{dx} = \frac{2(6^x)\log 6}{1+(36)^x} = \frac{2.2^x.3^x\log 6}{1+(36)^x}$$
$$\frac{dy}{dx} = \left(\frac{2^{x+1}.3^x}{1+(36)^x}\right)\log 6. \qquad \textbf{Ans.}$$

16. **Find the value of k, for which**

$$f(x) = \begin{cases} \dfrac{\sqrt{1+kx}-\sqrt{1-kx}}{x}, & \text{if } -1 \le x < 0 \\[2mm] \dfrac{2x+1}{x-1}, & \text{if } 0 \le x < 1 \end{cases}$$

continuous at $x = 0$. **[4]**

Solution : At $x = 0$,

$$\text{L.H.L.} = \lim_{x\to0^-} f(x) = \lim_{h\to0} f(0-h)$$
$$= \lim_{h\to0}\frac{\left(\sqrt{1-kh}-\sqrt{1+kh}\right)}{-h} \times \frac{\left(\sqrt{1-kh}+\sqrt{1+kh}\right)}{\left(\sqrt{1-kh}+\sqrt{1+kh}\right)}$$
$$= \lim_{h\to0}\frac{(1-kh-1-kh)}{-h\left(\sqrt{1-kh}+\sqrt{1+kh}\right)}$$
$$= \frac{2k}{\sqrt{1-0}+\sqrt{1-0}} = \frac{2k}{2} = k$$
$$\text{R.H.L} = \lim_{x\to0^+} f(x) = \lim_{h\to0} f(0+h)$$
$$= \lim_{h\to0}\frac{2h+1}{h-1} = \frac{0+1}{0-1} = -1$$

Since, $f(x)$ is continuous,

$$\therefore \quad \text{L.H.L} = \text{R.H.L}$$
$$\Rightarrow \quad k = -1. \qquad \textbf{Ans.}$$

OR

If $x = a\cos^3\theta$ and $y = a\sin^3\theta$, then find the value of $\dfrac{d^2y}{dx^2}$ at $\theta = \dfrac{\pi}{6}$.

Solution :

Given, $$x = a\cos^3\theta$$
$$\Rightarrow \quad \frac{dx}{d\theta} = 3a\cos^2\theta(-\sin\theta)$$

$$\Rightarrow \quad \frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta \quad ...(i)$$

and $\quad y = a\sin^3\theta$

$$\Rightarrow \quad \frac{dx}{d\theta} = 3a\sin^2\theta\cos\theta$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta}$$

$$\Rightarrow \quad \frac{dy}{dx} = -\tan\theta$$

and $\quad \frac{d^2y}{dx^2} = -\sec^2\theta.\frac{d\theta}{dx}$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{\sec^2\theta}{3a\cos^2\theta\sin\theta} \quad \text{[using (i)]}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{1}{3a\cos^4\theta\sin\theta}$$

$$\left(\frac{d^2y}{dx^2}\right)_{at\,\theta=\frac{\pi}{6}} = \frac{1}{3a\sin\frac{\pi}{6}\cos^4\frac{\pi}{6}}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{1}{3a\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)^4}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{32}{27a}$$ **Ans.**

17. Evaluate : $\int\frac{\cos 2x - \cos 2\alpha}{\cos x - \cos\alpha}dx$. **[4]**

Solution : $\frac{\cos 2x - \cos 2\alpha}{\cos x - \cos\alpha}$

$$= \frac{-2\sin\left(\frac{2x+2\alpha}{2}\right)\sin\left(\frac{2x-2\alpha}{2}\right)}{-2\sin\left(\frac{x+\alpha}{2}\right)\sin\left(\frac{x-\alpha}{2}\right)}$$

$$= \frac{\sin(x+\alpha)\sin(x-\alpha)}{\sin\left(\frac{x+\alpha}{2}\right)\sin\left(\frac{x-\alpha}{2}\right)}$$

$$= \frac{\left[\left\{2\sin\left(\frac{x+\alpha}{2}\right)\cos\left(\frac{x+\alpha}{2}\right)\right\}\left\{2\sin\left(\frac{x-\alpha}{2}\right)\cos\left(\frac{x-\alpha}{2}\right)\right\}\right]}{\sin\left(\frac{x+\alpha}{2}\right)\sin\left(\frac{x-\alpha}{2}\right)}$$

$[\because \sin 2\theta = 2\sin\theta\cos\theta]$

$$= 4\cos\left(\frac{x+\alpha}{2}\right)\cos\left(\frac{x-\alpha}{2}\right)$$

$$= 2\left[\cos\left(\frac{x+\alpha}{2}+\frac{x-\alpha}{2}\right)+\cos\left(\frac{x+\alpha}{2}-\frac{x-\alpha}{2}\right)\right]$$

$[\because \cos(A+B) + \cos(A-B) = 2\cos A\cos B]$

$$= 2\cos x + 2\cos\alpha$$

Now,

$$\int\frac{\cos 2x - \cos 2\alpha}{\cos x - \cos\alpha} = \int(2\cos x + 2\cos\alpha)dx$$

$$= 2\sin x + 2x\cos\alpha + C.$$ **Ans.**

OR

Evaluate : $\int\frac{x+2}{\sqrt{x^2+2x+3}}dx$

Solution : $\int\frac{x+2}{\sqrt{x^2+2x+3}}dx$

Let, $\quad x+2 = A\frac{d}{dx}(x^2+2x+3)+B$

$$\Rightarrow \quad x+2 = A(2x+2)+B$$

$$\Rightarrow \quad x+2 = 2Ax+2A+B$$

On equating the coefficient of x and constant term on both sides, we get

$$1 = 2A$$

$$\Rightarrow \quad A = \frac{1}{2}$$

and $\quad 2 = 2A + B$

$$\Rightarrow \quad 2 = 2\times\frac{1}{2} + B \Rightarrow B = 2-1 = 1$$

$$\therefore \quad \int\frac{x+2}{\sqrt{x^2+2x+3}}dx$$

$$= \int\frac{\frac{1}{2}(2x+2)+1}{\sqrt{x^2+2x+3}}dx$$

$$= \frac{1}{2}\int\frac{\frac{1}{2}(2x+2)+1}{\sqrt{x^2+2x+3}}dx$$

$$+ \int\frac{1}{\sqrt{x^2+2x+3}}dx$$

$$= I_1 + I_2 \quad ...(i)$$

Now,

$$I_1 = \frac{1}{2}\int\frac{2x+2}{\sqrt{x^2+2x+3}}dx$$

Let $\quad x^2+2x+3 = t$

$$\Rightarrow \quad (2x+2)\,dx = dt$$

$$\therefore \quad I_1 = \frac{1}{2}\int\frac{1}{\sqrt{t}}dt = \frac{1}{2}\int t^{-1/2}dt = \frac{1}{2}.\frac{t^{-1/2+1}}{-1/2+1}+C_1$$

$$= \frac{1}{2}(2\sqrt{t})+C_1$$

$$= \sqrt{x^2+2x+3}+C_1 \quad ...(ii)$$

and

$$I_2 = \int \frac{1}{\sqrt{x^2+2x+3}}dx$$
$$= \int \frac{1}{\sqrt{x^2+2x+1+2}}dx$$
$$= \int \frac{1}{\sqrt{(x+1)^2+(\sqrt{2})^2}}dx$$
$$= \log\left|(x+1)+\sqrt{x^2+2x+3}\right|+C_2 \quad ...(iii)$$

Put the value of I_1 and I_2 in equation (i), we get

$$\int \frac{x+2}{\sqrt{x^2+2x+3}}dx = \sqrt{x^2+2x+3} + \log\left|(x+1)+\sqrt{x^2+2x+3}\right|+C$$

(where $C = C_1 + C_2$) **Ans.**

18. Evaluate $\int \frac{dx}{x(x^5+3)}$. **[4]**

Solution : $\int \frac{dx}{x(x^5+3)} = \int \frac{x^4}{x^5(x^5+3)}dx$

Put, $(x^5+3) = t$

$\Rightarrow \quad 5x^4 dx = dt$

$\Rightarrow \quad x^4\,dx = \frac{dt}{5}$

$\therefore \int \frac{dx}{x(x^5+3)} = \frac{1}{5}\int \frac{dt}{t(t-3)}$

Let $\frac{1}{t(t-3)} = \frac{A}{t}+\frac{B}{t-3}$

$1 = A(t-3) + Bt$

$\Rightarrow \quad 1 = At - 3A + Bt$

$\Rightarrow \quad 1 = (A+B)\,t - 3A$

On comparing the co-efficient, we get

$1 = -3A$

$\Rightarrow \quad A = -1/3$

$A + B = 0$

$\Rightarrow \quad B = 1/3$

Now,

$$\frac{1}{5}\int \frac{dt}{t(t-3)} = \frac{1}{5}\int\left(\frac{-1}{3t}+\frac{1}{3(t-3)}\right)dt$$
$$= \frac{-1}{15}\log t + \frac{1}{15}\log(t-3)+C$$
$$= \frac{-1}{15}\left(\log\left(\frac{t}{t-3}\right)\right)+C$$

$[\because \log(a/b) = \log a - \log b]$

$$= \frac{-1}{15}\left(\log\frac{(x^5+3)}{(x^5+3-3)}\right)+C$$
$$= \frac{1}{15}\left(\log\frac{x^5}{x^5+3}\right)+C. \quad \textbf{Ans.}$$

19. Evaluate : $\int_0^{2\pi} \frac{1}{1+e^{\sin x}}dx.$ **[4]**

Solution : Let, $I = \int_0^{2\pi} \frac{1}{1+e^{\sin x}}dx$

We know that,

$$\int_0^{2a} f(x)dx = \int_0^{a}\{f(x)+f(2a-x)\}dx$$

$$\therefore \quad I = \int_0^{\pi}\left(\frac{1}{1+e^{\sin x}}+\frac{1}{1+e^{\sin(2\pi-x)}}\right)dx$$
$$= \int_0^{\pi}\left(\frac{1}{1+e^{\sin x}}+\frac{1}{1+e^{-\sin x}}\right)dx$$
$$= \int_0^{\pi}\left(\frac{1}{1+e^{\sin x}}+\frac{e^{\sin x}}{1+e^{\sin x}}\right)dx$$
$$= \int_0^{\pi}\left(\frac{1+e^{\sin x}}{1+e^{\sin x}}\right)dx$$
$$= \int_0^{\pi} dx = [x]_0^{\pi}$$
$$= \pi. \quad \textbf{Ans.}$$

20. If $\vec{a} = \hat{i}-\hat{j}+7\hat{k}$ **and** $\vec{b} = 5\hat{i}-\hat{j}+\lambda\hat{k}$, **then find the value of** λ, **so that** $\vec{a}+\vec{b}$ **and** $\vec{a}-\vec{b}$ **are perpendicular vectors.** **[4]**

Solution : Given $\vec{a} = \hat{i}-\hat{j}+7\hat{k}$

and $\vec{b} = 5\hat{i}-\hat{j}+\lambda\hat{k}$

$\therefore \quad \vec{a}+\vec{b} = \hat{i}-\hat{j}+7\hat{k}+5\hat{i}-\hat{j}+\lambda\hat{k}$

$= 6\hat{i}-2\hat{j}+(7+\lambda)\hat{k}$

and $\vec{a}-\vec{b} = \hat{i}-\hat{j}+7\hat{k}-5\hat{i}+\hat{j}-\lambda\hat{k}$

$= -4\hat{i}+(7-\lambda)\hat{k}$

Since $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$ are perpendicular vectors.

$\therefore \quad (\vec{a}+\vec{b}).(\vec{a}-\vec{b}) = 0$

$[6\hat{i}-2\hat{j}+(7+\lambda)\hat{k}]\,[-4\hat{i}+(7-\lambda)\hat{k}] = 0$

$\Rightarrow -24+(7+\lambda)(7-\lambda) = 0$

$\Rightarrow \quad -24+49-\lambda^2 = 0$

$\Rightarrow \quad \lambda^2 = 25$

$\therefore \quad \lambda = \pm 5.$ **Ans.**

21. Show that the lines

$$\vec{r} = 3\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(\hat{i} + 2\hat{j} + 2\hat{k})$$
$$\vec{r} = 5\hat{i} - 2\hat{j} + \mu(3\hat{i} + 2\hat{j} + 6\hat{k})$$

are intersecting. Hence find their point of intersection. **[4]**

Solution : Consider,

$$\vec{r} = 3\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(\hat{i} + 2\hat{j} + 2\hat{k}) \quad ...(i)$$

$\therefore \quad \vec{r} = (3+\lambda)\hat{i} + (2+2\lambda)\hat{j} + (-4+2\lambda)\hat{k}$

Put $\quad \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ in above equation

$\therefore x\hat{i} + y\hat{j} + z\hat{k} = (3+\lambda)\hat{i} + (2+2\lambda)\hat{j} + (-4+2\lambda)\hat{k}$

Equating coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$, we get

$x = 3 + \lambda, y = 2 + 2\lambda, z = -4 + 2\lambda$

$\therefore$ Coordinates of any point on line (i) are

$(3 + \lambda, 2 + 2\lambda, -4 + 2\lambda)$

Consider,

$$\vec{r} = 5\hat{i} - 2\hat{j} + \mu(3\hat{i} + 2\hat{j} + 6\hat{k}) \quad ...(ii)$$

$\therefore \quad \vec{r} = (5+3\mu)\hat{i} + (-2+2\mu)\hat{j} + 6\mu\hat{k}$

Put $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ in the above equation, we get

$x\hat{i} + y\hat{j} + z\hat{k} = (5+3\mu)\hat{i} + (-2+2\mu)\hat{j} + 6\mu\hat{k}$

Equating coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$, we get

$x = 5 + 3\mu, y = -2 + 2\mu, z = 6\mu$

$\therefore$ Coordinates of any point on line (ii) are

$(5 + 3\mu, -2 + 2\mu, 6\mu)$

Line (i) and (ii) intersect if

$3 + \lambda = 5 + 3\mu \Rightarrow \lambda - 3\mu = 2 \quad ...(iii)$

$2 + 2\lambda = -2 + 2\mu \Rightarrow 2\lambda - 2\mu = -4 \quad ...(iv)$

$-4 + 2\lambda = 6\mu \Rightarrow 2\lambda - 6\mu = 4 \quad ...(v)$

Subtracting equations (iv) from (v), we get

$4\mu = -8$

$\Rightarrow \quad \mu = -2$

From equation (iv),

$2\lambda - 2(-2) = -4$

$\Rightarrow \quad 2\lambda = -4 - 4$

$\Rightarrow \quad \lambda = \frac{-8}{2} = -4$

Put the value of λ and μ in equation (iii),

$\therefore \quad \lambda - 3\mu = -4 - 3(-2)$

$= -4 + 6$

$= 2$

Putting the value of λ in $(3 + \lambda, 2 + 2\lambda, -4 + 2\lambda)$, we get point $(-1, -6, -12)$

$\therefore$ Point of intersection is $(-1, -6, -12)$. **Ans.**

OR

Find the vector equation of the plane through the points (2, 1, –1) and (–1, 3, 4) and perpendicular to the plane $x - 2y + 4z = 10$.

Solution : Equation of plane passing through (2, 1, –1) is

$$a(x-2) + b(y-1) + c(z+1) = 0 \quad ...(i)$$

This plane passes through (–1, 3, 4)

Thus, $a(-1-2) + b(3-1) + c(4+1) = 0$

$\Rightarrow \quad -3a + 2b + 5c = 0 \quad ...(ii)$

Also, the above plane is perpendicular to the plane

$x - 2y + 4z = 10$

$\therefore \quad a(1) + b(-2) + c(4) = 0$

$\Rightarrow \quad a - 2b + 4c = 0 \quad ...(iii)$

Now, we have

$-3a + 2b + 5c = 0$

and $\quad a - 2b + 4c = 0$

Solving the above equation by cross multiplication, we get

$$\frac{a}{8+10} = \frac{b}{5+12} = \frac{c}{6-2}$$

$\Rightarrow \quad \frac{a}{18} = \frac{b}{17} = \frac{c}{4} = \lambda$

$\Rightarrow \quad a = 18\lambda, \quad b = 17\lambda, \quad c = 4\lambda$

$\therefore$ Eq. (i) becomes

$18\lambda(x-2) + 17\lambda(y-1) + 4\lambda(z+1) = 0$

$\Rightarrow \quad 18x + 17y + 4z - 36 - 17 + 4 = 0$

$\Rightarrow \quad 18x + 17y + 4z - 49 = 0$

which is cartesian equation of the plane.

$\therefore$ The required vector equation of plane is

$\vec{r}.(18\hat{i} + 17\hat{j} + 4\hat{k}) = 49.$ **Ans.**

22. The probabilities of two students A and B coming to the school in time are $\frac{3}{7}$ and $\frac{5}{7}$ respectively.

Assuming that the events, A coming in time and B coming in time are independent. Find the probability of only one of them coming to the school in time. Write atleast one advantages of coming to school in time. **[4]**

Solution : Let probability that A comes on time

$= P(A)$

Let probability that B comes on time

$= P(B)$

$$\therefore \quad P(A) = \frac{3}{7}$$

$$P(B) = \frac{5}{7}$$

Probability of only one of them coming to school on time

$$= P(A\,\bar{B} \text{ or } \bar{A}\,B)$$
$$= P(A)\,P(\bar{B}) + P(\bar{A})\,P(B)$$
$$= P(A)\,[1-P(B)] + P(B)\,[1-P(A)]$$
$$= \frac{3}{7}\left(1-\frac{5}{7}\right)+\frac{5}{7}\left(1-\frac{3}{7}\right)$$
$$= \frac{3}{7}.\frac{2}{7}+\frac{5}{7}.\frac{4}{7}$$
$$= \frac{6}{49}+\frac{20}{49}=\frac{26}{49}.$$

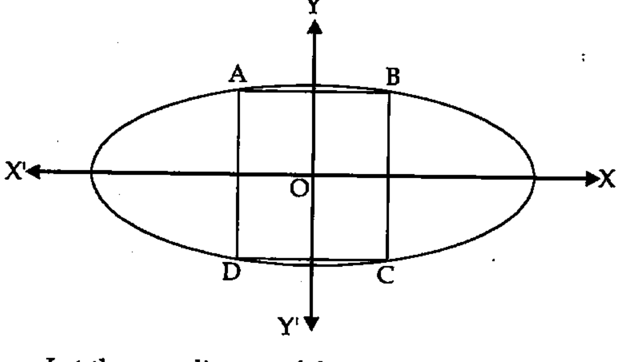
Student will not get punishment if he reach on time. **Ans.**

SECTION — C

23. Find the area of the greatest rectangle that can be inscribed in an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. **[6]**

Solution : Equation of ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1 \qquad \text{...(i)}$$



Let the coordinates of the points be

$$A = (-a\cos\theta, b\sin\theta)$$
$$B = (a\cos\theta, b\sin\theta)$$
$$C = (a\cos\theta, -b\sin\theta)$$

and $\quad D = (-a\cos\theta, -b\sin\theta)$

Thus,

$$AB = 2a\cos\theta$$
$$BC = 2b\sin\theta$$

Area of rectangle ABCD,

$$S = AB \times BC$$
$$= (2a\cos\theta).(2b\sin\theta)$$
$$\Rightarrow \quad S = 2ab\sin 2\theta$$

On differentiating, we get

$$\frac{dS}{d\theta} = 2ab(2\cos 2\theta)$$
$$\Rightarrow \quad \frac{dS}{d\theta} = 4ab\cos 2\theta \qquad \text{...(ii)}$$

For greatest area,

$$\therefore \quad \frac{dS}{d\theta} = 0$$
$$\Rightarrow \quad 4ab\cos 2\theta = 0$$
$$\Rightarrow \quad \cos 2\theta = 0$$
$$\Rightarrow \quad \cos 2\theta = \cos \pi / 2$$
$$\Rightarrow \quad 2\theta = \pi/2$$
$$\Rightarrow \quad \theta = \frac{\pi}{4}$$

On further differentiation of equation (ii), we get

$$\frac{d^2S}{d\theta^2} = -8ab\sin 2\theta$$

Put $\theta = \frac{\pi}{4}$

$$\therefore \quad \left(\frac{d^2S}{d\theta^2}\right)_{\theta=\frac{\pi}{4}} = -8ab < 0$$

$\therefore$ Area is maximum when $\theta = \frac{\pi}{4}$

$S_{max} = 2ab\sin 2\left(\frac{\pi}{4}\right) = 2ab$ sq. units. **Ans.**

OR

Find the equations of tangents to the curve $3x^2 - y^2 = 8$ which pass through the point $\left(\frac{4}{3},0\right)$.

Solution : The equation of the curve is $3x^2-y^2=8$.

On differentiating, we get

$$6x-2y.\frac{dy}{dx} = 0$$
$$\Rightarrow \quad \frac{dy}{dx} = \frac{3x}{y}$$

Let (x_1, y_1) be the point on the curve at which tangent passes through the point $\left(\frac{4}{3},0\right)$.

$\therefore \quad 3x_1^2 - y_1^2 = 8$

$y_1^2 = 3x_1^2 - 8 \quad$ ...(i)

Slope of the tangent

$$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{3x_1}{y_1}$$

Equation of the tangent passing through the point (x_1, y_1) with slope $\frac{3x_1}{y_1}$ is

$$y - y_1 = \frac{3x_1}{y_1}(x - x_1)$$

Point $\left(\frac{4}{3}, 0\right)$ lies on the above tangent,

$$\therefore \quad 0 - y_1 = \frac{3x_1}{y_1}\left(\frac{4}{3} - x_1\right)$$

$\Rightarrow y_1^2 - 3x_1^2 + 4x_1 = 0$

$\Rightarrow \quad y_1^2 = 3x_1^2 - 4x_1 \quad$ ...(ii)

Now, take equation (i) and (ii), we get

$$3x_1^2 - 8 = 3x_1^2 - 4x_1$$

$\Rightarrow \quad 4x_1 = 8$

$\Rightarrow \quad x_1 = 2$

From equation (ii), we get

$$y_1^2 = 3(2)^2 - 4(2)$$

$\Rightarrow \quad y_1^2 = 12 - 8$

$\Rightarrow \quad y_1^2 = 4$

$\Rightarrow \quad y_1 = \pm 2$

Now, equation of tangent at the point (2, 2) is given by,

$$y - 2 = \frac{3 \times 2}{2}(x - 2)$$

$\Rightarrow \quad y - 2 = 3x - 6$

$\Rightarrow \quad y - 3x + 4 = 0$

and equation of tangent at the point (2, – 2) is given by,

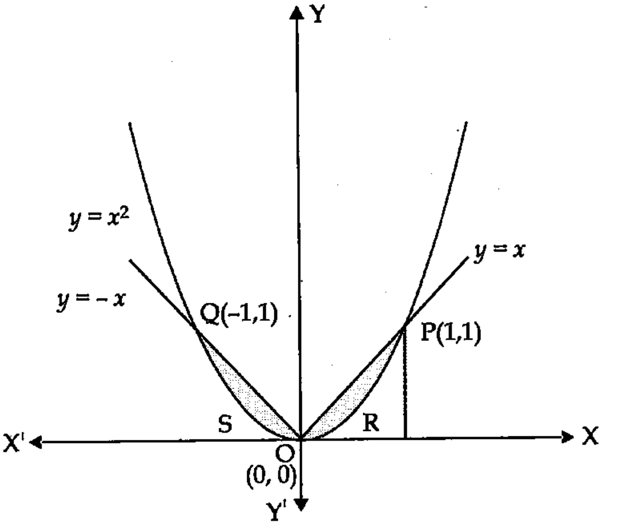
$$y + 2 = \frac{3 \times 2}{(-2)}(x - 2)$$

$\Rightarrow \quad y + 2 = -3x + 6$

$\Rightarrow \quad y + 3x - 4 = 0.$ **Ans.**

24. Find the area of the region bounded by the parabola $y = x^2$ and $y = |x|$. **[6]**

Solution : Given parabola $y = x^2$ which is symmetrical about Y-axis and passes through O(0, 0) and the curve $y = |x|$. The point of intersection of parabola, $y = x^2$ and line, $y = x$ in the first quadrant is P(1, 1). The given area is symmetrical about Y–axis.



$\therefore$ Area of OPRO = Area of OQSO.

$\therefore$ Required area = 2 (Area of shaded region in the first quadrant)

$$= 2\int_0^1 (x - x^2)dx$$

$$= 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= 2\left[\frac{1}{2} - \frac{1}{3}\right]$$

$$= 2 \times \frac{1}{6}$$

$= \frac{1}{3}$ sq. units. **Ans.**

25. Find the particular solution of the differential equation $(\tan^{-1} y - x)dy = (1 + y^2)dx$, given that when $x = 0, y = 0$. **[6]**

Solution : The given differential equation is,

$$(\tan^{-1} y - x)dy = (1 + y^2)dx,$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{(\tan^{-1} y - x)}{(1 + y^2)}$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{\tan^{-1} y}{1 + y^2} - \frac{x}{1 + y^2}$$

$$\Rightarrow \quad \frac{dx}{dy} + \frac{x}{1 + y^2} = \frac{\tan^{-1} y}{1 + y^2}$$

On comparing with the form $\frac{dx}{dy} + Px = Q$, we get

$$P = \frac{1}{1 + y^2} \text{ and } Q = \frac{\tan^{-1} y}{1 + y^2}$$

Integrating factor (I.F.) = $e^{\int P dy}$

$$= e^{\int \frac{dy}{1+y^2}} = e^{\tan^{-1} y}$$

The solution is

$$x(\text{I.F.}) = \int Q.(\text{I.F.})dy$$

$$\Rightarrow \quad x.e^{\tan^{-1} y} = \int e^{\tan^{-1} y}.\frac{\tan^{-1} y}{1+y^2} dy$$

Let $t = \tan^{-1} y$ for R.H.S

$$\Rightarrow \quad \frac{dt}{dy} = \frac{1}{1+y^2}$$

$$\Rightarrow \quad dt = \frac{dy}{1+y^2}$$

$$\Rightarrow \quad x.e^{\tan^{-1} y} = \int e^t.t\, dt$$

Integrating by parts, we get

$$x.e^{\tan^{-1} y} = t.\int e^t dt - \int\left[\frac{d}{dt}t.\int e^t dt\right]dt$$

$$\Rightarrow \quad x.e^{\tan^{-1} y} = t(e^t) - \int e^t\, dt$$

$$\Rightarrow \quad x.e^{\tan^{-1} y} = te^t - e^t + C$$

$$\Rightarrow \quad x.e^{\tan^{-1} y} = \tan^{-1} y\, e^{\tan^{-1} y} - e^{\tan^{-1} y} + C$$

When $x = 0, y = 0$

$$0 = \tan^{-1} 0\left(e^{\tan^{-1} 0}\right) - e^{\tan^{-1} 0} + C$$

$$\Rightarrow \quad 0 = 0.\, e^0 - e^0 + C$$

$$\Rightarrow \quad 0 = 0 - 1 + C$$

$$\Rightarrow \quad C = 1$$

Thus,

$$x.e^{\tan^{-1} y} = \tan^{-1} y\left(e^{\tan^{-1} y}\right) - e^{\tan^{-1} y} + 1$$

$$\Rightarrow \quad x = \tan^{-1} y - 1 + e^{-\tan^{-1} y}.$$ **Ans.**

26. Find the equation of the plane passing through the line of intersection of the planes $\vec{r}.(\hat{i}+3\hat{j})-6=0$ and $\vec{r}.(3\hat{i}-\hat{j}-4\hat{k})=0$ whose perpendicular distance from origin is unity. [6]

Solution : The equation of the plane passes through the intersection of given planes is

I[st] plane + λ (2[nd] plane) = 0

$$[\vec{r}\ (\hat{i}+3\hat{j})-6] + \lambda[\vec{r}.(3\hat{i}-\hat{j}-4\hat{k})] = 0$$

$$\Rightarrow \quad \vec{r}\ [(1+3\lambda)\hat{i} + (3-\lambda)\hat{j} - 4\lambda\hat{k}] - 6 = 0 \quad ...(i)$$

Length of perpendicular from the origin is

$$\left|\frac{-6}{\sqrt{(1+3\lambda)^2+(3-\lambda)^2+(-4\lambda)^2}}\right| = 1$$

$$\Rightarrow \quad (1+3\lambda)^2 + (3-\lambda)^2 + (-4\lambda)^2 = 36$$

$$\Rightarrow \quad 1 + 9\lambda^2 + 6\lambda + 9 + \lambda^2 - 6\lambda + 16\lambda^2 = 36$$

$$\Rightarrow \quad 26\lambda^2 = 26$$

$$\Rightarrow \quad \lambda^2 = 1$$

$$\therefore \quad \lambda = \pm 1$$

Put the value of 'λ' in equation (i), we get

When, $\lambda = 1$

$$\vec{r}.[(1+3)\hat{i} + (3-1)\hat{j} - 4\hat{k}] - 6 = 0$$

$$\vec{r}.(4\hat{i} + 2\hat{j} - 4\hat{k}) - 6 = 0$$

$$\vec{r}.(2\hat{i} + \hat{j} - 2\hat{k}) = 3$$

When, $\lambda = -1$

$$\vec{r}.[(1-3)\hat{i} + (3+1)\hat{j} + 4\hat{k}] - 6 = 0$$

$$\Rightarrow \quad \vec{r}.(-2\hat{i} + 4\hat{j} + 4\hat{k}) - 6 = 0$$

$$\Rightarrow \quad \vec{r}.(-\hat{i} + 2\hat{j} + 2\hat{k}) = 3$$

Hence, equations of the planes are $\vec{r}.(2\hat{i}+\hat{j}-2\hat{k}) = 3$ and $\vec{r}.(-\hat{i}+2\hat{j}+2\hat{k}) = 3.$ **Ans.**

OR

Find the vector equation of the line passing through the point (1, 2, 3) and parallel to the planes $\vec{r}.(\hat{i}-\hat{j}+2\hat{k})=5$ and $\vec{r}.(3\hat{i}-\hat{j}+\hat{k})=6$.

Solution : The equation of line passing through the point (1, 2, 3) is

$$\frac{x-1}{a} = \frac{y-2}{b} = \frac{z-3}{c} \quad ...(i)$$

The given planes are

$$\vec{r}.(\hat{i}-\hat{j}+2\hat{k}) = 5$$

$$\therefore \quad x - y + 2z = 5 \quad ...(ii)$$

and $\vec{r}.(3\hat{i}+\hat{j}+\hat{k}) = 6$

$$\therefore \quad 3x + y + z = 6 \quad ...(iii)$$

Since line (i) is parallel to both planes (ii) and (iii),

∴ It is perpendicular to the normal to the planes

$$a.1 + b(-1) + c.2 = 0$$

$$\Rightarrow \quad a - b + 2c = 0$$

and $a.3 + b.1 + c.1 = 0$

$$\Rightarrow \quad 3a + b + c = 0$$

Solving these equations, we get

$$\frac{a}{-1-2} = \frac{b}{6-1} = \frac{c}{1+3}$$

$$\Rightarrow \quad \frac{a}{-3} = \frac{b}{5} = \frac{c}{4}$$

∴ From (i), the required line is

$$\frac{x-1}{-3} = \frac{y-2}{5} = \frac{z-3}{4}$$

∴ The vector equation of the given line is

$$\vec{r} = \vec{a} + \lambda \vec{b}$$

$$\Rightarrow \quad \vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(-3\hat{i} + 5\hat{j} + 4\hat{k}).$$ **Ans.**

27. In a hockey match, both teams A and B scored same number of goals up to the end of the game, so to decide the winner, the referee asked both the captains to throw a die alternately and decided that the team, whose captain gets a six first, will be declared the winner. If the captain of team A was asked to start, find their respective probabilities of winning the match and state whether the decision of the referee was fair or not. [6]

Solution : Probability of getting a six by the captains of both the teams A and B is

$$P(A) = \frac{1}{6}$$

and $$P(B) = \frac{1}{6}$$

$$\therefore \quad P(\bar{A}) = P(\bar{B}) = 1 - \frac{1}{6} = \frac{5}{6}$$

Since A starts the game, he can throw a six in the following mutually exclusive ways :

$(A), (\bar{A}\bar{B}A), (\bar{A}\bar{B}\bar{A}\bar{B}A), ...$

By the theorem of Total Probability,

∴ Probability that A wins

$$= P(A) + P(\bar{A}\bar{B}A) + P(\bar{A}\bar{B}\bar{A}\bar{B}A) + ...$$

$$= P(A) + P(\bar{A})P(\bar{B})P(A) + P(\bar{A})P(\bar{B})P(\bar{A})P(\bar{B})P(A) + ...$$

$$= \frac{1}{6} + \frac{5}{6}\cdot\frac{5}{6}\cdot\frac{1}{6} + \frac{5}{6}\cdot\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{1}{6} + ...$$

$$= \frac{1}{6} + \frac{1}{6}\cdot\left(\frac{5}{6}\right)^2 + \frac{1}{6}\cdot\left(\frac{5}{6}\right)^4 + ...$$

This is an infinite G.P.

$$\therefore \quad a = \frac{1}{6}$$

and $$r = \left(\frac{5}{6}\right)^2$$

Hence, the probability of the team A winning the match

$$= \frac{\frac{1}{6}}{1 - \left(\frac{5}{6}\right)^2} \quad \left(\text{Using } S_\infty = \frac{a}{1-r}\right)$$

$$= \frac{\frac{1}{6}}{\frac{11}{36}} = \frac{6}{11}$$

Since, the total probability is unity, the probability of team B winning the match $1 - \frac{6}{11} = \frac{5}{11}$.

The decision of the referee was not fair as whosoever starts throwing the die gets an upper hand. **Ans.**

28. A manufacturer considers that men and women workers are equally efficient and so he pays them at the same rate. He has 30 and 17 units of workers (male and female) and capital respectively; which he uses to produce two types of goods A and B. To produce one unit of A, 2 workers and 3 units of capital are required while 3 workers and 1 unit of capital is required to produce one unit of B. If A and B are priced at ₹ 100 and ₹ 120 per unit respectively, how should he use his resources to maximize the total revenue ? Form the above as an LPP and solve graphically.

Do you agree with this view of the manufacturer that men and women workers are equally efficient and so should be paid at the same rate? [6]

Solution : Let x units of the goods A and y units of goods B be produced to maximize the total revenue. Then the total revenue is $Z = 100x + 120y$. This is a linear function which is to be maximized. Hence it is the objective function. The constraints are as per the following table :

	Unit A	Unit B	Total Units
Workers	2	3	30
Capital	3	1	17

From the table, the constraints are

$$2x + 3y \le 30$$
$$3x + y \le 17$$

and $$x \ge 0, y \ge 0$$

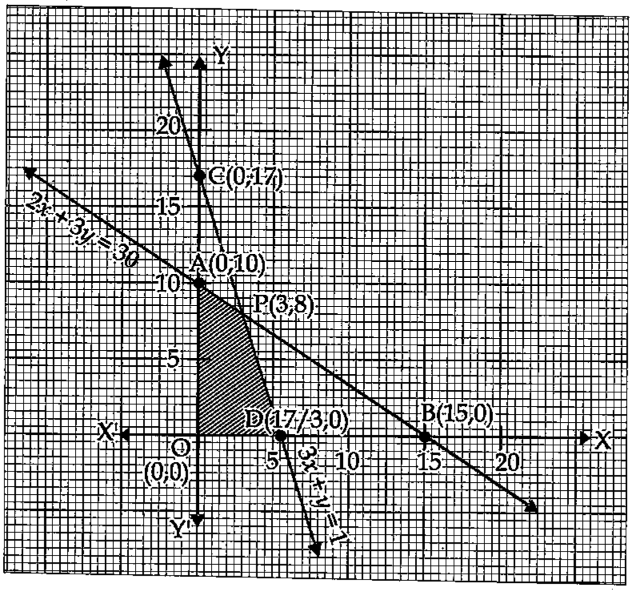
First we draw the line AB and CD whose equations are

$$2x + 3y = 30 \quad ...(i)$$

	A	B
x	0	15
y	10	0

and $$3x + y = 17 \quad ...(ii)$$

	C	D
x	0	17/3
y	17	0



The feasible region is ODPAO which is shaded in the figure.

P is the point of intersection of the lines

$$2x + 3y = 30$$

and $$3x + y = 17$$

Solving these equations, we get point P(3, 8).

The vertices of the feasible region are O(0, 0), D(17/3), P (3, 8) and A(0, 10).

The value of objective function $Z = 100x + 120y$ at these vertices are as follows :

Corner Point	Total Revenue $Z = 100x + 120y$
At O (0, 0)	$Z = 0$
At D (17/3, 0)	$Z = \frac{1700}{3}$
At P(3, 8)	$Z = 1260 \leftarrow$ maximum
At A (0, 10)	$Z = 1200$

$\therefore$ The maximum revenue ₹ 1260 at the point P(3, 8) *i.e.*, when 3 units of goods A and 8 units of goods B are produced.

Yes, I agree with the view of the manufacturer.

Ans.

29. The management committee of a residential colony decided to award some of its members (say x) for honesty, some (say y) for helping others and some others (say z) for supervising the workers to keep the colony neat and clean. The sum of all the awardees is 12. Three times the sum of awardees for cooperation and supervision added to two times the number of awardees for honesty is 33. If the sum of the number of awardees for honesty and supervision is twice the number of awardees for helping others using matrix method, find the number of awardees of each category. Apart from these values namely honesty, cooperation and supervision, suggest one more value which the management of the colony must include for awards. [6]

Solution : From question,

$$x + y + z = 12 \quad \text{...(i)}$$

$$2x + 3(y + z) = 33 \Rightarrow 2x + 3y + 3z = 33 \quad \text{...(ii)}$$

$$x + z = 2y \Rightarrow x - 2y + z = 0 \quad \text{...(iii)}$$

$\therefore$ The given equations can be written in matrix form

$$AX = B \quad \text{...(iv)}$$

Here $$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 1 & -2 & 1 \end{bmatrix}; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}; B = \begin{bmatrix} 12 \\ 33 \\ 0 \end{bmatrix}$$

$$\therefore \quad |A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 1 & -2 & 1 \end{vmatrix}$$

$$= 1(3 + 6) - 1(2 - 3) + 1(-4 - 3)$$

$$= 9 + 1 - 7 = 3 \neq 0$$

$\Rightarrow A^{-1}$ exists.

For adj A,

$A_{11} = (3+6) = 9, A_{12} = -(2-3) = 1, A_{13} = (-4-3) = -7$

$A_{21} = -(1+2) = -3, A_{22} = (1-1) = 0, A_{23} = -(-2-1) = 3$

$A_{31} = (3-3) = 0, A_{32} = -(3-2) = -1, A_{33} = (3-2) = 1$

$$\therefore \quad \text{adj } A = \begin{bmatrix} 9 & 1 & -7 \\ -3 & 0 & 3 \\ 0 & -1 & 1 \end{bmatrix}^T = \begin{bmatrix} 9 & -3 & 0 \\ 1 & 0 & -1 \\ -7 & 3 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj. } A = \frac{1}{3}\begin{bmatrix} 9 & -3 & 0 \\ 1 & 0 & -1 \\ -7 & 3 & 1 \end{bmatrix}$$

$\therefore$ From (iv), $X = A^{-1}B$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 9 & -3 & 0 \\ 1 & 0 & -1 \\ -7 & 3 & 1 \end{bmatrix}\begin{bmatrix} 12 \\ 33 \\ 0 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 108 - 99 + 0 \\ 12 + 0 + 0 \\ -84 + 99 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 9 \\ 12 \\ 15 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

$$\Rightarrow \quad x = 3, \ y = 4, z = 5$$

The colony management must includes cleanliness for awards. **Ans.**

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Mathematics 2013 (Outside Delhi) SET II

Time allowed : 3 hours **Maximum marks : 100**

Note : Except for the following questions, all the remaining questions have been asked in previous set

SECTION — A

9. If matrix A = $\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$ and $A^2 = pA$, then write the value of p. [1]

Solution : Given that

$$A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 4+4 & -4-4 \\ -4-4 & 4+4 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & -8 \\ -8 & 8 \end{bmatrix} = 4\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = 4A$$

Therefore, the value of p is 4. **Ans.**

10. A and B are two points with position vectors $2\vec{a} - 3\vec{b}$ and $6\vec{b} - \vec{a}$ respectively. Write the position vector of a point P which divides the line segment AB internally in the ratio 1 : 2. [1]

Solution : Position vector of point

$$P = \frac{2(2\vec{a}-3\vec{b})+1(6\vec{b}-\vec{a})}{1+2}$$

$$= \frac{4\vec{a}-6\vec{b}+6\vec{b}-\vec{a}}{3}$$

$$= \frac{3\vec{a}}{3} = \vec{a}$$ **Ans.**

SECTION — B

19. If $x^y = e^{x-y}$, prove that $\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$. [4]

Solution : Given, $x^y = e^{x-y}$

Taking log on both sides, we get

$y \log x = x - y \Rightarrow x = y(1 + \log x)$

$$\Rightarrow \quad y = \frac{x}{1+\log x}$$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(1+\log x).1 - x.\frac{1}{x}}{(1+\log x)^2}$$

$$= \frac{1+\log x - 1}{(1+\log x)^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}.$$ **Hence Proved**

20. Evaluate : $\int \frac{dx}{x(x^3+8)}$. [4]

Solution :

$$\int \frac{dx}{x(x^3+8)} = \int \frac{\frac{1}{x^3}}{\frac{x(x^3+8)}{x^3}}dx = \int \frac{dx}{x^4\left(1+\frac{8}{x^3}\right)}$$

Let, $\left(1+\frac{8}{x^3}\right) = t \Rightarrow -\frac{24}{x^4}dx = dt$

$$\Rightarrow \quad \frac{dx}{x^4} = -\frac{dt}{24}$$

$$\therefore \quad I = -\frac{1}{24}\int\frac{dt}{t} = -\frac{1}{24}\log|t|$$

$$= -\frac{1}{24}\log\left|\left(1+\frac{8}{x^3}\right)\right| + C$$

$$= \frac{1}{24}\log\left|\frac{x^3}{x^3+8}\right| + C.$$ **Ans.**

21. Evaluate : $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}dx$. [4]

Solution :

Let $\quad I = \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$ ...(i)

and $\quad I = \int_0^{\pi} \frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}dx$

$\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$

$\therefore \quad I = \int_0^{\pi} \frac{(\pi-x)\sin x}{1+\cos^2 x}dx$ ...(ii)

Adding (i) and (ii), we get

$$2I = \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx + \int_0^{\pi}\frac{(\pi-x)\sin x}{1+\cos^2 x}dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi}\frac{\pi \sin x}{1+\cos^2 x}dx$$

Let, $\cos x = t \Rightarrow -\sin x\, dx = dt$

$\Rightarrow \quad \sin x\, dx = -dt$

when $x = 0, t = 1$ and $x = \pi, t = -1$

$$\Rightarrow \quad 2I = -\int_1^{-1} \frac{\pi}{1+t^2} dt = -\pi\left[\tan^{-1} t\right]_1^{-1}$$

$$= -\pi\,[\tan^{-1}(-1) - \tan^{-1}(1)]$$

$$= -\pi\left(-\frac{\pi}{4} - \frac{\pi}{4}\right) = -\pi\left(-\frac{\pi}{2}\right)$$

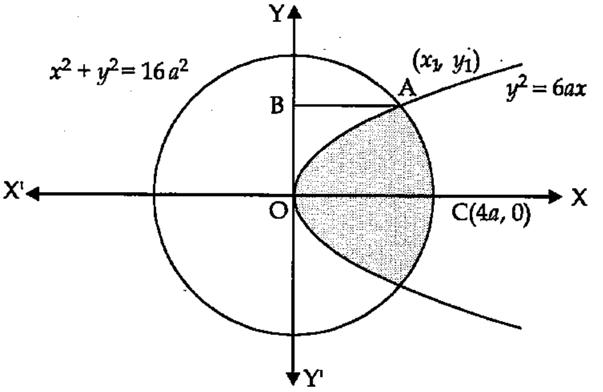
$$= \frac{\pi^2}{2}$$

$$\therefore \quad I = \frac{\pi^2}{4}.$$ **Ans.**

SECTION — C

28. Find the area of the region $\{(x, y) : y^2 \le 6ax$ and $x^2 + y^2 \le 16\,a^2\}$ using method of integration. [6]

Solution :



Coordinates of point C is $(4a, 0)$. Let the coordinates of point A be (x_1, y_1).

$\therefore x_1^2 + 6ax_1 = 16a^2 \Rightarrow (x_1 - 2a)(x_1 + 8a) = 0$

$\Rightarrow x_1 = 2a, x_1 \ne -8a$ as x_1 lies in the first quadrant.

$\therefore y_1^2 = 6ax_1 = 12a^2 \quad \Rightarrow y_1 = 2\sqrt{3}.a$

Required Area = Area of the shaded region

$$= 2\left(\left|\int_0^{2\sqrt{3}.a}\left(\sqrt{16a^2 - y^2}\right)dy\right| - \left|\int_0^{2\sqrt{3}.a}\left(\frac{y^2}{6a}\right)dy\right|\right)$$

$$= 2\left[\left(2\sqrt{3}a^2 + \frac{8\pi a^2}{3}\right) - \left(\frac{4\sqrt{3}}{3}a^2\right)\right]$$

$$= \frac{4a^2}{3}(4\pi + \sqrt{3}) \text{ sq. units.}$$ **Ans.**

29. Show that the differential equation $\left[x\sin^2\left(\frac{y}{x}\right) - y\right] dx + xdy = 0$ is homogeneous.

Find the particular solution of this differential equation, given that $y = \frac{\pi}{4}$ when $x = 1$. [6]

Solution : Given, $\left[x\sin^2\left(\frac{y}{x}\right) - y\right]dx + xdy = 0$

$$\Rightarrow \quad x\frac{dy}{dx} = y - x\sin^2\left(\frac{y}{x}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x} - \sin^2\frac{y}{x}$$

which is a homogeneous differential equation.

Put, $\quad y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$,

we get

$$v + x\frac{dv}{dx} = v - \sin^2 v$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\sin^2 v$$

Separating variables and integrating

$$\int\frac{dv}{\sin^2 v} = -\int\frac{dx}{x}$$

$$\Rightarrow \quad -\cot v = -\log x + C$$

$$\Rightarrow \quad \log x - \cot(y/x) = C$$

When $x = 1, y = \frac{\pi}{4}$

$$\log 1 - \cot\frac{\pi}{4} = C$$

$$\Rightarrow \quad 0 - 1 = C \Rightarrow C = -1$$

Hence required particular solution is

$$\log x - \cot\left(\frac{y}{x}\right) + 1 = 0$$ **Ans.**

Mathematics 2013 (Outside Delhi) SET III

Time allowed : 3 hours **Maximum marks : 100**

Note : Except for the following questions, all the remaining questions have been asked in previous sets.

SECTION — A

9. If matrix $A = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$ and $A^2 = \lambda A$, then write the value of λ. [1]

Solution : Given,

$$A = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} \text{ and } A^2 = \lambda A$$

$$\Rightarrow \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} = \lambda\begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 18 & -18 \\ -18 & 18 \end{bmatrix} = \lambda \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$$

$$\Rightarrow 6\begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} = \lambda \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$$

$$\Rightarrow 6A = \lambda A$$

$$\Rightarrow \lambda = 6.$$

Ans.

10. L and M are two points with position vectors $2\vec{a} - \vec{b}$ and $\vec{a} + 2\vec{b}$ respectively. Write the position vector of a point N which divides the line segment LM in the ratio 2 : 1 externally. **[1]**

Solution : The position vectors of the points are L $(2\vec{a} - \vec{b})$ and M $(\vec{a} + 2\vec{b})$.

We have to divide segment LM at N externally in the ratio 2 : 1.

Position of vector of point N

$$= \frac{2(\vec{a}+2\vec{b}) - 1(2\vec{a} - \vec{b})}{2-1}$$

$$= 2\vec{a} + 4\vec{b} - 2\vec{a} + \vec{b}$$

$$= 5\vec{b}$$

Ans.

SECTION — B

19. Using vectors, find the area of the triangle ABC, whose vertices are A(1, 2, 3), B(2, –1, 4) and C(4, 5, –1). **[4]**

Solution : Given,

$$A(1,2,3) \Rightarrow \vec{OA} = \hat{i} + 2\hat{j} + 3\hat{k}$$

$$B(2,-1,4) \Rightarrow \vec{OB} = 2\hat{i} - \hat{j} + 4\hat{k}$$

$$C(4,5,-1) \Rightarrow \vec{OC} = 4\hat{i} + 5\hat{j} - \hat{k}$$

$$\therefore \quad \vec{AB} = \vec{OB} - \vec{OA}$$

$$= (2\hat{i} - \hat{j} + 4\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k})$$

$$= \hat{i} - 3\hat{j} + \hat{k}$$

$$\vec{AC} = \vec{OC} - \vec{OA}$$

$$= (4\hat{i} + 5\hat{j} - \hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k})$$

$$= 3\hat{i} + 3\hat{j} - 4\hat{k}$$

$$\text{Now } (\vec{AB} \times \vec{AC}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 1 \\ 3 & 3 & -4 \end{vmatrix}$$

$$= \hat{i}(12-3) - \hat{j}(-4-3) + \hat{k}(3+9)$$

$$= 9\hat{i} + 7\hat{j} + 12\hat{k}$$

$$\therefore \quad |\vec{AB} \times \vec{AC}| = \sqrt{(9)^2 + (7)^2 + (12)^2}$$

$$= \sqrt{81+49+144}$$

$$= \sqrt{274}$$

$$\therefore \text{Area of } \Delta ABC = \frac{1}{2}|\vec{AB} \times \vec{AC}|$$

$$= \frac{1}{2}\sqrt{274} \text{ sq. units}$$

Ans.

20. Evaluate : $\int \frac{dx}{x(x^3+1)}$. **[4]**

Solution :

Let
$$I = \int \frac{1}{x(x^3+1)} dx$$

$$= \int \frac{x^2}{x^3(x^3+1)} dx$$

Put $x^3 + 1 = t$

$$\Rightarrow \quad x^3 = t - 1$$

$$\Rightarrow \quad 3x^2\, dx = dt$$

$$\Rightarrow \quad x^2 dx = \frac{dt}{3}$$

$$\therefore \quad I = \frac{1}{3}\int \frac{1}{t(t-1)} dt$$

$$= \frac{1}{3}\int \frac{1}{t^2 - t} dt$$

$$= \frac{1}{3}\int \frac{1}{t^2 - t + \frac{1}{4} - \frac{1}{4}} dt$$

$$= \frac{1}{3}\int \frac{1}{\left(t - \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}$$

$$= \frac{1}{3} \times \frac{1}{2\left(\frac{1}{2}\right)} \log\left|\frac{t - \frac{1}{2} - \frac{1}{2}}{t - \frac{1}{2} + \frac{1}{2}}\right| + C$$

$$= \frac{1}{3}\log\left|\frac{t-1}{t}\right| + C$$

$$\therefore \quad I = \frac{1}{3}\log\left|\frac{x^3}{x^3+1}\right| + C.$$

Ans.

21. If $x \sin(a+y) + \sin a \cos(a+y) = 0$, prove that $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$. **[4]**

Solution : Here, $x \sin(a+y) + \sin a \cos(a+y) = 0$

$$\Rightarrow \quad x = -\sin a . \frac{\cos(a+y)}{\sin(a+y)}$$

$$\Rightarrow \quad x = -\sin a . \cot(a+y)$$

Differentiating w.r.t. y, we get

$$\frac{dx}{dy} = -\sin a . \{-\operatorname{cosec}^2(a+y).(0+1)\}$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{\sin a}{\sin^2(a+y)}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{\sin^2(a+y)}{\sin a}.$$

Hence Proved.

22. Using properties of determinants prove the following : **[4]**

$$\begin{vmatrix} 3x & -x+y & -x+z \\ x-y & 3y & z-y \\ x-z & y-z & 3z \end{vmatrix} = 3(x+y+z)(xy+yz+xz)$$

Solution :

Let $\quad \Delta = \begin{vmatrix} 3x & -x+y & -x+z \\ x-y & 3y & z-y \\ x-z & y-z & 3z \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} x+y+z & -x+y & -x+z \\ x+y+z & 3y & -y+z \\ x+y+z & y-z & 3z \end{vmatrix}$$

Taking $(x+y+z)$ common from C_1, we get

$$= (x+y+z)\begin{vmatrix} 1 & -x+y & -x+z \\ 1 & 3y & -y+z \\ 1 & y-z & 3z \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$\therefore \quad = (x+y+z)\begin{vmatrix} 1 & -x+y & -x+z \\ 0 & 2y+x & -y+x \\ 0 & -z+x & 2z+x \end{vmatrix}$$

Expanding along C_1, we get

$$= (x+y+z)\begin{vmatrix} 2y+x & -y+x \\ -z+x & 2z+x \end{vmatrix}$$

$$= (x+y+z)[(2y+x)(2z+x) - (-y+x)(-z+x)]$$

$$= (x+y+z)[(4yz+2xy+2xz+x^2) - (yz-xy-xz+x^2)]$$

$$= (x+y+z)(3yz+3xy+3xz)$$

$$\therefore \quad = 3(x+y+z)(xy+yz+zx).$$

Hence Proved.

28. Find the area of the region, $\{(x, y) : y^2 \le 4x, 4x^2 + 4y^2 \le 9\}$ using method of integration. **[6]**

Solution : Given

$$R = \{(x, y) : y^2 \le 4x, 4x^2 + 4y^2 \le 9\}$$

$$\Rightarrow \quad R = \{(x, y) : y^2 \le 4x\} \cap \{(x, y) : 4x^2 + 4y^2 \le 9\}$$

$$\Rightarrow \quad R = R_1 \cap R_2,$$

Where $R_1 = \{(x, y) : y^2 \le 4x\}$ ...(i)

$R_2 = \{(x, y) : 4x^2 + 4y^2 \le 9\}$ ...(ii)

$$y^2 = 4x$$

$$4x^2 + 4y^2 = 9$$

$$\Rightarrow \quad 4x^2 + 16x - 9 = 0$$

$$\Rightarrow \quad (2x+9)(2x-1) = 0$$

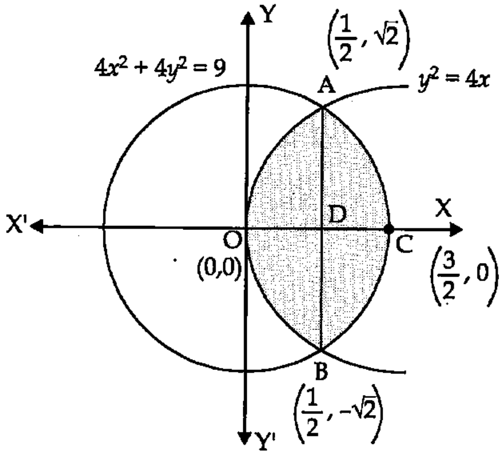
$$\Rightarrow \quad x = \frac{-9}{2} \text{ or } x = \frac{1}{2}$$

From equation (i), we find that $x = \frac{1}{2}$

$$\Rightarrow \quad y = \pm\sqrt{2} \text{ and } x = \frac{-9}{2}$$

$\Rightarrow y$ is imaginary.

So the two curves intersect at $\left(\frac{1}{2}, \sqrt{2}\right)$ and $\left(\frac{1}{2}, -\sqrt{2}\right)$



Required area = Area of shaded region

= 2 (Area in Ist Quadrant)

= 2 (Area of OADO + Area of ADCA)

$$A = 2\int_0^{\frac{1}{2}} 2\sqrt{x}\,dx + 2\int_{\frac{1}{2}}^{\frac{3}{2}} \sqrt{\frac{9}{4} - x^2}\,dx$$

$$= 4 \times \frac{2}{3}\left[x^{3/2}\right]_0^{1/2} + 2\left[\frac{1}{2}x\sqrt{\frac{9}{4} - x^2} + \frac{1}{2}.\frac{9}{4}\sin^{-1}\frac{2x}{3}\right]_{\frac{1}{2}}^{\frac{3}{2}}$$

$$= \frac{8}{3}\left(\frac{1}{2\sqrt{2}} - 0\right) + \left[\left\{\frac{9}{4}\sin^{-1}(1)\right\} - \left\{\frac{1}{\sqrt{2}} + \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)\right\}\right]$$

$$= \frac{2\sqrt{2}}{3} + \left[\frac{9}{8}\pi - \frac{1}{\sqrt{2}} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)\right]$$

$$\therefore \quad A = \left[\frac{\sqrt{2}}{6}+\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)\right] \text{ sq. units.}$$

Ans.

29. Find the particular solution of the differential equation $\frac{dx}{dy}+x\cot y = 2y+y^2\cot y, (y \neq 0)$, given that $x = 0$ when $y = \frac{\pi}{2}$. **[6]**

Solution : The given differential equation is

$$\frac{dx}{dy}+x\cot y = 2y+y^2\cot y, (y\neq 0)$$

Given equation is of the form $\frac{dx}{dy}+Px = Q$

Here, $P = \cot y$

and $Q = 2y + y^2 \cot y$

$$\Rightarrow \quad \text{I.F.} = e^{\int P dy} = e^{\int \cot y\, dy}$$

$$= e^{\log|\sin y|} = \sin y$$

Now, solution is given by

$$x\,(\text{I.F.}) = \int (\text{I.F.})\, Q\, dy + C$$

$$\Rightarrow \quad x \sin y = \int \sin y(2y+y^2\cot y)\,dy + C$$

$$= \int (2y\sin y + y^2 \cot y \sin y)\,dy + C$$

$$= \int 2y \sin y\, dy + \int y^2 \cos y\, dy + C$$

$$= 2\sin y.\frac{y^2}{2} - 2\int \cos y.\frac{y^2}{2}\,dy + \int y^2 \cos y\, dy + C$$

$$= y^2 \sin y - \int y^2\cos y\, dy + \int y^2 \cos y\, dy + C$$

$$= y^2 \sin y + C$$

$$\therefore x\sin y = y^2 \sin y + C \qquad \text{...(i)}$$

Given that $x = 0$, when $y = \frac{\pi}{2}$.

$$\Rightarrow (0).\sin\left(\frac{\pi}{2}\right) = \left(\frac{\pi}{2}\right)^2 \sin\left(\frac{\pi}{2}\right) + C$$

$$\Rightarrow \quad 0 = \frac{\pi^2}{4}+C$$

$$\therefore \quad C = \frac{-\pi^2}{4}$$

Hence the required particular solution of given differential equation is

$$x\sin y = y^2 \sin y - \frac{\pi^2}{4}.$$

Ans.

Mathematics 2013 (Delhi) SET I

Time allowed : 3 hours **Maximum marks : 100**

SECTION — A

1. Write the principal value of $\tan^{-1}(1)+\cos^{-1}\left(-\frac{1}{2}\right)$. **[1]**

Solution : $\tan^{-1}(1)+\cos^{-1}\left(-\frac{1}{2}\right)$

$$= \frac{\pi}{4}+\left[\pi - \cos^{-1}\left(\frac{1}{2}\right)\right]$$

$$[\because \cos^{-1}(-x) = \pi - \cos^{-1}x]$$

$$= \frac{\pi}{4}+\pi-\frac{\pi}{3} = \frac{\pi}{4}+\frac{2\pi}{3} = \frac{11}{12}\pi.$$ **Ans.**

2. Write the value of $\tan\left(2\tan^{-1}\frac{1}{5}\right)$. **[1]**

Solution : $\tan\left(2\tan^{-1}\frac{1}{5}\right) = \tan\left(\tan^{-1}\frac{2\times\frac{1}{5}}{1-\left(\frac{1}{5}\right)^2}\right)$

$$\left(\because 2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}\right)$$

$$= \tan\left(\tan^{-1}\frac{\frac{2}{5}}{\frac{24}{25}}\right)$$

$$= \tan\left(\tan^{-1}\left(\frac{5}{12}\right)\right)$$

$$= \frac{5}{12}.$$ **Ans.**

3. Find the value of a if $\begin{bmatrix} a-b & 2a+c \\ 2a-b & 3c+d\end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13\end{bmatrix}$. **[1]**

Solution : Given, $\begin{bmatrix} a-b & 2a+c \\ 2a-b & 3c+d\end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13\end{bmatrix}$

Comparing the corresponding elements, we get

$$a-b = -1 \qquad \text{...(i)}$$

$$2a-b = 0 \qquad \text{...(ii)}$$

$$2a+c = 5 \qquad \text{...(iii)}$$

$$3c+d = 13 \qquad \text{...(iv)}$$

Solving equation (i) and (ii), we get

$$a = 1.$$ **Ans.**

4. **If** $\begin{vmatrix} x+1 & x-1 \\ x-3 & x+2 \end{vmatrix} = \begin{vmatrix} 4 & -1 \\ 1 & 3 \end{vmatrix}$**, then write the value of** x. **[1]**

Solution : Given, $\begin{vmatrix} x+1 & x-1 \\ x-3 & x+2 \end{vmatrix} = \begin{vmatrix} 4 & -1 \\ 1 & 3 \end{vmatrix}$

$\Rightarrow (x+1)(x+2)-(x-3)(x-1)$
$= 4 \times 3 - 1 \times (-1)$
$\Rightarrow (x^2+3x+2)-(x^2-4x+3)$
$= 12+1$
$\Rightarrow \quad 3x+2+4x-3 = 13$
$\Rightarrow \quad 7x-14 = 0$
$\Rightarrow \quad x = 2.$ **Ans.**

5. **If** $\begin{bmatrix} 9 & -1 & 4 \\ -2 & 1 & 3 \end{bmatrix} = A + \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 9 \end{bmatrix}$**, then find the matrix A.** **[1]**

Solution : Given $\begin{bmatrix} 9 & -1 & 4 \\ -2 & 1 & 3 \end{bmatrix} = A + \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 9 \end{bmatrix}$

$$\Rightarrow A = \begin{bmatrix} 9 & -1 & 4 \\ -2 & 1 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 9-1 & -1-2 & 4+1 \\ -2-0 & 1-4 & 3-9 \end{bmatrix} = \begin{bmatrix} 8 & -3 & 5 \\ -2 & -3 & -6 \end{bmatrix}.$$

Ans.

6. **Write the degree of the differential equation** $x^3\left(\frac{d^2y}{dx^2}\right)^2 + x\left(\frac{dy}{dx}\right)^4 = 0.$ **[1]**

Solution : The degree of the given differential equation is 2. **Ans.**

7. **If** $\vec{a} = x\hat{i}+2\hat{j}-z\hat{k}$ **and** $\vec{b} = 3\hat{i}-y\hat{j}+\hat{k}$ **are two equal vectors, then write the value of** $x+y+z$. **[1]**

Solution : Given ; $\vec{a} = \vec{b}$

$\Rightarrow \quad x\hat{i}+2\hat{j}-z\hat{k} = 3\hat{i}-y\hat{j}+\hat{k}$

Comparing the corresponding element, we get

$x = 3, y = -2, z = -1$

$\therefore \quad x+y+z = 3+(-2)+(-1) = 0.$

Ans.

8. **If a unit vector** $\vec{a}$ **makes angle** $\frac{\pi}{3}$ **with** $\hat{i}$, $\frac{\pi}{4}$ **with** $\hat{j}$ **and an acute angle θ with** $\hat{k}$ **then find the value of θ.** **[1]**

Solution : $l = \cos\alpha = \cos\frac{\pi}{3} = \frac{1}{2}$

$m = \cos\beta = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$n = \cos\gamma = \cos\theta$

Since, $l^2+m^2+n^2 = 1$

$\Rightarrow \quad \left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \cos^2\theta = 1$

$\Rightarrow \quad \cos^2\theta = 1 - \frac{1}{4} - \frac{1}{2} = \frac{1}{4}$

$\Rightarrow \quad \cos\theta = \frac{1}{2}$

$\Rightarrow \quad \theta = \frac{\pi}{3}.$ **Ans.**

9. **Find the cartesian equation of the line which passes through the point (– 2, 4, – 5) and is parallel to the line** $\frac{x+3}{3} = \frac{4-y}{5} = \frac{z+8}{6}$. **[1]**

Solution : Direction ratios of the line parallel to

$\frac{x+3}{3} = \frac{4-y}{5} = \frac{z+8}{6}$ *i.e.,* $\frac{x+3}{3} = \frac{y-4}{-5} = \frac{z+8}{6}$ ...(i)

are 3, – 5, 6.

$\therefore$ Cartesian equation of the lines passes through (– 2, 4, – 5) and parallel to line (i) is

$\frac{x+2}{3} = \frac{y-4}{-5} = \frac{z+5}{6}.$ **Ans.**

10. **The amount of pollution content added in air in a city to** x**-diesel vehicles is given by** $P(x) = 0.005x^3 + 0.02x^2 + 30x$. **Find the marginal increase in pollution content when 3 diesel vehicles are added and write which value is indicated in the above question.** **[1]**

Solution : Here, pollution content is given by

$P(x) = 0.005x^3 + 0.02x^2 + 30x$

where x is the number of diesel vehicles.

$\Rightarrow \quad \frac{dP}{dx} = 0.015x^2 + 0.04x + 30$

$\therefore$ The marginal increase in pollution content (when $x = 3$)

$= 0.015 \times (3)^2 + 0.04 \times 3 + 30$

$= 0.135 + 0.12 + 30$

$= 30.255$

The value indicated in the question is diesel vehicles causes environmental pollution. **Ans.**

SECTION — B

11. **Show that the function** f **in** $A = R - \left\{\frac{2}{3}\right\}$ **defined as** $f(x) = \frac{4x+3}{6x-4}$ **is one-one and onto. Hence find** f^{-1}. **[4]**

Solution : Given, $f(x) = \frac{4x+3}{6x-4}$ where $x \in A$

$= R - \left\{\frac{2}{3}\right\}$

$f(x_1) = f(x_2)$ where $(x_1, x_2 \in A)$

$\Rightarrow \quad \frac{4x_1+3}{6x_1-4} = \frac{4x_2+3}{6x_2-4}$

$\Rightarrow \quad (4x_1 + 3)(6x_2 - 4) = (6x_1 - 4)(4x_2 + 3)$

$\Rightarrow 24x_1x_2 - 16x_1 + 18x_2 - 12 = 24x_1x_2 + 18x_1 - 16x_2 - 12$

$\Rightarrow \quad -34x_1 = -34x_2 \Rightarrow x_1 = x_2.$

$\therefore f$ is one-one. **Hence Proved.**

For $\quad y \in A = R - \left\{\frac{2}{3}\right\}$

$f(x) = y$

$\Rightarrow \quad \frac{4x+3}{6x-4} = y \Rightarrow (6x-4)y = 4x + 3$

$\Rightarrow \quad 6xy - 4y = 4x + 3$

$\Rightarrow \quad (6y-4)x = 4y + 3$

$\Rightarrow \quad x = \frac{4y+3}{6y-4}$

$\Rightarrow f$ is onto. **Hence Proved.**

Hence f is invertible as it is one-one and onto.

Now, $\quad f(x) = y \Leftrightarrow x = f^{-1}(y)$

$\Rightarrow \quad f^{-1}(y) = \frac{4y+3}{6y-4} \forall y \in A.$ **Ans.**

12. Find the value of the following :

$$\tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2} + \cos^{-1}\frac{1-y^2}{1+y^2}\right], \ |x| < 1, y > 0$$

and $xy < 1$. **[4]**

Solution : $\tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2} + \cos^{-1}\frac{1-y^2}{1+y^2}\right],$

Putting $x = \tan\alpha$ and $y = \tan\beta$

$$\tan\frac{1}{2}\left[\sin^{-1}\left(\frac{2\tan\alpha}{1+\tan^2\alpha}\right) + \cos^{-1}\left(\frac{1-\tan^2\beta}{1+\tan^2\beta}\right)\right]$$

$$= \tan\frac{1}{2}[\sin^{-1}(\sin 2\alpha) + \cos^{-1}(\cos 2\beta)]$$

$$\left[\because \sin 2\alpha = \frac{2\tan\alpha}{1+\tan^2\alpha} \text{ and } \cos 2\alpha = \frac{1-\tan^2\alpha}{1+\tan^2\alpha}\right]$$

$$= \tan\frac{1}{2}(2\alpha + 2\beta) = \tan(\alpha + \beta)$$

$$= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \frac{x+y}{1-xy}.$$ **Ans.**

OR

Prove that :

$$\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{8}\right) = \frac{\pi}{4}.$$

Solution : L.H.S.

$$= \left[\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{5}\right)\right] + \tan^{-1}\left(\frac{1}{8}\right)$$

$$= \tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{5}}{1-\frac{1}{2}\cdot\frac{1}{5}}\right) + \tan^{-1}\left(\frac{1}{8}\right)$$

$$\left(\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-x.y}\right)\right)$$

$$= \tan^{-1}\left(\frac{7}{10}\cdot\frac{10}{9}\right) + \tan^{-1}\left(\frac{1}{8}\right)$$

$$= \tan^{-1}\left(\frac{7}{9}\right) + \tan^{-1}\left(\frac{1}{8}\right) = \tan^{-1}\left(\frac{\frac{7}{9}+\frac{1}{8}}{1-\frac{7}{9}\cdot\frac{1}{8}}\right)$$

$$= \tan^{-1}\left(\frac{\frac{65}{72}}{\frac{65}{72}}\right)$$

$$= \tan^{-1}(1) = \frac{\pi}{4} = \text{R.H.S.}$$ **Hence Proved.**

13. Using properties of determinants, prove the following :

$$\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = (1-x^3)^2.$$

[4]

Solution : L.H.S. $= \begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$

Applying $C_1 \rightarrow C_1 - C_2$, $C_2 \rightarrow C_2 - C_3$, we get

$$= \begin{vmatrix} 1-x & x-x^2 & x^2 \\ x^2-1 & 1-x & x \\ x-x^2 & x^2-1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1-x & x(1-x) & x^2 \\ (x-1)(x+1) & 1-x & x \\ x(1-x) & (x-1)(x+1) & 1 \end{vmatrix}$$

Taking $(1-x)$ common from C_1 and C_2, we get.

$$= (1-x).(1-x)\begin{vmatrix} 1 & x & x^2 \\ -(x+1) & 1 & x \\ x & -(x+1) & 1 \end{vmatrix}$$

Applying $R_1 \rightarrow R_1 + R_2 + R_3$, we get

$$= (1-x)^2\begin{vmatrix} 0 & 0 & x^2+x+1 \\ -(x+1) & 1 & x \\ x & -(x+1) & 1 \end{vmatrix}$$

Expanding along R_1, we get

$$= (1-x)^2.(1+x+x^2)\,[(x+1)^2-x]$$
$$= (1-x)^2\,(1+x+x^2)\,(1+x+x^2)$$
$$= [(1-x)\,(1+x+x^2)]^2$$
$$= (1-x^3)^2 = \text{R.H.S.}$$
$$[\because a^3-b^3=(a-b)\ (a^2+ab+b^2)]$$

Hence Proved.

14. Differentiate the following function with respect to x:

$$(\log x)^x + x^{\log x} \qquad [4]$$

Solution : Let, $y = (\log x)^x + x^{\log x}$

$= u + v$, where $u = (\log x)^x$ and $v = x^{\log x}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{du}{dx}+\frac{dv}{dx} \qquad ...(i)$$

Now, $u = (\log x)^x$

Taking log on both sides, we get

$$\log u = x.\log(\log x)$$

Differentiating w.r. t. x, we get

$$\frac{1}{u}\frac{du}{dx} = 1.\log(\log x)+x.\frac{1}{\log x}.\frac{1}{x}$$

$$\Rightarrow \quad \frac{du}{dx} = u\left[\log(\log x)+\frac{1}{\log x}\right] \qquad ...(ii)$$

and $v = x^{\log x}$

Taking log on both sides, we get

$$\log v = \log x.\log x = (\log x)^2$$

Differentiating w.r. t. x, we get

$$\frac{1}{v}\frac{dv}{dx} = 2.\log x.\frac{1}{x}$$

$$\Rightarrow \quad \frac{dv}{dx} = v\left[2\log x.\frac{1}{x}\right]$$

$$\Rightarrow \quad \frac{dv}{dx} = x^{\log x}.\frac{2\log x}{x} \qquad ...(iii)$$

From (i), (ii) and (iii), we get

$$\frac{dy}{dx} = (\log x)^x\left[\log(\log x)+\frac{1}{\log x}\right] + x^{\log x}.\frac{2\log x}{x}. \quad \textbf{Ans.}$$

15. If $y = \log\left[x+\sqrt{x^2+a^2}\right]$, show that

$$(x^2+a^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=0. \qquad [4]$$

Solution : Given, $y = \log\left[x+\sqrt{x^2+a^2}\right]$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{x+\sqrt{x^2+a^2}}.\frac{d}{dx}\left[x+\sqrt{x^2+a^2}\right]$$
$$= \frac{1}{x+\sqrt{x^2+a^2}}\left[1+\frac{1}{2}(x^2+a^2)^{-1/2}.2x\right]$$
$$= \frac{1}{x+\sqrt{x^2+a^2}}\left[1+\frac{x}{\sqrt{x^2+a^2}}\right]$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{x+\sqrt{x^2+a^2}}.\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}} = \frac{1}{\sqrt{x^2+a^2}}$$

$$\Rightarrow \sqrt{x^2+a^2}.\frac{dy}{dx} = 1$$

Again differentiating w.r. t. x, we get

$$\sqrt{x^2+a^2}.\frac{d^2y}{dx^2}+\frac{x}{\sqrt{x^2+a^2}}.\frac{dy}{dx} = 0$$

Multiply by $\sqrt{x^2+a^2}$, we get

$$(x^2+a^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx} = 0$$

Hence Proved.

16. Show that the function $f(x) = |x-3|, x \in \mathbf{R}$, is continuous but not differentiable at $x = 3$. [4]

Solution : Given, $f(x) = |x-3|, x \in R$

$$= \begin{cases} x-3, & \text{if } x \geq 3 \\ 3-x, & \text{if } x < 3 \end{cases}$$

When $x > 3$, $f(x) = x-3$ and it is a polynomial, so it is continuous.

When $x < 3$, $f(x) = 3-x$. Again it is a polynomial, so it is continuous.

Also $f(3-0) = 0 = f(3+0) = f(3)$

$\Rightarrow f(x)$ is continuous at $x = 3$.

$$\text{Now, LHD} = f'(3-0) = \lim_{x\to 3-h}\frac{f(x)-f(3)}{-h}$$
$$= \lim_{h\to 0}\frac{3-(3-h)-0}{-h}$$
$$= \lim_{h\to 0}\frac{h}{-h} = -1.$$

$$\text{and} \quad \text{RHD} = f'(3+0) = \lim_{x\to 3+h}\frac{f(x)-f(3)}{h}$$
$$= \lim_{h\to 0}\frac{(3+h)-3-0}{h}$$
$$= \lim_{h\to 0}\frac{h}{h} = 1.$$

Since LHD $\neq$ RHD

$\Rightarrow f$ is not differentiable at $x = 3$. **Hence Proved.**

OR

If $x = a \sin t$ and $y = a(\cos t + \log \tan t/2)$, find $\frac{d^2y}{dx^2}$.

Solution : Given, $x = a \sin t$

Differentiating w.r. t. x, we get

$$\frac{dx}{dt} = a \cos t \quad \text{...(i)}$$

and $$y = a\left(\cos t + \log \tan \frac{t}{2}\right)$$

Differentiating w.r. t. t, we get

$$\frac{dy}{dt} = a\left(-\sin t + \frac{1}{\tan \frac{t}{2}} \sec^2 \frac{t}{2} \cdot \frac{1}{2}\right)$$

$$\Rightarrow \frac{dy}{dt} = a\left[-\sin t + \frac{\cos t/2}{\sin t/2} \times \frac{1}{2\cos^2 t/2}\right]$$

$$\Rightarrow \frac{dy}{dt} = a\left[-\sin t + \frac{1}{2\sin t/2 \cos t/2}\right]$$

$$= a\left(-\sin t + \frac{1}{\sin t}\right)$$

$[\because \sin 2\theta = 2 \sin \theta \cos \theta]$

$$= a.\left(\frac{1-\sin^2 t}{\sin t}\right)$$

$$= a.\frac{\cos^2 t}{\sin t} \quad \text{...(ii)}$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= a.\frac{\cos^2 t}{\sin t} / a\cos t$$

[Using (i) and (ii)]

$$\therefore \frac{dy}{dx} = \cot t.$$

Differentiating w.r. t. x, we get

$$\frac{d^2y}{dx^2} = -\text{cosec}^2 t.\frac{dt}{dx}$$

$$= -\text{cosec}^2 t / \frac{dx}{dt}$$

$$= -\text{cosec}^2 t / a \cos t \quad \text{[Using (i)]}$$

$$= -\frac{1}{a} \sec t \ \text{cosec}^2 t.$$ **Ans.**

17. Evaluate : $\int \frac{\sin(x-a)}{\sin(x+a)} dx$. **[4]**

Solution : Let, $I = \int \frac{\sin(x-a)}{\sin(x+a)} dx$

$$= \int \frac{\sin(x+a-a-a)}{\sin(x+a)} dx$$

$$= \int \frac{\sin[(x+a)-2a]}{\sin(x+a)} dx$$

$$= \int \frac{\sin(x+a)\cos 2a - \cos(x+a)\sin 2a}{\sin(x+a)} dx$$

[Using formula of sin (A – B)]

$$= \int [\cos 2a - \sin 2a.\cot(x+a)] dx$$

$$= \cos 2a \int 1.dx - \sin 2a \int \cot(x+a) dx$$

$$= x\cos 2a - \sin 2a \log|\sin(x+a)| + C.$$ **Ans.**

OR

Evaluate : $\int \frac{5x-2}{1+2x+3x^2} dx$.

Solution : Let $I = \int \frac{5x-2}{1+2x+3x^2} dx$

Now, $$5x - 2 = A\frac{d}{dx}(1+2x+3x^2) + B$$

$$\Rightarrow 5x - 2 = A(2+6x) + B$$

$$\Rightarrow 5x - 2 = 6Ax + 2A + B$$

On equating the coefficient of x and constant on both sides, we get

$$5 = 6A \Rightarrow A = 5/6$$

and $$2A + B = -2 \Rightarrow 2 \times \frac{5}{6} + B = -2$$

$$\Rightarrow B = -2 - \frac{5}{3} = -\frac{11}{3}$$

$$\therefore 5x - 2 = \frac{5}{6}(2+6x) + \left(-\frac{11}{3}\right)$$

$$\therefore I = \int \frac{5x-2}{1+2x+3x^2} dx$$

$$= \int \frac{\frac{5}{6}(2+6x) - \frac{11}{3}}{1+2x+3x^2} dx$$

$$= \frac{5}{6} \int \frac{2+6x}{1+2x+3x^2} dx - \frac{11}{3} \int \frac{1}{1+2x+3x^2} dx$$

$$= \frac{5}{6} \log|1+2x+3x^2| - \frac{11}{3.3} \int \frac{dx}{x^2 + \frac{2}{3}x + \frac{1}{3}}$$

$$=\frac{5}{6}\log|1+2x+3x^2|-\frac{11}{9}\int\frac{dx}{\left(x+\frac{1}{3}\right)^2+\left(\frac{\sqrt{2}}{3}\right)^2}$$

$$=\frac{5}{6}\log|1+2x+3x^2|-\frac{11}{9}\cdot\frac{1}{\sqrt{2}/3}\tan^{-1}\left(\frac{x+\frac{1}{3}}{\sqrt{2}/3}\right)+C$$

$$=\frac{5}{6}\log|1+2x+3x^2|-\frac{11}{3\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)+C$$ **Ans.**

18. Evaluate : $\int\frac{x^2}{(x^2+4)(x^2+9)}dx.$ **[4]**

Solution : Here, $\frac{x^2}{(x^2+4)(x^2+9)}=\frac{y}{(y+4)(y+9)}$, where $y=x^2$

Let $\frac{y}{(y+9)(y+4)}=\frac{A}{y+4}+\frac{B}{y+9}$

$\Rightarrow y = A(y+9)+B(y+4)$

Putting $y=-9$ and -4, we get

$-4 = A(-4+9)$ and $-9 = B(-9+4)$

$\Rightarrow B=\frac{9}{5}$ and $A=-\frac{4}{5}$

$\therefore \frac{y}{(y+4)(y+9)}=-\frac{4}{5}\cdot\frac{1}{y+4}+\frac{9}{5}\cdot\frac{1}{y+9}$

$\Rightarrow \frac{x^2}{(x^2+4)(x^2+9)}=-\frac{4}{5}\cdot\frac{1}{x^2+4}+\frac{9}{5}\cdot\frac{1}{x^2+9}$

On integrating both sides w.r.t. x, we get

$$\int\frac{x^2}{(x^2+4)(x^2+9)}dx=-\frac{4}{5}\int\frac{dx}{x^2+2^2}+\frac{9}{5}\int\frac{dx}{x^2+3^2}$$

$$=-\frac{4}{5}\cdot\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+\frac{9}{5}\cdot\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)+C$$

$$=-\frac{2}{5}\tan^{-1}\left(\frac{x}{2}\right)+\frac{3}{5}\tan^{-1}\left(\frac{x}{3}\right)+C.$$ **Ans.**

19. Evaluate : $\int_0^4(|x|+|x-2|+|x-4|)dx.$ **[4]**

Solution : Let $I=\int_0^4(|x|+|x-2|+|x-4|)dx$

$$=\int_0^2(|x|+|x-2|+|x-4|)dx+\int_2^4(|x|+|x-2|+|x-4|)dx$$

$$=\int_0^2[x-(x-2)-(x-4)]dx+\int_2^4[x+(x-2)-(x-4)]dx$$

$$=\int_0^2(x-x+2-x+4)dx+\int_2^4(x+x-2-x+4)dx$$

$$=\int_0^2(-x+6)dx+\int_2^4(x+2)dx$$

$$=\left[-\frac{x^2}{2}+6x\right]_0^2+\left[\frac{x^2}{2}+2x\right]_2^4$$

$= (-2+12-0)+[8+8-(2+4)]$

$= 10+10 = 20.$ **Ans.**

20. If $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a}+\vec{b}|=|\vec{a}|$, then prove that vector $2\vec{a}+\vec{b}$ is perpendicular to vector $\vec{b}$. **[4]**

Solution : Given,

$$|\vec{a}+\vec{b}| = |\vec{a}|$$

$\Rightarrow$ $|\vec{a}+\vec{b}|^2 = |\vec{a}|^2$

$\Rightarrow$ $(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) = \vec{a}\cdot\vec{a}$

$\Rightarrow$ $\vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b} = \vec{a}\cdot\vec{a}$

$\Rightarrow$ $2\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{b} = 0$

$(\because \vec{b}\cdot\vec{a}=\vec{a}\cdot\vec{b})$

$\Rightarrow$ $(2\vec{a}+\vec{b})\cdot\vec{b} = 0$

$\Rightarrow$ $(2\vec{a}+\vec{b})\perp\vec{b}$.

$\therefore$ The vector $2\vec{a}+\vec{b}$ is perpendicular to vector $\vec{b}$.

Hence Proved.

21. Find the coordinates of the point, where the line $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{2}$ intersects the plane $x-y+z-5=0$. Also find the angle between the line and the plane. **[4]**

Solution : Equation of the line is

$$\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{2}=k \text{ (say)} \quad ...(i)$$

$\therefore$ Point on line is $(3k+2, 4k-1, 2k+2)$.

If it lies on the plane $x-y+z-5=0$ ...(ii)

then $(3k+2)-(4k-1)+(2k+2)-5=0$

$\Rightarrow$ $3k+2-4k+1+2k-3=0$

$\Rightarrow$ $k=0$

$\therefore$ Point is $(2,-1,2)$

$\therefore$ $(2,-1,2)$ is the point on line (i), where it intersects (ii).

If θ is the angle between line (i) and plane (ii), then

$$\sin\theta = \frac{3.1+4.(-1)+2.1}{\sqrt{3^2+4^2+2^2}.\sqrt{1^2+(-1)^2+1^2}}$$

$$\Rightarrow \quad \sin\theta = \frac{3-4+2}{\sqrt{29}\sqrt{3}} = \frac{1}{\sqrt{87}}$$

$$\Rightarrow \quad \theta = \sin^{-1}\left(\frac{1}{\sqrt{87}}\right)$$ **Ans.**

OR

Find the vector equation of the plane which contains the line of intersection of the planes $\vec{r}.(\hat{i}+2\hat{j}+3\hat{k})-4=0$ and $\vec{r}.(2\hat{i}+\hat{j}-\hat{k})+5=0$ and which is perpendicular to the plane $\vec{r}.(5\hat{i}+3\hat{j}-6\hat{k})+8=0.$

Solution : Given planes are

$$\vec{r}.(\hat{i}+2\hat{j}+3\hat{k})-4 = 0 \quad ...(i)$$

$$\vec{r}.(2\hat{i}+\hat{j}-\hat{k})+5 = 0 \quad ...(ii)$$

The vector equation of the plane which contains the line of intersection of the planes (i) and (ii) is

$$\vec{r}.(\hat{i}+2\hat{j}+3\hat{k})-4+\lambda[\vec{r}.(2\hat{i}+\hat{j}-\hat{k})+5] = 0$$

$$\Rightarrow \vec{r}.[(1+2\lambda)\hat{i}+(2+\lambda)\hat{j}+(3-\lambda)\hat{k}]+(5\lambda-4)=0 \quad ...(iii)$$

Now plane (iii) is perpendicular to plane

$$\vec{r}.(5\hat{i}+3\hat{j}-6\hat{k})+8 = 0 \quad ...(iv)$$

$$\therefore (1+2\lambda)\,5 + (2+\lambda)\,3 + (3-\lambda)\,(-6) = 0$$

$$\Rightarrow \quad 19\lambda - 7 = 0 \Rightarrow \lambda = \frac{7}{19}.$$

Substituting the value of λ in (iii), we get

$$\vec{r}.\left[\left(1+2\times\frac{7}{19}\right)\hat{i}+\left(2+\frac{7}{19}\right)\hat{j}+\left(3-\frac{7}{19}\right)\hat{k}\right] + \left(5\times\frac{7}{19}-4\right)=0$$

$$\Rightarrow \vec{r}(33\hat{i}+45\hat{j}+50\hat{k}) = 41.$$ **Ans.**

22. A speaks truth in 60% of the cases, while B in 90% of the cases. In what percent of cases are they likely to contradict each other in stating the same fact ? In the cases of contradiction do you think, the statement of B will carry more weight as he speaks truth in more number of cases then A ? **[4]**

Solution : Probability of A speaking the truth is

$$P(A) = \frac{60}{100} = \frac{6}{10}$$

$$\Rightarrow \quad (\bar{A}) = 1 - P(A) = \frac{4}{10}$$

Probability of B speaking the truth is

$$P(B) = \frac{90}{100} = \frac{9}{10}$$

$$\Rightarrow \quad P(\bar{B}) = 1 - P(B) = \frac{1}{10}$$

Now A and B will contradict each other in the following mutually exclusive cases :

(i) A speaks the truth and B does not.
(ii) B speaks the truth and A does not.
By the theorem of total probability
Probability that A and B will contradict each other

$$= P(A).(\bar{B}) + P(B).(\bar{A})$$

$$= \frac{6}{10}.\frac{1}{10} + \frac{9}{10}.\frac{4}{10} = \frac{42}{100}$$

∴ They will contradict each other in 42% of the cases.

Yes, the statement of B will carry more weight. **Ans.**

SECTION — C

23. A school wants to award its students for the values of Honesty, Regularity and Hard work with a total cash award of ₹ 6,000. Three times the award money for Hard work added to that given for honesty amounts to ₹ 11,000. The award money given for Honesty and Hard work together is double the one given for Regularity. Represent the above situation algebraically and find the award money for each value, using matrix method. Apart from these values namely, Honesty, Regularity and Hard work, suggest one more value which the school must include for awards. **[6]**

Solution : Let award for honesty be ₹ x

award for regularity be ₹ y

and award for hardwork be ₹ z.

According to question,

$$x+y+z = 6000$$

$$x+0y+3z = 11000$$

$$x+z = 2y \Rightarrow x-2y+z=0$$

The given equations can be written in matrix form

$$AX = B \quad ...(i)$$

$$\begin{bmatrix}1 & 1 & 1\\1 & 0 & 3\\1 & -2 & 1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}6000\\11000\\0\end{bmatrix}$$

Here $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & -2 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 6000 \\ 11000 \\ 0 \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & -2 & 1 \end{vmatrix}$

$= 1(0+6) - 1(1-3) + 1(-2-0)$

$= 6 \neq 0$

$\therefore A^{-1}$ exists.

For adj A,

$A_{11} = 0 + 6 = 6,$ $\quad A_{12} = -(1-3) = 2,$

$A_{13} = -2 - 0 = -2,$

$A_{21} = -(1+2) = -3,$ $\quad A_{22} = 1 - 1 = 0,$

$A_{23} = -(-2-1) = 3$

$A_{31} = 3 - 0 = 3,$ $\quad A_{32} = -(3-1) = -2,$

$A_{33} = 0 - 1 = -1$

$$\therefore \text{adj } A = \begin{bmatrix} 6 & 2 & -2 \\ -3 & 0 & 3 \\ 3 & -2 & -1 \end{bmatrix}^T = \begin{bmatrix} 6 & -3 & 3 \\ 2 & 0 & -2 \\ -2 & 3 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj. } A = \frac{1}{6}\begin{bmatrix} 6 & -3 & 3 \\ 2 & 0 & -2 \\ -2 & 3 & -1 \end{bmatrix}$$

From (1), $X = A^{-1}B$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 6 & -3 & 3 \\ 2 & 0 & -2 \\ -2 & 3 & -1 \end{bmatrix}\begin{bmatrix} 6000 \\ 11000 \\ 0 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 3000 \\ 12000 \\ 21000 \end{bmatrix}$$

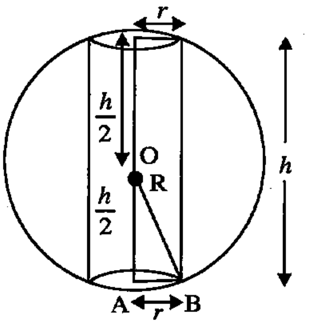
$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 500 \\ 2000 \\ 3500 \end{bmatrix}$$

$\Rightarrow x = 500, y = 2000, z = 3500.$

Apart from honesty, regularity and hard work, the school must include an award for a student to be well behaved. **Ans.**

24. Show that the height of the cylinder of maximum volume, that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also find the maximum volume. **[6]**

Solution : From the figure:



In Δ OAB,

$$\left(\frac{h}{2}\right)^2 + r^2 = R^2 \quad \text{...(i)}$$

$$\Rightarrow \quad r^2 = R^2 - \frac{h^2}{4}$$

Let, V be Volume of the cylinder inscribed in a sphere

$$\therefore \quad V = \pi r^2 h$$

$$= \pi h\left(R^2 - \frac{h^2}{4}\right) \text{ [using (i)]} \quad \text{...(ii)}$$

$$\therefore \quad V = \pi\left(R^2 h - \frac{h^3}{4}\right)$$

Differentiating w.r. t. h, we get

$$\frac{dV}{dh} = \pi\left(R^2 - \frac{3h^2}{4}\right) \quad \text{...(iii)}$$

and $$\frac{d^2V}{dh^2} = \pi\left(0 - \frac{3}{4}.2h\right) \quad \text{...(iv)}$$

For maxima or minima

$$\frac{dV}{dh} = 0$$

From (iii),

$$R^2 - \frac{3}{4}h^2 = 0$$

$$\Rightarrow \quad h^2 = \frac{4}{3}R^2$$

$$\Rightarrow \quad h = \frac{2R}{\sqrt{3}}$$

For the value of h, from (iv),

$$\frac{d^2V}{dh^2} = -\frac{3}{2}\pi.\frac{2R}{\sqrt{3}} = -\sqrt{3}.\pi R < 0(-\text{ve})$$

$\Rightarrow$ V is maximum.

Also maximum value of V

$$= \pi.\frac{2R}{\sqrt{3}}\left(R^2 - \frac{1}{4}.\frac{4}{3}R^2\right)$$

$$= \pi.\frac{2R}{\sqrt{3}}.\frac{2}{3}R^2 = \frac{4\pi}{3\sqrt{3}}R^3$$

$$= \frac{4\sqrt{3}}{9}\pi R^3 \text{ cu.units} \qquad \textbf{Ans.}$$

OR

Find the equation of the normal at a point on the curve $x^2 = 4y$ which passes through the point (1, 2). Also find the equation of the corresponding tangent.

Solution : The given curve is

$$x^2 = 4y \qquad ...(i)$$

Let (x_1, y_1) be the required point on curve

$$\therefore \quad x_1^2 = 4y_1 \qquad ...(ii)$$

Differentiating eq. (i) w.r.t. x, we get

$$2x = 4\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{x}{2}$$

$$\therefore \text{ Slope of tangent} = m = \left.\frac{dy}{dx}\right]_{(x_1, y_1)} = \frac{x_1}{2}$$

$$\text{and slope of normal} = m' = -\frac{1}{m} = -\frac{2}{x_1}$$

$\therefore$ Eq. of normal at (x_1, y_1) is

$$y - y_1 = m'(x - x_1)$$

It passes through (1, 2)

$$2 - y_1 = -\frac{2}{x_1}(1 - x_1)$$

$$\Rightarrow \quad 2x_1 - x_1 y_1 = -2 + 2x_1$$

$$\Rightarrow \quad x_1 y_1 = 2$$

Put value of y_1 from eq. (ii)

$$x_1 \cdot \frac{x_1^2}{4} = 2$$

$$\Rightarrow \quad x_1^3 = 8 \Rightarrow x_1 = 2$$

$$\therefore \text{ From (ii)}, \quad y_1 = \frac{x_1^2}{4} = \frac{2^2}{4} = 1$$

$\therefore$ Point on curve is (2, 1) and $m = \frac{2}{2} = 1, m' = -1$

Eq. of normal at (2, 1) and $m' = -1$ is

$$y - 1 = -(x - 2)$$

$$\Rightarrow \quad x + y = 3$$

Eq. of tangent at (2, 1) and $m = 1$

$$y - 1 = (x - 2)$$

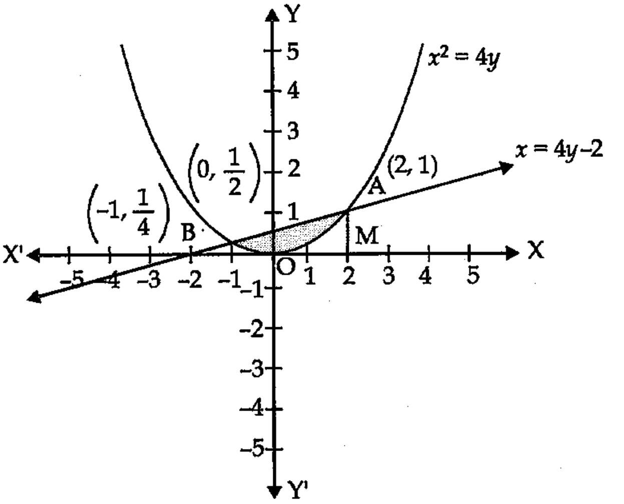
$$\Rightarrow \quad x - y = 1. \qquad \textbf{Ans.}$$

25. Using integration, find the area bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$. **[6]**

Solution : The equation of the given curves are :

$$x^2 = 4y \qquad ...(i)$$

$$x = 4y - 2 \qquad ...(ii)$$



The points of intersection of (i) and (ii) are A (2, 1) and B$\left(-1, \frac{1}{4}\right)$

Required area = Area of shaded region = Area under line – area under parabola.

$$= \int_{-1}^{2} (y_2 - y_1)\,dx$$

$$= \int_{-1}^{2} \left(\frac{x+2}{4} - \frac{x^2}{4}\right)dx$$

$$= \frac{1}{4}\left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2}$$

$$= \frac{1}{4}\left[\left(2 + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)\right]$$

$$= \frac{1}{4}\left[6 - \frac{8}{3} + \frac{3}{2} - \frac{1}{3}\right] = \frac{1}{4}\left(\frac{9}{2}\right) = \frac{9}{8} \text{ sq. units.} \qquad \textbf{Ans.}$$

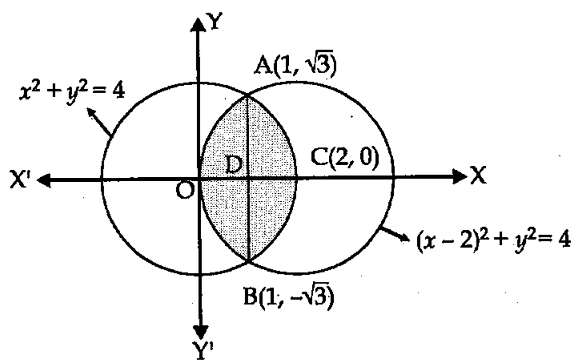
OR

Using integration, find the area of the region enclosed between the two circles $x^2 + y^2 = 4$ and $(x - 2)^2 + y^2 = 4$.

Solution : The given circles are :

$$x^2 + y^2 = 4 \qquad ...(i)$$

$$\text{and} \qquad (x - 2)^2 + y^2 = 4 \qquad ...(ii)$$



They meet at $A(1,\sqrt{3})$ and $B(1,-\sqrt{3})$

Required Area = Area of shaded region

= 2 (Area in Ist Quadrant)

= {Area under circle (2) + Area under circle (1)}

$$= 2\left[\int_0^1 y_2 dx + \int_1^2 y_1 dx\right]$$

$$= 2\left[\int_0^1 \sqrt{4-(x-2)^2}\,dx + \int_1^2 \sqrt{4-x^2}\,dx\right]$$

$$= 2\left[\left\{\frac{(x-2)\sqrt{4-(x-2)^2}}{2} + \frac{4}{2}\sin^{-1}\left(\frac{x-2}{2}\right)\right\}_0^1 + \left\{\frac{x\sqrt{4-x^2}}{2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right\}_1^2\right]$$

$$= \left\{-\sqrt{3} + 4\sin^{-1}\left(\frac{-1}{2}\right) - (0 + 4\sin^{-1}(-1))\right\} + \left\{0 + 4\sin^{-1}(1) - \left(\sqrt{3} + 4\sin^{-1}\frac{1}{2}\right)\right\}$$

$$= -\sqrt{3} - 4\sin^{-1}\left(\frac{1}{2}\right) + 4\sin^{-1}(1) + 4\sin^{-1}(1) - \sqrt{3} - 4\sin^{-1}\left(\frac{1}{2}\right)$$

$$= -2\sqrt{3} + 8\sin^{-1}(1) - 8\sin^{-1}\left(\frac{1}{2}\right)$$

$$= 8\left(\frac{\pi}{2} - \frac{\pi}{6}\right) - 2\sqrt{3} = \frac{8}{3}\pi - 2\sqrt{3} \text{ sq. units.}$$ **Ans.**

26. Show that the differential equation $2ye^{x/y}\,dx + (y - 2x e^{x/y})\,dy = 0$ is homogeneous. Find the particular solution of this differential equation, given that $x = 0$, when $y = 1$. **[6]**

Solution : The given differential equation is

$$2ye^{x/y}\,dx + (y - 2x\,e^{x/y})\,dy = 0$$

Separate the given differential equation, we get

$$\frac{dx}{dy} = \frac{2xe^{x/y} - y}{2ye^{x/y}} = \frac{2\frac{x}{y}e^{x/y} - 1}{2e^{x/y}}$$

$$= \lambda^0 f\left(\frac{x}{y}\right)$$

This is a homogeneous differential equation.

Putting $x = vy$

$$\Rightarrow \quad \frac{dx}{dy} = 1.v + y.\frac{dv}{dy}$$

$$\Rightarrow \quad v + y\frac{dv}{dy} = \frac{2ve^v - 1}{2e^v}$$

$$= v\ \frac{1}{2e^v}$$

$$\Rightarrow \quad v + y\frac{dv}{dy} = v - \frac{1}{2e^v}$$

$$\Rightarrow \quad 2e^v\,dv = -\frac{dy}{y}$$

On integrating, we get

$$2\int e^v dv = -\int \frac{dy}{y}$$

$$\Rightarrow \quad 2e^v + \log y = C$$

$$\Rightarrow 2e^{x/y} + \log y = C$$

Put $x = 0, y = 1$, we get

$$2e^0 + \log 1 = C$$

$$\Rightarrow \quad C = 2$$

∴ The required particular solution is

$$2\,e^{x/y} + \log y = 2.$$ **Ans.**

27. Find the vector equation of the plane passing through three points with position vectors $\hat{i}+\hat{j}-2\hat{k}, 2\hat{i}-\hat{j}+\hat{k}$ and $\hat{i}+2\hat{j}+\hat{k}$. Also find the coordinates of the point of intersection of this plane and the line $\vec{r} = 3\hat{i}-\hat{j}-\hat{k}+\lambda(2\hat{i}-2\hat{j}+\hat{k})$. **[6]**

Solution : The given points are

$$A(\hat{i}+\hat{j}-2\hat{k}) \equiv (1, 1, -2)$$

$$B(2\hat{i}-\hat{j}+\hat{k}) \equiv (2, -1, 1)$$

and $C(\hat{i}+2\hat{j}+\hat{k}) \equiv (1, 2, 1)$

Equation of any plane passing through A is

$a(x-1) + b(y-1) + c(z+2) = 0$ ...(i)

As it is to pass through B and C respectively

$$a - 2b + 3c = 0$$

$$0.a + b + 3c = 0$$

Solving these equations, we get

$$\frac{a}{-6-3} = \frac{b}{0-3} = \frac{c}{1-0}$$

$$\Rightarrow \quad \frac{a}{9} = \frac{b}{3} = \frac{c}{-1}$$

$\therefore$ From (i),

$$9(x-1)+3(y-1)-(z+2)=0$$

$$\Rightarrow \quad 9x+3y-z-14=0 \quad \text{....(ii)}$$

$\therefore$ Vector equation of the plane is

$$\vec{r}.(9\hat{i}+3\hat{j}-\hat{k}) = 14$$

The given line is

$$\vec{r} = 3i-\hat{j}-\hat{k}+\lambda(2\hat{i}-2\hat{j}+\hat{k})$$

$$\frac{x-3}{2} = \frac{y+1}{-2} = \frac{z+1}{1} = \lambda \text{ (say)}$$

$(2\lambda+3, -2\lambda-1, \lambda-1)$

If it lies on the above plane (ii), then

$$9(2\lambda+3)+3(-2\lambda-1)-(\lambda-1)-14=0$$

$$\Rightarrow \quad 11\lambda+11=0 \Rightarrow \lambda=-1$$

$\therefore$ Their point of intersection is

$(-2+3, 2-1, -1-1) = (1, 1, -2)$. **Ans.**

28. A cooperative society of farmers has 50 hectares of land to grow two crops A and B. The profits from crops A and B per hectare are estimated as ₹ 10,500 and ₹ 9,000 respectively. To control weeds, a liquid herbicide has to be used for crops A and B at the rate of 20 litre and 10 litre per hectare, respectively. Further not more than 800 litres of herbicide should be used in order to protect fish and wildlife using a pond which collects drainage from this land. Keeping in mind that the protection of fish and other wildlife is more important than earning profit, how much land should be allocated to each crop so as to maximize the total profit ? Form an LPP from the above and solve it graphically. Do you agree with the message that the protection of wildlife is utmost necessary to preserve the balance in environment ? [6]

Solution : Let x hectare and y hectare be allotted to grow crops A and B respectively. Then the L.P.P. is maximize :

$Z = 10{,}500\,x + 9{,}000\,y$ subject to constraints,

$$x+y \le 50$$

$$20x+10y \le 800$$

$$\Rightarrow \quad 2x+y \le 80$$

and $\quad x \ge 0, y \ge 0$

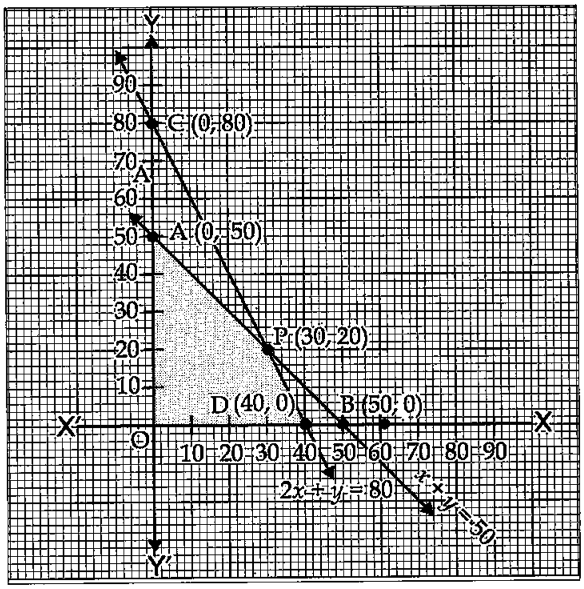
First we draw the line AB and CD whose equations are

$$x+y=50 \quad \text{...(i)}$$

	A	B
x	0	50
y	50	0

and $\quad 2x+y=80 \quad$ ...(ii)

	C	D
x	0	40
y	80	0



$\therefore$ The feasible region is ODPAO which is shaded in the figure.

P is the point of intersection of the lines

$$x+y=50$$
$$2x+y=80$$

Solving these equations, we get point P (30, 20).

The vertices of the feasible region are (0, 0), D (40, 0), P (30, 20) and A (0, 50).The value of objective function $Z = 10{,}500\,x + 9{,}000\,y$ at these vertices are as follows :

Corner Points	$Z = 10{,}500x + 9{,}000y$
At O (0, 0)	$Z = 0$
At D (40, 0)	$Z = 4{,}20{,}000$
At P (30, 20)	$Z = 4{,}95{,}000$ maximum
At A (0, 50)	$Z = 4{,}50{,}000$

$\therefore$ The maximum profit is 4,95,000 at point P (30, 20). Yes, I agree with the message in the question. **Ans.**

29. Assume that the chances of a patient having a heart attack is 40%. Assuming that a meditation and yoga course reduces the risk of heart attack by 30% and prescription of certain drug reduces its chance by 25%. At a time a patient can choose any one of the two options with equal possibilities. It is given that after going through one of the two options, the patient selected at random suffers a heart attack. Find the probability that the patient followed a course of meditation and yoga. Interpret the result and state which of the above stated methods is more

beneficial for the patient. **[6]**

Solution : Let E_1 be taking a course of meditation and yoga and E_2 be taking a course of drugs.

A be the patient gets a heart attack.

Here $P(E_1) = \frac{1}{2}; P(E_2) = \frac{1}{2}$

$$P(A/E_1) = \frac{70}{100} \times \frac{40}{100}$$

$$P(A/E_2) = \frac{75}{100} \times \frac{40}{100}$$

By Bayes' theorem,

$$P(E_1/A) = \frac{P(E_1)P(A/E_1)}{P(E_1).P(A/E_1)+P(E_2)P(A/E_2)}$$

$$= \frac{\frac{1}{2}\times\frac{70}{100}\times\frac{40}{100}}{\frac{1}{2}\times\frac{70}{100}\times\frac{40}{100}+\frac{1}{2}\times\frac{75}{100}\times\frac{40}{100}} = \frac{14}{29}$$

Now, $P(E_2/A)$

$$= \frac{P(E_2)P(A/E_2)}{P(E_1)P(A/E_1)+P(E_2)P(A/E_2)}$$

$$= \frac{\frac{1}{2}\times\frac{75}{100}\times\frac{40}{100}}{\frac{1}{2}\times\frac{75}{100}\times\frac{40}{100}+\frac{1}{2}\times\frac{70}{100}\times\frac{40}{100}} = \frac{15}{29}$$

Since, $P(E_1/A) < P(E_2/A)$ the course of yoga and meditation is more beneficial for a person having chances of heart attack. **Ans.**

Mathematics 2013 (Delhi) SET II

Time allowed : 3 hours **Maximum marks : 100**

> **Note : Except for the following questions, all the remaining questions have been asked in previous set.**

SECTION — A

3. Find the value of b if

$$\begin{bmatrix} a-b & 2a+c \\ 2a-b & 3c+d \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}.$$

[1]

Solution : Given, $\begin{bmatrix} a-b & 2a+c \\ 2a-b & 3c+d \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$

Comparing the corresponding, we get

$$a-b = -1 \quad \text{...(i)}$$

and $2a-b = 0 \Rightarrow a = \frac{b}{2}$ ...(ii)

From (i) and (ii),

$$\frac{b}{2}-b = -1 \Rightarrow \frac{-b}{2} = -1 \Rightarrow b = 2.$$ **Ans.**

9. Write the degree of the differential equation $\left(\frac{dy}{dx}\right)^4 + 3x\frac{d^2y}{dx^2} = 0.$ **[1]**

Solution : The degree of the differential equation $\left(\frac{dy}{dx}\right)^4 + 3x\frac{d^2y}{dx^2} = 0$ is 1. **Ans.**

SECTION — B

16. P speaks truth in 70% of the cases and Q in 80% of the cases. In what percent of cases are they likely to agree in stating the same fact ? Do you think, when they agree, means both are speaking truth ? **[4]**

Solution : Probability of P speaking the truth is

$$P(P) = \frac{70}{100} = \frac{7}{10} \Rightarrow P(\bar{P}) = 1-\frac{7}{10} = \frac{3}{10}$$

Probability of Q speaking the truth is

$$P(Q) = \frac{80}{100} = \frac{8}{10} \Rightarrow P(\bar{Q}) = 1-\frac{8}{10} = \frac{2}{10}$$

Now P and Q are likely to agree with each other in the following mutually exclusive cases.

(i) Both speak the truth

(ii) Both do not speak the truth

By the theorem of total probability,

$\therefore$ Probability that both P and Q agree

$= P(PQ \text{ or } \bar{P}\,\bar{Q})$

$= P(P)\,P(Q) + P(\bar{P})P(\bar{Q})$

$= \frac{7}{10}.\frac{8}{10}+\frac{3}{10}.\frac{2}{10} = \frac{56+6}{100} = \frac{62}{100}$

Hence, they are likely to agree in 62% of the cases.

No, not necessarily when they agree there may be case in which they both does not speak truth.

Ans.

18. **If $\vec{a} = \hat{i}+\hat{j}+\hat{k}$ and $\vec{b} = \hat{j}-\hat{k}$, find a vector $\vec{c}$, such that $\vec{a} \times \vec{c} = \vec{b}$ and $\vec{a}.\vec{c} = 3$.** **[4]**

Solution : Given, $\vec{a} = \hat{i}+\hat{j}+\hat{k}$ and $\vec{b} = \hat{j}-\hat{k}$

Let $\vec{c} = x_1\hat{i}+x_2\hat{j}+x_3\hat{k}$

$\therefore \quad \vec{a}.\vec{c} = 3$ (Given)

$(\hat{i}+\hat{j}+\hat{k}).(x_1\hat{i}+x_2\hat{j}+x_3\hat{k}) = 3$

$\Rightarrow \quad x_1+x_2+x_3 = 3$ ...(i)

and $\quad \vec{a}\times\vec{c} = \vec{b}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ x_1 & x_2 & x_3 \end{vmatrix} = \hat{j}-\hat{k}$$

$(x_3-x_2)\hat{i}-(x_3-x_1)\hat{j}+(x_2-x_1)\hat{k} = \hat{j}-\hat{k}$

$\Rightarrow \quad x_3-x_2 = 0$ ...(ii)

$x_1-x_3 = 1$ ...(iii)

$x_1-x_2 = 1$ ...(iv)

Solving the equations (i) to (iv), we get

$$x_1 = \frac{5}{3}, x_2 = x_3 = \frac{2}{3}.$$

$$\therefore \quad \vec{c} = \frac{5}{3}\hat{i}+\frac{2}{3}\hat{j}+\frac{2}{3}\hat{k}$$

$$\therefore \quad \vec{c} = \frac{1}{3}(5\hat{i}+2\hat{j}+2\hat{k}).$$

19. **Evaluate : $\int_1^3 [|x-1|+|x-2|+|x-3|]dx$.** **[4]**

Solution : Let $I = \int_1^3 [|x-1|+|x-2|+|x-3|]dx$

$$= \int_1^2 [|x-1|+|x-2|+|x-3|]dx + \int_2^3 [|x-1|+|x-2|+|x-3|]dx$$

$$= \int_1^2 [x-1-(x-2)-(x-3)].dx + \int_2^3 [(x-1)+(x-2)-(x-3)].dx$$

$$= \int_1^2 (-x+4)dx + \int_2^3 (x)dx$$

$$= \left[-\frac{x^2}{2}+4x\right]_1^2 + \left[\frac{x^2}{2}\right]_2^3$$

$$= \left[(-2+8)-\left(-\frac{1}{2}+4\right)+\left(\frac{9}{2}-2\right)\right]$$

$$= 6-\frac{7}{2}+\frac{5}{2} = 6-1 = 5.$$ **Ans.**

20. **Evaluate : $\int \frac{x^2+1}{(x^2+4)(x^2+25)}dx$.** **[4]**

Solution : Let $I = \int \frac{x^2+1}{(x^2+4)(x^2+25)}dx$

Putting $x^2 = y$,

$$I = \frac{y+1}{(y+4)(y+25)} = \frac{A}{y+4}+\frac{B}{y+25}$$

$\Rightarrow \quad y+1 = A(y+25)+B(y+4)$

Putting $y = -25$, we get

$$-24 = B(-25+4) \Rightarrow B = \frac{8}{7}$$

and putting $y = -4$, we get

$$-3 = A(-4+25) \Rightarrow A = -\frac{1}{7}.$$

$$\therefore \quad \frac{y+1}{(y+4)(y+25)} = -\frac{1}{7}\left(\frac{1}{y+4}\right)+\frac{8}{7}\left(\frac{1}{y+25}\right)$$

$$\Rightarrow \frac{x^2+1}{(x^2+4)(x^2+25)} = -\frac{1}{7}\left(\frac{1}{x^2+4}\right)+\frac{8}{7}\left(\frac{1}{x^2+25}\right)$$

On integrating both sides w.r.t. x, we get

$$\int \frac{x^2+1}{(x^2+4)(x^2+25)}dx = -\frac{1}{7}\int\frac{dx}{x^2+2^2}+\frac{8}{7}\int\frac{dx}{x^2+5^2}$$

$$= -\frac{1}{7}\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+\frac{8}{7}\frac{1}{5}\tan^{-1}\left(\frac{x}{5}\right)+C$$

$$\left(\text{Using } \int\frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}+C\right)$$

$$= -\frac{1}{14}\tan^{-1}\left(\frac{x}{2}\right)+\frac{8}{35}\tan^{-1}\left(\frac{x}{5}\right)+C.$$ **Ans.**

SECTION — C

28. **Show that the differential equation $x\frac{dy}{dx}\sin\left(\frac{y}{x}\right)+x-y\sin\left(\frac{y}{x}\right) = 0$ is homogeneous. Find the particular solution of this differential equation given that $x = 1$ when $y = \frac{\pi}{2}$.** **[6]**

Solution : The given differential equation is

$$x\frac{dy}{dx}\sin\left(\frac{y}{x}\right)+x-y\sin\left(\frac{y}{x}\right)=0$$

$$\Rightarrow \quad x\frac{dy}{dx}\sin\left(\frac{y}{x}\right)=y\sin\left(\frac{y}{x}\right)-x$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{y\sin\left(\frac{y}{x}\right)-x}{x\sin\left(\frac{y}{x}\right)}=\frac{\frac{y}{x}\sin\frac{y}{x}-1}{\sin\frac{y}{x}} \quad \text{...(i)}$$

which is clearly a homogeneous differentiable equation.

Putting $\frac{y}{x}=v \Rightarrow y=vx$

$$\Rightarrow \quad \frac{dy}{dx}=v+x\frac{dv}{dx}$$

$\therefore$ Eq. (i) becomes

$$v+x\frac{dv}{dx}=\frac{v\sin v-1}{\sin v}=v-\frac{1}{\sin v}$$

$$\Rightarrow \quad x\frac{dv}{dx}=-\frac{1}{\sin v}$$

$$\Rightarrow \quad \sin v\, dv=\frac{-dx}{x}$$

On integrating, we get

$$\Rightarrow \quad \int\sin v\,dv=-\int\frac{1}{x}dx$$

$$\Rightarrow \quad -\cos v=-\log x+C$$

$$\Rightarrow \quad \log x-\cos\left(\frac{y}{x}\right)=C \quad \text{...(ii)}$$

Given at $x=1$ and $y=\frac{\pi}{2}$

$$\Rightarrow \quad \log 1-\cos\left(\frac{\pi}{2}\right)=C$$

$$\therefore \quad C=0$$

$\therefore$ The particular solution of the given differential equation is

$$\log x-\cos\left(\frac{y}{x}\right)=0. \quad \textbf{Ans.}$$

29. Find the vector equation of the plane determined by the points A (3, –1, 2), B (5, 2, 4) and C (–1, –1, 6). Also find the distance of point P (6, 5, 9) from this plane. [6]

Solution : The given points are

A (3, – 1, 2); B (5, 2, 4); C (– 1, – 1, 6)

Equation of any plane passes through A (3, – 1, 2) is

$$a(x-3)+b(y+1)+c(z-2)=0 \quad \text{...(i)}$$

Points B and C lie on it.

$$\therefore \quad 2a+3b+2c=0$$
$$-4a+0b+4c=0$$

Solving these equations, we get

$$\frac{a}{12-0}=\frac{b}{-8-8}=\frac{c}{0+12}$$

$$\Rightarrow \quad \frac{a}{3}=\frac{b}{-4}=\frac{c}{3}$$

$\therefore$ Equation of the required plane is

$$3(x-3)-4(y+1)+3(z-2)=0$$

$$\Rightarrow \quad 3x-4y+3z-19=0 \quad \text{...(ii)}$$

In vector form, it becomes

$$\vec{r}.(3\hat{i}-4\hat{j}+3\hat{k})-19=0$$

Now distance of P (6, 5, 9) from (ii),

$$\therefore \quad d=\left|\frac{3\times 6-4\times 5+3\times 9-19}{\sqrt{3^2+(-4)^2+3^2}}\right|$$

$$=\left|\frac{18-20+27-19}{\sqrt{34}}\right|=\frac{6}{\sqrt{34}} \quad \textbf{Ans.}$$

Mathematics 2013 (Delhi) SET III

Time allowed : 3 hours **Maximum marks : 100**

> **Note : Except for the following questions, all the remaining questions have been asked in previous sets.**

SECTION — A

2. Write, unit vector in the direction of the sum of vectors $\vec{a}=2\hat{i}-\hat{j}+2\hat{k}$ and $\vec{b}=-\hat{i}+\hat{j}+3\hat{k}$. [1]

Solution : $\vec{a}=2\hat{i}-\hat{j}+2\hat{k}$, $\vec{b}=-\hat{i}+\hat{j}+3\hat{k}$

$$\therefore \vec{a}+\vec{b}=(2-1)\hat{i}+(-1+1)\hat{j}+(2+3)\hat{k}=\hat{i}+5\hat{k}$$

$$|\vec{a}+\vec{b}|=\sqrt{1^2+5^2}=\sqrt{26}$$

Therefore, the required unit vector is

$$=\frac{\vec{a}+\vec{b}}{|\vec{a}+\vec{b}|}=\frac{\hat{i}+5\hat{k}}{\sqrt{26}}. \quad \textbf{Ans.}$$

4. Write the degree of the differential equation

$$x\left(\frac{d^2y}{dx^2}\right)^3+y\left(\frac{dy}{dx}\right)^4+x^3=0$$ **[1]**

Solution : The given differential equation is

$$x\left(\frac{d^2y}{dx^2}\right)^3+y\left(\frac{dy}{dx}\right)^4+x^3=0$$

The highest order derivative present in the given differential equation is $\frac{d^2y}{dx^2}$. The power raised to $\frac{d^2y}{dx^2}$ is three.

So, the degree of the given differential equation is 3. **Ans.**

SECTION — B

11. A speaks truth in 75% of the cases, while B in 90% of the cases. In what percent of cases are they likely to contradict each other in stating the same fact ? Do you think that statement of B is true ? [4]

Solution : Let, the probability that A and B speak truth be P(A) and P(B) respectively.

Therefore, $P(A)=\frac{75}{100}=\frac{3}{4}$ and $P(B)=\frac{90}{100}=\frac{9}{10}$

They can agree in stating the fact when both are speaking the truth or when both are not speaking the truth.

Case 1 : When A is not speaking the truth and B is speaking the truth.

Required probability

$$=(1-P(A))\times P(B)=\left(1-\frac{3}{4}\right)\times\frac{9}{10}=\frac{1}{4}\times\frac{9}{10}=\frac{9}{40}.$$

Case 2 : When A is speaking the truth and B is not speaking the truth.

Required probability

$$=P(A)\times(1-P(B))=\frac{3}{4}\times\left(1-\frac{9}{10}\right)=\frac{3}{4}\times\frac{1}{10}=\frac{3}{40}$$

Therefore, the percent of cases in which they are likely to agree in stating the same fact is equal to

$$\left(\frac{9}{40}+\frac{3}{40}\right)\times100=30\%.$$ **Ans.**

13. Using vectors, find the area of the triangle ABC with vertices A (1, 2, 3), B (2, – 1, 4) and C (4, 5, – 1). [4]

Solution : The vertices of Δ ABC are A (1, 2, 3), B (2, – 1, 4) and C (4, 5, – 1).

$\therefore \vec{AB}$ = Position vector of B – Position vector of A

$$=(2\hat{i}-\hat{j}+4\hat{k})-(\hat{i}+2\hat{j}+3\hat{k})$$
$$=\hat{i}-3\hat{j}+\hat{k}$$

$\vec{AC}$ = Position vector of C – Position vector of A

$$=(4\hat{i}+5\hat{j}-\hat{k})-(\hat{i}+2\hat{j}+3\hat{k})$$
$$=3\hat{i}+3\hat{j}-4\hat{k}$$

Now,

$$\vec{AB}\times\vec{AC}=(\hat{i}-3\hat{j}+\hat{k})\times(3\hat{i}+3\hat{j}-4\hat{k})$$

$$\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -3 & 1\\ 3 & 3 & -4\end{vmatrix}=\hat{i}(12-3)-\hat{j}(-4-3)+\hat{k}(3+9)$$

$$=9\hat{i}+7\hat{j}+12\hat{k}$$

$$\therefore\ |\vec{AB}\times\vec{AC}|=\sqrt{(9)^2+(7)^2+(12)^2}=\sqrt{274}$$

$$\therefore \text{Area of the } \Delta\, ABC=\frac{1}{2}|\vec{AB}\times\vec{AC}|$$

$$=\frac{1}{2}\sqrt{274}\text{ sq. units.}$$ **Ans.**

14. Evaluate : $\int_2^5[|x-2|+|x-3|+|x-5|]dx.$ [4]

Solution : Let, $f(x)=|x-2|+|x-3|+|x-5|$

$$I=\int_2^5 f(x)dx=\int_2^3 f(x)dx+\int_3^5 f(x)dx$$

$$\Rightarrow I=\int_2^3(x-2+3-x+5-x)dx+\int_3^5(x-2+x-3+5-x)dx$$

$$\Rightarrow I=\int_2^3(6-x)dx+\int_3^5 x\,dx=\left[6x-\frac{x^2}{2}\right]_2^3+\left[\frac{x^2}{2}\right]_3^5$$

$$=\left[18-\frac{9}{2}-12+2\right]+\left[\frac{25}{2}-\frac{9}{2}\right]=\frac{23}{2}.$$ **Ans.**

15. Evaluate : $\int\frac{2x^2+1}{x^2(x^2+4)}dx.$ [4]

Solution :

$$I=\int\frac{2x^2+1}{x^2(x^2+4)}dx=\int\frac{2x^2+8-7}{x^2(x^2+4)}dx$$

$$=\int\frac{2(x^2+4)-7}{x^2(x^2+4)}dx$$

[Multiplying & dividing by 4 and then adding and subtracting x^2 in 2nd Integral]

$$= \int \frac{2}{x^2}dx - 7\int \frac{1}{x^2(x^2+4)}dx$$

$$\Rightarrow \quad I = \int \frac{2}{x^2}dx - \frac{7}{4}\int \frac{(x^2+4)-x^2}{x^2(x^2+4)}dx$$

$$= \int \frac{2}{x^2}dx - \frac{7}{4}\int \frac{1}{x^2}dx + \frac{7}{4}\int \frac{1}{(x^2+4)}dx$$

$$\Rightarrow \quad I = \frac{1}{4}\int \frac{1}{x^2}dx + \frac{7}{4}\int \frac{1}{(x^2+4)}dx$$

$$= \frac{1}{4}\left(-\frac{1}{x}\right) + \frac{7}{4} \times \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)$$

$$= -\frac{1}{4x} + \frac{7}{8}\tan^{-1}\left(\frac{x}{2}\right) + C. \quad \textbf{Ans.}$$

SECTION — C

25. Find the coordinates of the point where the line through (3, – 4, – 5) and (2, – 3, 1) crosses the plane, passing through the points (2, 2, 1), (3, 0, 1) and (4, – 1, 0). [6]

Solution : The equation of the straight line passing through the point (3, – 4, – 5) and (2, – 3, 1) is

$$\frac{x-3}{2-3} = \frac{y-(-4)}{-3-(-4)} = \frac{z-(-5)}{1-(-5)}$$

$$\Rightarrow \quad \frac{x-3}{-1} = \frac{y+4}{1} = \frac{z+5}{6} = \lambda \text{ (say)} \quad ...(i)$$

The coordinates of any point on line (i) is

$(-\lambda + 3, \lambda - 4, 6\lambda - 5)$.

We know that, the equation of the plane passing through three points

$(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)$ is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

So, the equation of the plane passing through the points (2, 2, 1), (3, 0, 1) and (4, – 1, 0) is

$$\begin{vmatrix} x-2 & y-2 & z-1 \\ 1 & -2 & 0 \\ 2 & -3 & -1 \end{vmatrix} = 0$$

$$\Rightarrow \quad (x-2)(2-0) - (y-2)(-1-0) + (z-1)(-3+4) = 0$$

$$\Rightarrow \quad 2x - 4 + y - 2 + z - 1 = 0$$

$$\Rightarrow \quad 2x + y + z - 7 = 0 \quad ..(ii)$$

If the point $(-\lambda + 3, \lambda - 4, 6\lambda - 5)$ lies on the plane (ii), then

$$2(-\lambda + 3) + (\lambda - 4) + (6\lambda - 5) - 7 = 0$$

$$\Rightarrow \quad 5\lambda - 10 = 0$$

$$\Rightarrow \quad \lambda = 2$$

Putting $\lambda = 2$ in $(-\lambda + 3, \lambda - 4, 6\lambda - 5)$, we have $(-2+3, 2-4, 6 \times 2 - 5) = (1, -2, 7)$

Thus, the coordinates of the point where the line (i) crosses the plane (ii) is (1, – 2, 7). **Ans.**

26. Show that the differential equation $(xe^{y/x} + y)$ $dx = xdy$ is homogeneous. Find the particular solution of this differential equation, given that $x = 1$ when $y = 1$. [6]

Solution : The given differential equation is

$$(xe^{\frac{y}{x}} + y)dx = xdy$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{xe^{\frac{y}{x}} + y}{x} = f(x, y) \quad ...(i)$$

Replacing x by λx and y by λy in (i), we have

$$f(\lambda x, \lambda y) = \frac{(\lambda x)e^{\lambda x/\lambda y} + \lambda y}{\lambda x}$$

$$= \lambda^0 \left(\frac{xe^{\frac{y}{x}} + y}{x}\right) = \lambda^0 f(x, y)$$

Therefore, the given differential equation is homogeneous whose degree is 0.

$$\frac{dy}{dx} = \frac{xe^{\frac{y}{x}} + y}{x}$$

Put $y = vx$ so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$, we get

$$v + x\frac{dv}{dx} = e^v + v$$

$$\Rightarrow \quad x\frac{dv}{dx} = e^v \Rightarrow e^{-v}\,dv = \frac{dv}{dx}$$

Integrating both sides

$$\int e^{-v}dv = \int \frac{dx}{x}$$

$$\Rightarrow \quad -e^{-v} = \log|x| + c$$

$$\Rightarrow \quad -e^{-y/x} = \log|x| + c$$

It is given that at $x = 1, y = 1$

$$-e^{-1} = \log 1 + c \Rightarrow c = \frac{1}{e}$$

∴ Particular solution is

$$-e^{-y/x} = \log|x| - \frac{1}{e}$$

$$\Rightarrow \quad -\frac{y}{x} = \log\left(\frac{1}{e} - \log|x|\right)$$

$$= \log\left(\frac{1 - e\log|x|}{e}\right)$$

$$= \log(1 - e\log|x|) - \log e$$

$$\Rightarrow \quad y = x - x\log(1 - e\log|x|). \quad \textbf{Ans.}$$

Mathematics 2014 (Outside Delhi) SET I

Time allowed : 3 hours **Maximum marks : 100**

SECTION — A

1. If $R = [(x, y) : x + 2y = 8]$ is a relation on N, write the range of R. [1]

Solution : The given relation on N is

$$R = [(x, y) : x + 2y = 8]$$

Since both $x, y \in N$

$\therefore$ x can take values 2, 4, 6 for other values of $y \in N$.

For $x = 2$, $2 + 2y = 8$

$\Rightarrow$ $y = 3$

For $x = 4$, $4 + 2y = 8$

$\Rightarrow$ $y = 2$

For $x = 6$, $6 + 2y = 8$

$\Rightarrow$ $y = 1$

$\therefore$ $R = \{(2, 3), (4, 2), (6, 1)\}$

$\therefore$ The range of R = Set of second element's

$= \{1, 2, 3\}$. **Ans.**

2. If $\tan^{-1} x + \tan^{-1} y = \frac{\pi}{4}, xy < 1$, then write the value of $x + y + xy$. [1]

Solution : Given,

$$\tan^{-1} x + \tan^{-1} y = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1} \frac{x+y}{1-xy} = \frac{\pi}{4}$$

$$\Rightarrow \frac{x+y}{1-xy} = \tan\frac{\pi}{4}$$

$$\Rightarrow \frac{x+y}{1-xy} = 1$$

$$\Rightarrow x + y = 1 - xy$$

$$\therefore x + y + xy = 1.$$ **Ans.**

3. If A is a square matrix such that $A^2 = A$, then write the value of $7A - (I + A)^3$, where I is an identity matrix. [1]

Solution :

$7A - (I + A)^3 = 7A - (I^3 + A^3 + 3I^2A + 3IA^2)$

$[\because (a + b)^3 = a^3 + b^3 + 3a^2b + 3ab^2]$

$= 7A - (I + A^2.A + 3IA + 3IA^2)$

$[\because I^n = I \ \forall n \in N]$

$= 7A - (I + A^2 + 3A + 3IA)$

$[\because A^2 = A, I A = A]$

$= 7A - (I + A + 3A + 3A)$

$= 7A - I - 7A = - I.$ **Ans.**

4. If $\begin{bmatrix} x-y & z \\ 2x-y & w \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 0 & 5 \end{bmatrix}$, find the value of $x + y$. [1]

Solution : Given,

$$\begin{bmatrix} x-y & z \\ 2x-y & w \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 0 & 5 \end{bmatrix}$$

Comparing the corresponding elements, we get

$$x - y = -1, z = 4$$

$$2x - y = 0, w = 5$$

Solving these equations, we get

$$x = 1, y = 2$$

$\therefore$ $x + y = 1 + 2 = 3.$ **Ans.**

5. If $\begin{vmatrix} 3x & 7 \\ -2 & 4 \end{vmatrix} = \begin{vmatrix} 8 & 7 \\ 6 & 4 \end{vmatrix}$, find the value of x. [1]

Solution : Given,

$$\begin{vmatrix} 3x & 7 \\ -2 & 4 \end{vmatrix} = \begin{vmatrix} 8 & 7 \\ 6 & 4 \end{vmatrix}$$

$\Rightarrow$ $3x \times 4 - (-2) \times 7 = 8 \times 4 - 6 \times 7$

$\Rightarrow$ $12x + 14 = 32 - 42$

$\Rightarrow$ $12x = -10 - 14 = -24$

$\therefore$ $x = -2.$ **Ans.**

6. If $f(x) = \int_0^x t \sin t \, dt$, then write the value of $f'(x)$. [1]

Solution : Given, $f(x) = \int_0^x t \sin t \, dt$,

Integrating by parts, we get

$$\int_0^x t \sin t \, dt = t(-\cos t)\Big]_0^x - \int_0^x 1.(-\cos t) dt$$

$$f(x) = [-t \cos t + \sin t]_0^x$$

$$= -x \cos x + \sin x$$

$\therefore$ $f(x) = -x \cos x + \sin x$

Differentiating w.r.t. x, we get

$$f'(x) = -[1.\cos x - x \sin x] + \cos x$$

$$= -\cos x + x \sin x + \cos x$$

$$= x \sin x$$ **Ans.**

7. Evaluate : $\int_2^4 \frac{x}{x^2+1} dx$. [1]

Solution : Let $I = \int_2^4 \frac{x}{x^2+1} dx$

Putting $x^2 + 1 = t$

$\Rightarrow$ $2x \, dx = dt$

$\Rightarrow$ $x \, dx = \frac{1}{2} dt$

Also, $x = 2 \Rightarrow t = 5$

and $x = 4 \Rightarrow t = 17$

$$\therefore \quad I = \frac{1}{2}\int_5^{17}\frac{dt}{t} = \frac{1}{2}[\log t]_5^{17}$$

$$= \frac{1}{2}[\log 17 - \log 5]$$

$$= \frac{1}{2}\log\left(\frac{17}{5}\right). \quad \textbf{Ans.}$$

8. Find the value of 'p' for which the vectors $3\hat{i}+2\hat{j}+9\hat{k}$ and $\hat{i}-2p\hat{j}+3\hat{k}$ are parallel. **[1]**

Solution : Let $\vec{a} = 3\hat{i}+2\hat{j}+9\hat{k}$ ...(i)

and $\vec{b} = \hat{i}-2p\hat{j}+3\hat{k}$ ...(ii)

Since $\vec{a}$ and $\vec{b}$ are parallel,

$$\therefore \quad \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

$$\Rightarrow \quad \frac{3}{1} = \frac{2}{-2p} = \frac{9}{3}$$

$$\Rightarrow \quad \frac{3}{1} = \frac{2}{-2p}$$

$$\Rightarrow \quad 3 = \frac{1}{-p}$$

$$\Rightarrow \quad p = \frac{-1}{3} \quad \textbf{Ans.}$$

9. Find $\vec{a}.(\vec{b}\times\vec{c})$, if $\vec{a} = 2\hat{i}+\hat{j}+3\hat{k}$, $\vec{b} = -\hat{i}+2\hat{j}+\hat{k}$ and $\vec{c} = 3\hat{i}+\hat{j}+2\hat{k}$. **[1]**

Solution : Given,

$$\vec{a} = 2\hat{i}+\hat{j}+3\hat{k}$$

$$\vec{b} = -\hat{i}+2\hat{j}+\hat{k}$$

$$\vec{c} = 3\hat{i}+\hat{j}+2\hat{k}$$

$$(\vec{b}\times\vec{c}) = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -1 & 2 & 1\\ 3 & 1 & 2\end{vmatrix}$$

$$= \hat{i}(4-1)-\hat{j}(-2-3)+\hat{k}(-1-6)$$

$$= 3\hat{i}+5\hat{j}-7\hat{k}$$

$$\therefore \quad \vec{a}\,.\,(\vec{b}\times\vec{c}) = (2\hat{i}+\hat{j}+3\hat{k}).(3\hat{i}+5\hat{j}-7\hat{k})$$

$$= 2\times3+1\times5+3\times(-7)$$

$$= 6+5-21 = -10. \quad \textbf{Ans.}$$

10. If the Cartesian equations of a line are $\frac{3-x}{5} = \frac{y+4}{7} = \frac{2z-6}{4}$, write the vector equation for the line. **[1]**

Solution : The Cartesian equations of a line are

$$\frac{3-x}{5} = \frac{y+4}{7} = \frac{2z-6}{4} \quad \text{...(i)}$$

$$\Rightarrow \quad \frac{x-3}{-5} = \frac{y-(-4)}{7} = \frac{z-3}{2} = \lambda(say)$$

$$\Rightarrow \quad x = 3-5\lambda,$$

$$y = -4+7\lambda,$$

$$z = 3+2\lambda$$

Now, $\vec{a} = 3\hat{i}-4\hat{j}+3\hat{k}$

and $\vec{b} = -5\hat{i}+7\hat{j}+2\hat{k}.$

$\therefore$ The vector equation of the line (i) is

$$\vec{r} = \vec{a}+\lambda\vec{b}$$

$$\Rightarrow \vec{r} = (3\hat{i}-4\hat{j}+3\hat{k})+\lambda(-5\hat{i}+7\hat{j}+2\hat{k}). \quad \textbf{Ans.}$$

SECTION — B

11. If the function $f: R \to R$ be given by $f(x) = x^2+2$ and $g: R \to R$ be given by $g(x) = \frac{x}{x-1}, x \neq 1$, find fog and gof and hence find fog (2) and $gof(-3)$. **[4]**

Solution : Given, $f: R \to R$

such that, $f(x) = x^2+2$ ...(i)

and $g: R \to R$

such that, $g(x) = \frac{x}{x-1}, x \neq 1$ ...(ii)

Now, $fog: R \to R$

such that, $(fog)(x) = f(g(x)) = f\left(\frac{x}{x-1}\right)$

$$= \left(\frac{x}{x-1}\right)^2+2 \; (x\neq1)$$

$$\therefore \quad (fog)(2) = \left(\frac{2}{2-1}\right)^2+2 = 6$$

Also, $gof: R \to R$

Such that, $(gof)(x) = g(f(x))$

$$= g(x^2+2) = \frac{x^2+2}{x^2+2-1}$$

$$= \frac{x^2+2}{x^2+1}$$

$$\therefore \quad (gof)(-3) = \frac{(-3)^2+2}{(-3)^2+1} = \frac{11}{10}. \quad \textbf{Ans.}$$

12. Prove that :

$$\tan^{-1}\left[\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right] = \frac{\pi}{4}-\frac{1}{2}\cos^{-1}x, \frac{-1}{\sqrt{2}} \leq x \leq 1.$$ **[4]**

Solution : L. H. S.

$$= \tan^{-1}\left[\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right]\left(\frac{-1}{\sqrt{2}}\leq x\leq 1\right)$$

Putting

$x = \cos\theta$, we get

$$= \tan^{-1}\left[\frac{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}\right],$$

$$= \tan^{-1}\left[\frac{\sqrt{2}\cos\frac{\theta}{2}-\sqrt{2}\sin\frac{\theta}{2}}{\sqrt{2}\cos\frac{\theta}{2}+\sqrt{2}\sin\frac{\theta}{2}}\right]\left(0\leq x\leq\frac{3\pi}{4}\right)$$

$$\left(\because \cos\theta = 2\cos^2\frac{\theta}{2}-1 = 1-2\sin^2\frac{\theta}{2}\right)$$

$$= \tan^{-1}\left[\frac{\sqrt{2}\cos\frac{\theta}{2}\left(1-\tan\frac{\theta}{2}\right)}{\sqrt{2}\cos\frac{\theta}{2}\left(1+\tan\frac{\theta}{2}\right)}\right]$$

$$= \tan^{-1}\left[\frac{\tan\frac{\pi}{4}-\tan\frac{\theta}{2}}{1+\tan\frac{\pi}{4}\tan\frac{\theta}{2}}\right]$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right] = \frac{\pi}{4}-\frac{1}{2}\theta$$

$$= \frac{\pi}{4}-\frac{1}{2}\cos^{-1}x = \text{R.H.S.}$$ **Hence Proved.**

OR

If $\tan^{-1}\left(\frac{x-2}{x-4}\right)+\tan^{-1}\left(\frac{x+2}{x+4}\right)=\frac{\pi}{4}$, **find the value of** x.

Solution : Given,

$$\tan^{-1}\left(\frac{x-2}{x-4}\right)+\tan^{-1}\left(\frac{x+2}{x+4}\right) = \frac{\pi}{4}$$

$$\Rightarrow \quad \tan^{-1}\left[\frac{\frac{x-2}{x-4}+\frac{x+2}{x+4}}{1-\frac{x-2}{x-4}\cdot\frac{x+2}{x+4}}\right] = \frac{\pi}{4}$$

$$\left(\because \tan^{-1}x+\tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}\right)$$

$$\Rightarrow \quad \frac{(x-2)(x+4)+(x+2)(x-4)}{(x^2-16)-(x^2-4)} = \tan\frac{\pi}{4}$$

$$\Rightarrow \quad \frac{x^2+2x-8+x^2-2x-8}{-12} = 1$$

$$\Rightarrow \quad 2x^2-16 = -12$$

$$\Rightarrow \quad 2x^2 = 4$$

$$\Rightarrow \quad x^2 = 2$$

$$\therefore \quad x = \pm\sqrt{2}.$$ **Ans.**

13. Using properties of determinants, prove that:

$$\begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix} = x^3$$ **[4]**

Solution : Taking L.H.S.

Let $$\Delta = \begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix}$$

Applying $R_3 \to R_3 - 2R_2$, we get

$$\Delta = \begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 0 & 0 & -x \end{vmatrix}$$

Taking x common from C_2 and C_3, we get

$$\Delta = x^2\begin{vmatrix} x+y & 1 & 1 \\ 5x+4y & 4 & 2 \\ 0 & 0 & -1 \end{vmatrix}$$

Expanding along R_3, we get

$$\Delta = x^2(-1)\begin{vmatrix} x+y & 1 \\ 5x+4y & 4 \end{vmatrix}$$

$$= -x^2[4(x+y)-(5x+4y)]$$

$$= -x^2(4x+4y-5x-4y)$$

$$= -x^2.(-x) = x^3 = \text{R. H. S.}$$

Hence Proved.

14. Find the value of $\frac{dy}{dx}$ **at** $\theta = \frac{\pi}{4}$ **if** $x = ae^{\theta}$ **(sin θ – cos θ) and** $y = ae^{\theta}$ **(sin θ + cos θ).** **[4]**

Solution : Given,

$$x = ae^{\theta}(\sin\theta-\cos\theta) \quad \text{...(i)}$$

Differentiating w. r. t. θ, we get

$$\frac{dx}{d\theta} = a[e^{\theta}(\sin\theta-\cos\theta)+e^{\theta}(\cos\theta+\sin\theta)]$$

$$= 2ae^{\theta}\sin\theta$$

and $$y = ae^{\theta}(\sin\theta+\cos\theta) \quad \text{...(ii)}$$

Differentiating w.r.t. θ, we get

$$\frac{dy}{d\theta} = a[e^{\theta}(\sin\theta+\cos\theta)+e^{\theta}(\cos\theta-\sin\theta)]$$

$$= 2ae^{\theta}\cos\theta$$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{2ae^{\theta}\cos\theta}{2ae^{\theta}\sin\theta}$$

$\therefore \quad \frac{dy}{dx} = \cot\theta$

At $\theta = \frac{\pi}{4}; \frac{dy}{dx} = \cot\frac{\pi}{4} = 1.$ **Ans.**

15. If $y = Pe^{ax} + Qe^{bx}$, Show that

$$\frac{d^2y}{dx^2} - (a+b)\frac{dy}{dx} + aby = 0. \qquad \textbf{[4]}$$

Solution : Given,

$$y = Pe^{ax} + Qe^{bx} \quad \text{...(i)}$$

Differentiating w. r. t. x, we get

$$\frac{dy}{dx} = P.e^{ax}\,a + Q.e^{bx}.b \quad \text{...(ii)}$$

Again differentiating, we get

$$\frac{d^2y}{dx^2} = aPe^{ax}.\,a + bQe^{bx}.b$$
$$= a^2Pe^{ax} + b^2Qe^{bx}$$

$$\therefore \text{ L. H. S.} = \frac{d^2y}{dx^2} - (a+b)\frac{dy}{dx} + aby$$
$$= a^2\,Pe^{ax} + b^2\,Q\,e^{bx} - (a+b)\,(a\,Pe^{ax} + b\,Q\,e^{bx}) + ab\,(Pe^{ax} + Q\,e^{bx}) \quad \text{...(iii)}$$

Using (i), (ii) and (iii),

$$= e^{ax}\,[a^2\,P - (a+b)a\,P + ab\,P] + e^{bx}\,[b^2Q - (a+b)\,bQ + ab\,Q]$$
$$= e^{ax}.0 + e^{bx}.0$$
$$= 0 = \text{R. H. S.}$$ **Hence Proved.**

16. Find the value (s) of x for which $y = [x(x-2)]^2$ is an increasing function. [4]

Solution : Given

$$y = [x(x-2)]^2$$
$$\Rightarrow \quad y = x^2\,(x-2)^2$$
$$= f(x) \text{ (Let)}$$

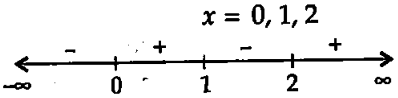
Differentiating w.r.t. x, we get

$$f'(x) = \frac{dy}{dx}$$
$$= 2x(x-2)^2 + 2x^2\,(x-2)$$
$$= 2x(x-2)\,(x-2+x_1)$$
$$= 4x(x-1)\,(x-2).$$

For y to be an increasing function, $\frac{dy}{dx} > 0$

$\Rightarrow \quad x(x-1)\,(x-2) > 0$

$\Rightarrow \quad x = 0, 1, 2$



Interval	Test Value	Sign of $f'(x)$ $f'(x) = 4x(x-1)(x-2)$	Nature of function $f(x) = y$
$(-\infty, 0)$	$x = -1$	$(-)(-)(-) = - < 0$	Strictly decreasing
$(0, 1)$	$x = 0.5$	$(+)(-)(-) = + > 0$	Strictly increasing
$(1, 2)$	$x = 1.5$	$(+)(+)(-) = - < 0$	Strictly decreasing
$(2, \infty)$	$x = 3$	$(+)(+)(+) = + > 0$	Strictly increasing

$\therefore$ y is an increasing function in $[0, 1] \cup [2, \infty)$ **Ans.**

OR

Find the equations of the tangent and normal to the curve $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $(\sqrt{2}a, b)$.

Solution : The given curve is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Differentiating w.r. t. x, we get

$$\frac{2x}{a^2} - \frac{2y}{b^2}.\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\frac{x}{a^2}}{\frac{y}{b^2}} = \frac{b^2}{a^2}.\frac{x}{y}$$

At $P(\sqrt{2}a, b)$

Slope of the tangent,

$$m_1 = \frac{b^2}{a^2}.\frac{\sqrt{2}a}{b} = \sqrt{2}.\frac{b}{a}$$

and slope of the normal,

$$m_2 = -\frac{a}{b\sqrt{2}}$$

$\therefore$ Equation of the tangent at P is

$$y - b = \sqrt{2}\frac{b}{a}(x - \sqrt{2}a)$$
$$\Rightarrow \quad y = \sqrt{2}\frac{b}{a}x - 2b + b$$
$$\Rightarrow \quad y = \sqrt{2}\frac{b}{a}x - b$$
$$\Rightarrow \quad \sqrt{2}bx - ay - ab = 0$$

Also equation of the normal at P is

$$y - b = \frac{-a}{\sqrt{2}b}(x - \sqrt{2}a)$$

$$\Rightarrow \quad y = \frac{-ax}{\sqrt{2}.b} + b + \frac{a^2}{b}.$$

$$\Rightarrow \quad \sqrt{2}\,by - \sqrt{2}b^2 = -ax + \sqrt{2}a^2$$

$$\Rightarrow \quad ax + \sqrt{2}\,by - \sqrt{2}(a^2 + b^2) = 0$$ **Ans.**

17. Evaluate : $\int_0^{\pi} \frac{4x\sin x}{1+\cos^2 x}dx.$ **[4]**

Solution : Let, $I = \int_0^{\pi} \frac{4x\sin x}{1+\cos^2 x}dx$ ...(i)

$$\Rightarrow \quad I = \int_0^{\pi} \frac{4(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)}dx$$

$$\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$\therefore \quad I = \int_0^{\pi} \frac{4(\pi - x)\sin x}{1+\cos^2 x}dx \quad ...(ii)$$

Adding (i) and (ii), we get

$$2I = 4\pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}dx$$

$$\Rightarrow \quad I = 2\pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}dx$$

Putting, $\cos x = t$

$\Rightarrow \quad \sin x\, dx = -dt$

Also, $x = 0 \Rightarrow t = 1$

and $x = \pi \Rightarrow t = -1$

$$\therefore \quad I = 2\pi\int_1^{-1} \frac{-dt}{1+t^2}$$

$$= 2\pi\int_{-1}^{1} \frac{dt}{t^2+1}$$

$$\left[\because \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx\right]$$

$$= 2\pi\left[\tan^{-1} t\right]_{-1}^{1}$$

$$= 2\pi[\tan^{-1}(1) - \tan^{-1}(-1)]$$

$$= 2\pi\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right]$$

$$= 2\pi\left[\frac{2\pi}{4}\right] = \pi^2$$ **Ans.**

OR

Evaluate : $\int \frac{x+2}{\sqrt{x^2+5x+6}}dx.$

Solution : Let, $x + 2 = A\frac{d}{dx}(x^2+5x+6) + B$

$$\Rightarrow \quad x + 2 = A(2x+5) + B \quad ...(i)$$

$$\Rightarrow \quad x + 2 = 2Ax + 5A + B$$

On equating the coefficients of x and constant term on both sides, we get

$$2A = 1$$

$$\Rightarrow \quad A = \frac{1}{2}$$

and $5A + B = 2$

$$\Rightarrow \quad \frac{5}{2} + B = 2$$

$$\Rightarrow \quad B = 2 - \frac{5}{2} = -\frac{1}{2}$$

$\therefore$ Equation (i) becomes,

$$x + 2 = \frac{1}{2}(2x+5) - \frac{1}{2}$$

$$\therefore \quad \int \frac{x+2}{\sqrt{x^2+5x+6}}dx = \int \frac{\frac{1}{2}(2x+5) - \frac{1}{2}}{\sqrt{x^2+5x+6}}dx$$

$$= \frac{1}{2}\int \frac{2x+5}{\sqrt{x^2+5x+6}}dx - \frac{1}{2}\int \frac{dx}{\sqrt{x^2+5x+6}}$$

Putting, $x^2 + 5x + 6 = t$

$\Rightarrow \quad (2x+5)\,dx = dt$

$$= \frac{1}{2}\int t^{\frac{-1}{2}}dt - \frac{1}{2}\int \frac{dx}{\sqrt{\left(x+\frac{5}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} + C$$

$$= \frac{1}{2}\frac{t^{\frac{1}{2}}}{\frac{1}{2}} - \frac{1}{2}\log\left|\left(x+\frac{5}{2}\right) + \sqrt{\left(x+\frac{5}{2}\right)^2 - \left(\frac{1}{2}\right)^2}\right| + C$$

$$\left[\because \int \frac{dx}{\sqrt{x^2-a^2}} = \log\left|x + \sqrt{x^2-a^2}\right|\right]$$

$$= \sqrt{x^2+5x+6} - \frac{1}{2}\log\left|\left(x+\frac{5}{2}\right) + \sqrt{x^2+5x+6}\right| + C$$ **Ans.**

18. Find the particular solution of the differential equation $\frac{dy}{dx} = 1 + x + y + xy$**, given that** $y = 0$ **when** $x = 1$. **[4]**

Solution : The given differential equation is

$$\frac{dy}{dx} = 1 + x + y + xy$$

$$\Rightarrow \quad \frac{dy}{dx} = (1+x) + (1+x)y$$

$$= (1+x)(1+y)$$

$$\Rightarrow \quad \frac{dy}{1+y} = (1+x)dx$$

Integrating both sides, we get

$$\int \frac{dy}{1+y} = \int (1+x)dx + C$$

$$\Rightarrow \quad \log(1+y) = x + \frac{x^2}{2} + C$$

Putting, $y = 0$,

when $x = 1$, we get

$$\log 1 = 1 + \frac{1}{2} + C$$

$$\Rightarrow \quad 0 = \frac{3}{2} + C$$

$$\Rightarrow \quad C = -\frac{3}{2}$$

$\therefore$ The particular solution of the given differential equation is

$$\log(1+y) = x + \frac{x^2}{2} - \frac{3}{2}. \quad \textbf{Ans.}$$

19. Solve the differential equation $(1+x^2)\frac{dy}{dx} + y = e^{\tan^{-1}x}$. [4]

Solution : The given differential equation is

$$(1+x^2)\frac{dy}{dx} + y = e^{\tan^{-1}x} \quad \text{...(i)}$$

Rewriting the given differential equation, we get

$$\frac{dy}{dx} + \frac{1}{1+x^2}\cdot y = \frac{e^{\tan^{-1}x}}{1+x^2},$$

which is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

Here, $P = \frac{1}{1+x^2}$

and $Q = \frac{e^{\tan^{-1}x}}{1+x^2}$

$$\therefore \quad \int P\,dx = \int \frac{dx}{1+x^2} = \tan^{-1}x$$

Now, $\text{I. F.} = e^{\int P\,dx} = e^{\tan^{-1}x}$

Hence the solution is

$$y(\text{I.F.}) = \int Q.(\text{I.F.})dx + C$$

$$y.e^{\tan^{-1}x} = \int \frac{e^{\tan^{-1}x}}{1+x^2}.e^{\tan^{-1}x}dx + C$$

$$= \int \frac{\left(e^{\tan^{-1}x}\right)^2 dx}{1+x^2} + C$$

Putting, $\tan^{-1}x = t$

$$\Rightarrow \quad \frac{1}{1+x^2}dx = dt$$

$$\Rightarrow \quad ye^{\tan^{-1}x} = \int e^{2t}dt + C$$

$$= \frac{1}{2}e^{2t} + C$$

$$y.e^{\tan^{-1}x} = \frac{1}{2}e^{2\tan^{-1}x} + C.$$

$$\therefore \quad y = \frac{1}{2}e^{\tan^{-1}x} + Ce^{-\tan^{-1}x} \quad \textbf{Ans.}$$

20. Show that the four points A, B, C and D with position vectors $4\hat{i}+5\hat{j}+\hat{k}$, $-\hat{j}-\hat{k}$, $3\hat{i}+9\hat{j}+4\hat{k}$ and $4(-\hat{i}+\hat{j}+\hat{k})$ respectively are coplanar. [4]

Solution : Given the position vectors are

$A(4\hat{i}+5\hat{j}+\hat{k})$; $B(-\hat{j}-\hat{k})$; $C(3\hat{i}+9\hat{j}+4\hat{k})$;

and $D\left[4(-\hat{i}+\hat{j}+\hat{k})\right]$.

These points will be coplanar if

$$\left[\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}\right] = 0$$

Now, $\overrightarrow{AB} = (-\hat{j}-\hat{k}) - (4\hat{i}+5\hat{j}+\hat{k})$

$= -4\hat{i} - 6\hat{j} - 2\hat{k}$

$\overrightarrow{AC} = (3\hat{i}+9\hat{j}+4\hat{k}) - (4\hat{i}+5\hat{j}+\hat{k})$

$= -\hat{i} + 4\hat{j} + 3\hat{k}$

$\overrightarrow{AD} = (-4\hat{i}+4\hat{j}+4\hat{k}) - (4\hat{i}+5\hat{j}+\hat{k})$

$= -8\hat{i} - \hat{j} + 3\hat{k}.$

$$\therefore \left[\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}\right] = \begin{vmatrix} -4 & -6 & -2 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix}$$

$= -4(12+3) + 6(-3+24) - 2(1+32)$

$= -60 + 126 - 66$

$= -126 + 126 = 0$

$\therefore$ The given points A, B, C and D are coplanar.

Hence Proved

OR

The scalar product of the vector $\vec{a} = \hat{i}+\hat{j}+\hat{k}$ with a unit vector along the sum of vectors $\vec{b} = 2\hat{i}+4\hat{j}-5\hat{k}$ and $\vec{c} = \lambda\hat{i}+2\hat{j}+3\hat{k}$ is equal to one. Find the value of λ and hence find the unit vector along $\vec{b}+\vec{c}$.

Solution : Given, $\vec{a} = \hat{i}+\hat{j}+\hat{k}$;

$\vec{b} = 2\hat{i}+4\hat{j}-5\hat{k}$

and $\vec{c} = \lambda\hat{i}+2\hat{j}+3\hat{k}$

$\Rightarrow \quad \vec{b}+\vec{c} = (2+\lambda)\hat{i}+6\hat{j}-2\hat{k}$

The unit vector along $\vec{b}+\vec{c}$ is

$$\vec{p} = \frac{\vec{b}+\vec{c}}{|\vec{b}+\vec{c}|}$$

$$\Rightarrow \quad \vec{p} = \frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{(2+\lambda)^2+6^2+(-2)^2}}$$

$$\Rightarrow \quad = \frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{\lambda^2+4\lambda+44}} \quad ...(i)$$

Given, $\quad \vec{a}.\vec{p} = 1$

$$\Rightarrow \quad \frac{(\hat{i}+\hat{j}+\hat{k}).[(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}]}{\sqrt{\lambda^2+4\lambda+44}} = 1$$

$$\Rightarrow \quad \frac{(2+\lambda)+6-2}{\sqrt{\lambda^2+4\lambda+44}} = 1$$

$$\Rightarrow \quad \sqrt{\lambda^2+4\lambda+44} = \lambda+6$$

Squaring on both sides, we get

$$\lambda^2+4\lambda+44 = \lambda^2+12\lambda+36$$

$$\Rightarrow \quad 8\lambda = 8$$

$$\Rightarrow \quad \lambda = 1 \quad ...(ii)$$

and the required unit vector is

$$\vec{p} = \frac{(2+1)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{1+4+44}}$$

[Using (i) and (ii)]

$$= \frac{1}{7}(3\hat{i}+6\hat{j}-2\hat{k}). \quad \textbf{Ans.}$$

21. A line passes through (2, –1, 3) and is perpendicular to the lines $\vec{r} = (\hat{i}+\hat{j}-\hat{k}) + \lambda(2\hat{i}-2\hat{j}+\hat{k})$ and $\vec{r} = (2\hat{i}-\hat{j}-3\hat{k}) + \mu(\hat{i}+2\hat{j}+2\hat{k})$. Obtain its equation in vector and Cartesian form. **[4]**

Solution : The given lines are

$$\vec{r} = (\hat{i}+\hat{j}-\hat{k})+\lambda(2\hat{i}-2\hat{j}+\hat{k}) \quad ...(i)$$

$$\vec{r} = (2\hat{i}-\hat{j}-3\hat{k})+\mu(\hat{i}+2\hat{j}+2\hat{k}) \quad ...(ii)$$

Equation of any line passes through (2, –1, 3) with direction cosines l, m, n is

$$\vec{r} = (2\hat{i}-\hat{j}+3\hat{k})+\lambda(l\hat{i}+m\hat{j}+n\hat{k}) \quad ...(iii)$$

Now line (i) and (ii) are perpendicular to (iii), we get

$$2l-2m+n = 0 \quad ...(iv)$$

$$l+2m+2n = 0 \quad ...(v)$$

Solving equations (iv) and (v), we get

$$\frac{l}{-4-2} = \frac{m}{1-4} = \frac{n}{4+2}$$

$$\Rightarrow \quad \frac{l}{2} = \frac{m}{1} = \frac{n}{-2}$$

∴ From (iii), the required line in vector form is

$$\vec{r} = (2\hat{i}-\hat{j}+3\hat{k})+\lambda(2\hat{i}+\hat{j}-2\hat{k})$$

Also cartesian Equation is

$$\frac{x-2}{2} = \frac{y+1}{1} = \frac{z-3}{-2} \quad \textbf{Ans.}$$

22. An experiment succeeds thrice as often as it fails. Find the probability that in the next five trials, there will be at least 3 successes. **[4]**

Solution : Let, p = Probability of success

$$= \frac{3}{3+1} = \frac{3}{4}$$

q = Probability of failure

$$= 1-\frac{3}{4} = \frac{1}{4}$$

Here, $\quad n = 5$

∴ Probability of at least 3 successes

$$= P(X \geq 3)$$

$$= P(X=3)+P(X=4)+P(X=5)$$

$$= {}^5C_3p^3q^2 + {}^5C_4p^4q^1 + {}^5C_5p^5q^0$$

$$= 10\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^2 + 5\left(\frac{3}{4}\right)^4\left(\frac{1}{4}\right) + \left(\frac{3}{4}\right)^5\left(\frac{1}{4}\right)^0$$

$$= \frac{270+405+243}{4^5}$$

$$= \frac{918}{1024} = \frac{459}{512}. \quad \textbf{Ans.}$$

SECTION — C

23. Two schools A and B want to award their selected students on the values of sincerity, truthfulness and helpfulness. The school A wants to award ₹ x each, ₹ y each and ₹ z each for the three respective values to 3, 2 and 1 students respectively with a total award money of ₹ 1,600. School B wants to spend ₹ 2,300 to award its 4, 1 and 3 students on the

respective values (by giving the same award money to the three values as before). If the total amount of award for one prize on each value is ₹ 900, using matrices, find the award money for each value. Apart from these three values, suggest one more value which should be considered for award. [6]

Solution : Given the awards for sincerity, truthfulness and helpfulness are ₹ x, ₹ y and ₹ z respectively.

$$\therefore \quad 3x+2y+z=1{,}600$$
$$4x+y+3z=2{,}300$$
$$x+y+z=900$$

The given equation can be written in matrix form,

$$AX = B \quad \text{...(i)}$$

$$\begin{bmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1{,}600 \\ 2{,}300 \\ 900 \end{bmatrix}$$

Here, $A = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$B = \begin{bmatrix} 1{,}600 \\ 2{,}300 \\ 900 \end{bmatrix}$$

Now, $|A| = \begin{vmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{vmatrix}$

$$= 3(1-3)-2(4-3)+1(4-1)$$
$$= -6-2+3$$
$$= -5 \neq 0$$

$\Rightarrow A^{-1}$ exists.

For adj A,

$A_{11} = (1-3) = -2$, $A_{12} = -(4-3) = -1$; $A_{13} = (4-1) = 3$

$A_{21} = -(2-1) = -1$, $A_{22} = (3-1) = 2$, $A_{23} = -(3-2) = -1$

$A_{31} = (6-1) = 5$, $A_{32} = -(9-4) = -5$, $A_{33} = (3-8) = -5$

$$\therefore \quad \text{adj } A = \begin{bmatrix} -2 & -1 & 3 \\ -1 & 2 & -1 \\ 5 & -5 & -5 \end{bmatrix}^T = \begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{|A|}.\text{adj } A$$

$$= -\frac{1}{5}\begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}$$

From (i), $X = A^{-1}B$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}\begin{bmatrix} 1{,}600 \\ 2{,}300 \\ 900 \end{bmatrix}$$

$$= -\frac{1}{5}\begin{bmatrix} -1{,}000 \\ -1{,}500 \\ -2{,}000 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 200 \\ 300 \\ 400 \end{bmatrix}$$

$\Rightarrow$ $x = ₹\,200$,
$y = ₹\,300$
and $z = ₹\,400$

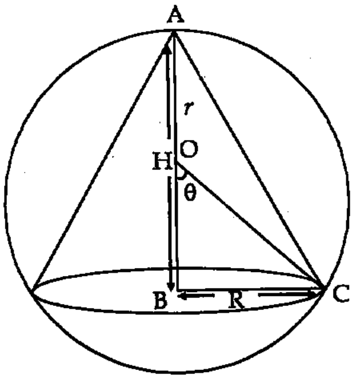
Apart from the three values, sincerity, truthfulness and helpfulness, another value for award should be discipline. **Ans.**

24. Show that the altitude of the right circular cone of maximum volume that can be described in a sphere of radius r is $\frac{4r}{3}$. Also show that the maximum volume of the cone is $\frac{8}{27}$ of the volume of the sphere. [6]

Solution : From the figure OA = OC = r (Radius of the sphere)

From right angled Δ OBC,

$$BC = r\sin\theta,$$
$$OB = r\cos\theta$$



V = Volume of the inscribed cone

$$= \frac{1}{3}\pi R^2 H = \frac{1}{3}\pi (BC)^2.AB$$

$$= \frac{1}{3}\pi r^2 \sin^2\theta(r + r\cos\theta) \quad \text{...(i)}$$

[∴ AB = AO + OB]

$$\Rightarrow \quad V = \frac{\pi}{3}r^3(\sin^2\theta + \sin^2\theta\cos\theta)$$

Differentiating w.r.t. θ, we get

$$\frac{dV}{d\theta} = \frac{\pi r^3}{3}[2\sin\theta\cos\theta + 2\sin\theta\cos^2\theta - \sin^3\theta]$$

For maxima or minima,

$$\frac{dV}{d\theta} = 0$$

$$\Rightarrow \sin\theta[2\cos\theta + 2\cos^2\theta - \sin^2\theta] = 0$$

$$\Rightarrow 2\cos\theta + 2\cos^2\theta - (1 - \cos^2\theta) = 0$$

$[\because \sin\theta \neq 0]$

$$\Rightarrow 3\cos^2\theta + 2\cos\theta - 1 = 0$$

$$\Rightarrow 3\cos^2\theta + 3\cos\theta - \cos\theta - 1 = 0$$

$$\Rightarrow 3\cos\theta(\cos\theta + 1) - 1(\cos\theta + 1) = 0$$

$$\Rightarrow (3\cos\theta - 1).(\cos\theta + 1) = 0$$

$$\Rightarrow 3\cos\theta - 1 = 0$$

$[\because \cos\theta \neq 0]$

$$\Rightarrow \cos\theta = \frac{1}{3}$$

$$\therefore H = r + r\cos\theta$$

$$= r + r.\frac{1}{3} = \frac{4r}{3}$$

Also V changes sign from + ve to – ve for this value of θ

⇒ V is maximum.

∴ Maximum volume of the inscribed cone,

$$V = \frac{\pi}{3}r^3(1-\cos^2\theta)(1+\cos\theta) \text{ [From (i)]}$$

$$= \frac{\pi}{3}r^3\left(1-\frac{1}{9}\right)\left(1+\frac{1}{3}\right)$$

$$= \frac{\pi}{3}r^3\frac{32}{27} = \frac{8}{27}\left(\frac{4}{3}\pi r^3\right)$$

$$= \frac{8}{27} \text{ Volume of the sphere.}$$

Hence Proved.

25. Evaluate : $\int \frac{1}{\cos^4 x + \sin^4 x}dx.$ **[6]**

Solution : Let,

$$I = \int \frac{1}{\cos^4 x + \sin^4 x}dx$$

$$= \int \frac{\sec^4 x}{1 + \tan^4 x}dx$$

$$= \int \frac{\sec^2 x.\sec^2 x}{1 + \tan^4 x}dx$$

$$= \int \frac{(\tan^2 + 1)\sec^2 x}{1 + \tan^4 x}dx$$

$(\because \sec^2 x = 1 + \tan^2 x)$

Putting $\tan x = t$

$$\Rightarrow \sec^2 x\, dx = dt$$

$$\therefore I = \int \frac{t^2+1}{t^4+1}dt = \int \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}}dt$$

$$= \int \frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+2}dt$$

Putting, $t - \frac{1}{t} = y$

$$\Rightarrow \left(1+\frac{1}{t^2}\right)dt = dy$$

$$\therefore I = \int \frac{dy}{y^2 + (\sqrt{2})^2}$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{y}{\sqrt{2}}\right) + C$$

$$\left(\because \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C\right)$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan x - \frac{1}{\tan x}}{\sqrt{2}}\right) + C$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan x - \cot x}{\sqrt{2}}\right) + C$$

Ans.

26. Using integration, find the area of the region bounded by the triangle whose vertices are (– 1, 2), (1, 5) and(3, 4). **[6]**

Solution : Let A (– 1, 2); B (1, 5) and C (3, 4)

Equation of AB is

$$y - 5 = \frac{5-2}{1+1}(x-1)$$

$$\Rightarrow \quad y = \frac{3}{2}x + \frac{7}{2} \quad ...(i)$$

Equation of BC is

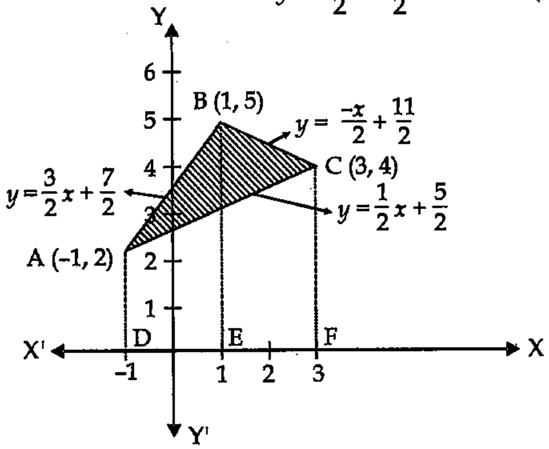
$$y - 4 = \frac{4-5}{3-1}(x-3)$$

$$\Rightarrow \quad y = -\frac{1}{2}x + \frac{11}{2} \quad ...(ii)$$

Equation of AC is

$$y - 2 = \frac{4-2}{3+1}(x+1)$$

$\Rightarrow \qquad y = \frac{1}{2}x + \frac{5}{2}$ ...(iii)



Area of the required triangular region, ABC

= Area of trapezium ADEB + Area of trapezium BEFC – Area of trapezium ADFC

$= \int_{-1}^{1} y_{AB}\,dx + \int_{1}^{3} y_{BC}\,dx - \int_{-1}^{3} y_{AC}\,dx$

$= \int_{-1}^{1}\left(\frac{3}{2}x+\frac{7}{2}\right)dx + \int_{1}^{3}\left(\frac{-1}{2}x+\frac{11}{2}\right)dx - \int_{-1}^{3}\left(\frac{1}{2}x+\frac{5}{2}\right)dx$

$= \left[\frac{3}{4}x^2+\frac{7}{2}x\right]_{-1}^{1} + \left[-\frac{x^2}{4}+\frac{11}{2}x\right]_{1}^{3} - \left[\frac{x^2}{4}+\frac{5}{2}x\right]_{-1}^{3}$

$= \left(\frac{3}{4}+\frac{7}{2}\right)-\left(\frac{3}{4}-\frac{7}{2}\right)+\left(-\frac{9}{4}+\frac{33}{2}\right)-\left(\frac{-1}{4}+\frac{11}{2}\right) - \left(\frac{9}{4}+\frac{15}{2}\right)+\left(\frac{1}{4}-\frac{5}{2}\right)$

$= \frac{3}{4}+\frac{7}{2}-\frac{3}{4}+\frac{7}{2}-\frac{9}{4}+\frac{33}{2}+\frac{1}{4}-\frac{11}{2}-\frac{9}{4}-\frac{15}{2}+\frac{1}{4}-\frac{5}{2}$

$= 7 - 4 + 1 = 4$ sq. units **Ans.**

27. Find the equation of the plane through the line of intersection of the planes $x + y + z = 1$ and $2x + 3y + 4z = 5$ which is perpendicular to the plane $x - y + z = 0$. Also find the distance of the plane obtained above, from the origin. [6]

Solution : Equation of any plane through the line of intersection of the planes.

$$x + y + z - 1 = 0$$

and $$2x + 3y + 4z - 5 = 0$$

$$x + y + z - 1 + \lambda(2x + 3y + 4z - 5) = 0$$

$$\Rightarrow (1 + 2\lambda)x + (1 + 3\lambda)y + (1 + 4\lambda)z - (1 + 5\lambda) = 0 \quad ...(i)$$

This plane is perpendicular to the plane

$$x - y + z = 0$$

$\therefore \quad (1 + 2\lambda) . 1 + (1 + 3\lambda)(-1) + (1 + 4\lambda) . 1 = 0$

$\Rightarrow \qquad 1 + 2\lambda - 1 - 3\lambda + 1 + 4\lambda = 0$

$\Rightarrow \qquad 1 + 3\lambda = 0$

$\Rightarrow \qquad \lambda = -\frac{1}{3}$

Substituting $\lambda = -\frac{1}{3}$ in (i), we get

$$\left(1-\frac{2}{3}\right)x + 0.y + \left(1-\frac{4}{3}\right)z - \left(1-\frac{5}{3}\right) = 0$$

$\Rightarrow \qquad x - z + 2 = 0$ ...(ii)

This is the equation of the required plane.

Distance of plane (2) from origin (0, 0, 0)

= Length of $\perp$ from (0, 0, 0) on plane

$$= \frac{0-0+2}{\sqrt{1^2+(-1)^2}}$$

$$= \frac{2}{\sqrt{2}} = \sqrt{2} \text{ units.}$$ **Ans.**

OR

Find the distance of the point (2, 12, 5) from the point of intersection of the line $\vec{r} = 2\hat{i} - 4\hat{j} + 2\hat{k} + \lambda\,(3\hat{i} + 4\hat{j} + 2\hat{k})$ and the plane $\vec{r}.(\hat{i} - 2\hat{j} + \hat{k}) = 0$.

Solution : The given line is

$$\vec{r} = 2\hat{i} - 4\hat{j} + 2\hat{k} + \lambda\,(3\hat{i} + 4\hat{j} + 2\hat{k}) \quad ...(i)$$

Writing equation in cartesian form

$$\frac{x-2}{3} = \frac{y+4}{4} = \frac{z-2}{2} = \lambda \text{ (say)}$$

$\therefore$ Point on line is $(3\lambda + 2, 4\lambda - 4, 2\lambda + 2)$

This lies on the plane

$$\vec{r}.(\hat{i} - 2\hat{j} + \hat{k}) = 0$$

$\Rightarrow \qquad (x\hat{i} + y\hat{j} + z\hat{k}).(\hat{i} - 2\hat{j} + \hat{k}) = 0$

$\Rightarrow \qquad x - 2y + z = 0$ ...(ii)

$\therefore \qquad 3\lambda + 2 - 2(4\lambda - 4) + 2\lambda + 2 = 0$

$\Rightarrow \qquad -3\lambda + 12 = 0$

$\Rightarrow \qquad \lambda = 4$

$\therefore$ The point of intersection of (i) and (ii) is

$$(3 \times 4 + 2, 4 \times 4 - 4, 2 \times 4 + 2) = (14, 12, 10).$$

Distance of the point (2, 12, 5) from the point (14, 12, 10)

$$= \sqrt{(14-2)^2 + (12-12)^2 + (10-5)^2}$$

$$= \sqrt{144+0+25} = \sqrt{169} = 13 \text{ units}$$ **Ans.**

28. A manufacturing company makes two types of teaching aids A and B of Mathematics for class

XII. Each type of A requires 9 labour hours of fabricating and 1 labour hour for finishing. Each type of B requires 12 labour hours for fabricating and 3 labour hours for finishing. For fabricating and finishing, the maximum labour hours available per week are 180 and 30 respectively. The company makes a profit of ` 80 on each piece of type A and ₹ 120 on each piece of type B. How many pieces of type A and type B should be manufactured per week to get a maximum profit ? Make it as an LPP and solve graphically. What is the maximum profit per week ? **[6]**

Solution : Let x and y be the number of teaching aids of type A and B respectively. Then the LPP is

Maximize $Z = 80x + 120y$

Subject to constraints :

$$9x + 12y \le 180$$
$$x + 3y \le 30$$

and $x \ge 0,\ y \ge 0,$

First we draw the lines AB and CD whose equations are

$$9x + 12y = 180$$
$$\Rightarrow \quad 3x + 4y = 60 \quad ...(i)$$

x	20	0
y	0	15

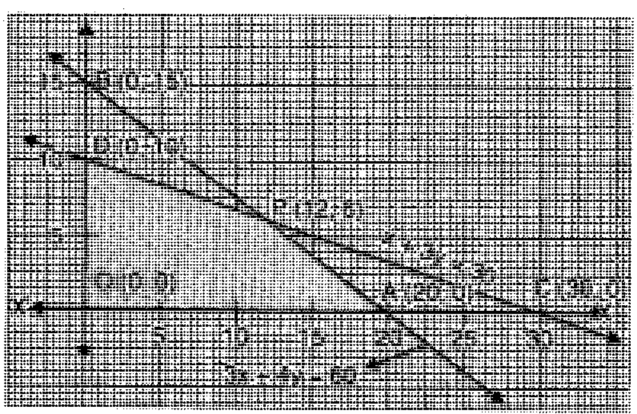
and
$$x + 3y = 30 \quad ...(ii)$$

x	30	0
y	0	10

Let P be the point of intersection of the lines

$$3x + 4y = 60$$
and
$$x + 3y = 30$$

Solving these equations, we get point P(12, 6).



The feasible region is OAPDO which is shaded in the figure.

The vertices of the feasible region are O (0, 0), A (20, 0), P(12, 6) and D (0, 10)

The value of objective function

$Z = 80x + 120y$ as follows :

Corner Points	Maximize $Z = 80x + 120y$
At O (0, 0)	0
At A (20, 0)	1600
At P (12, 6)	$960 + 720 = 1680$ maximum
At D (0, 10)	1200

$\therefore$ The profit is maximum at P(12, 6) *i.e.*, when the teaching aids of types A and B are 12 and 6 respectively.

Also maximum profit = ₹ 1680 per week. **Ans.**

29. There are three coins. One is a two-headed coin (having head on both faces), another is a biased coin that comes up heads 75% of the times and third is also a biased coin that comes up tails 40% of the times. One of the three coins is chosen at random and tossed, and it shows heads. What is the probability that it was the two-headed coin ? **[6]**

Solution : Let A be the two headed coin, B be the biased coin showing up heads 75% of the times and C be the biased coin showing up tails 40% (*i.e.*, showing up heads 60%) of the times.

Let E_1, E_2 and E_3 be the events of choosing coins of the type A, B, C respectively. Let S be the event of getting a head. Then,

$$P(E_1) = \frac{1}{3}, P(E_2) = \frac{1}{3}, P(E_3) = \frac{1}{3}$$

$$P(S/E_1) = 1, P(S/E_2) = 75\% = \frac{75}{100} = \frac{3}{4},$$

$$P(S/E_3) = 60\% = \frac{60}{100} = \frac{3}{5}.$$

$\therefore$ By Bayes' theorem, the required probability

$$P(E_1/S) = \frac{P(E_1).P(S/E_1)}{\sum_{i=1}^{3} P(E_i).P(S/E_i)}$$

$$= \frac{\frac{1}{3}\times 1}{\frac{1}{3}\times 1 + \frac{1}{3}\times\frac{3}{4} + \frac{1}{3}\times\frac{3}{5}}$$

$$= \frac{\frac{1}{3}}{\frac{1}{3}\left(1 + \frac{3}{4} + \frac{3}{5}\right)}$$

$$= \frac{1}{1+\frac{3}{4}+\frac{3}{5}}$$

$$= \frac{20}{20+15+12} = \frac{20}{47}.$$ **Ans.**

OR

Two numbers are selected at random (without replacement) from the first six positive integers. Let X denote the larger of the two numbers obtained. Find the probability distribution of the random variable X, and hence find the mean of the distribution.

Solution : Sample space

$$S = \{1, 2, 3, 4, 5, 6\}$$

Number of ways of selecting any two members of S is

$$^6C_2 = \frac{6\times5}{2!} = 15.$$

Now X denotes the larger of the two selected numbers.

$\therefore$ $P(X=1) = 0$

$P(X=2) = \frac{1}{15}$ $(2 > 1)$

$P(X=3) = \frac{2}{15}$ $(3 > 1, 3 > 2)$

$P(X=4) = \frac{3}{15}$ $(4 > 1, 4 > 2, 4 > 3)$

$P(X=5) = \frac{4}{15}$ $(5 > 1, 5 > 2, 5 > 3, 5 > 4)$

$P(X=6) = \frac{5}{15}$ $(6>1, 6>2, 6>3, 6>4, 6>5)$

$\therefore$ The probability distribution is

X	1	2	3	4	5	6
$P(X=x)$	0	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{4}{15}$	$\frac{5}{15}$

$\therefore$ Mean of the distribution $= \sum_{i=1}^{6} P_i X_i$

$$= 0\times1+\frac{1}{15}\times2+\frac{2}{15}\times3+\frac{3}{15}\times4+\frac{4}{15}\times5+\frac{5}{15}\times6$$

$$= \frac{1}{15}(0+2+6+12+20+30)$$

$$= \frac{70}{15} = \frac{14}{3}.$$ **Ans.**

●●

Mathematics 2014 (Outside Delhi) SET II

Time allowed : 3 hours **Maximum marks : 100**

Note : Except for the following questions, all the remaining questions have been asked in previous set.

SECTION — A

9. Evaluate : $\int_e^{e^2} \frac{dx}{x\log x}$ **[1]**

Solution : Let, $I = \int_e^{e^2} \frac{dx}{x\log x}$

Putting, $\log x = t$

$\Rightarrow$ $\frac{1}{x}dx = dt$

Also, when $x = e$

$\Rightarrow$ $t = \log e = 1$

and $x = e^2$

$\Rightarrow$ $t = \log e^2 = 2\log e = 2$

$\therefore$ $I = \int_1^2 \frac{dt}{t}$

$= [\log t]_1^2$

$= \log 2 - \log 1 = \log 2$ **Ans.**

10. Find a vector $\vec{a}$ of magnitude $5\sqrt{2}$, making an angle of $\frac{\pi}{4}$ with x-axis, $\frac{\pi}{2}$ with y-axis and an acute angle θ with z-axis. **[1]**

Solution : Here,

$$l = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}},$$

$$m = \cos\frac{\pi}{2} = 0, n = \cos\theta$$

Since, $l^2 + m^2 + n^2 = 1$

$\Rightarrow$ $\frac{1}{2}+0+\cos^2\theta = 1$

$\Rightarrow$ $\cos^2\theta = \frac{1}{2}$

$\Rightarrow$ $\cos\theta = \pm\frac{1}{\sqrt{2}}$

$\Rightarrow$ $\cos\theta = \frac{1}{\sqrt{2}}$ (Rejected –ve as θ is acute)

$$\Rightarrow \quad \theta = \frac{\pi}{4}$$

$\therefore$ The vector of magnitude $5\sqrt{2}$ is

$$\vec{a} = 5\sqrt{2}(l\hat{i}+m\hat{j}+n\hat{k})$$
$$= 5\sqrt{2}\left(\frac{1}{\sqrt{2}}\hat{i}+0\hat{j}+\frac{1}{\sqrt{2}}\hat{k}\right)$$
$$= 5(\hat{i}+\hat{k})$$ **Ans.**

SECTION — B

19. Using properties of determinants, prove that

$$\begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$$ **[4]**

Solution : Taking L. H. S.

Let, $$\Delta = \begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$ and take 2 common from C_1, we get

$$\Delta = 2\begin{vmatrix} a+b+c & c+a & a+b \\ p+q+r & r+p & p+q \\ x+y+z & z+x & x+y \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$; $C_3 \to C_3 - C_1$, we get

$$\Delta = 2\begin{vmatrix} a+b+c & -b & -c \\ p+q+r & -q & -r \\ x+y+z & -y & -z \end{vmatrix}$$

Taking -1 common from C_2 and C_3

$$= 2(-1)(-1)\begin{vmatrix} a+b+c & b & c \\ p+q+r & q & r \\ x+y+z & y & z \end{vmatrix}$$

Applying $C_1 \to C_1 - C_2 - C_3$, we get

$$\Delta = 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} = \text{R.H.S.}$$

Hence Proved.

20. If $x = a \sin 2t\ (1 + \cos 2t)$ and $y = b \cos 2t\ (1 - \cos 2t)$, show that at $t = \frac{\pi}{4}, \left(\frac{dy}{dx}\right) = \frac{b}{a}$. [4]

Solution : Here,

$$x = a \sin 2t\ (1 + \cos 2t) \quad \text{(i)}$$
$$y = b \cos 2t\ (1 - \cos 2t) \quad \text{(ii)}$$

Differentiating (i) w. r. t. 't', we get

$$\Rightarrow \frac{dx}{dt} = a[2\cos 2t(1+\cos 2t) + \sin 2t.(-2\sin 2t)]$$

Differentiating (ii) w. r. t. 't', we get

$$\frac{dy}{dt} = b[-2\sin 2t(1-\cos 2t) + \cos 2t.2\sin 2t]$$

$$\therefore \frac{dy}{dt} = \frac{dy/dt}{dx/dt}$$

$$= \frac{2b[-\sin 2t + 2\sin 2t\cos 2t]}{2a[\cos 2t + \cos^2 2t - \sin^2 2t]}$$

Putting $t = \frac{\pi}{4}$, we get

$$\frac{dy}{dx} = \frac{b}{a}\left[\frac{-\sin\frac{\pi}{2} + 2\sin\frac{\pi}{2}\cos\frac{\pi}{2}}{\cos\frac{\pi}{2} + \cos^2\frac{\pi}{2} - \sin^2\frac{\pi}{2}}\right]$$

$$= \frac{b}{a}\frac{[-1+0]}{[0+0-1]}$$

$$\therefore \frac{dy}{dx} = \frac{b}{a}$$ **Hence Proved.**

21. Find the particular solution of the differential equation $x(1 + y^2)\ dx - y(1 + x^2)\ dy = 0$, given that $y = 1$ when $x = 0$. [4]

Solution : The given differential equation is

$$x(1+y^2)dx - y(1+x^2)\ dy = 0 \quad \text{...(i)}$$

Separate the given differential equation, we get

$$\frac{x}{1+x^2}dx - \frac{y}{1+y^2}dy = 0$$

$$\Rightarrow \frac{2x}{1+x^2}dx - \frac{2y}{1+y^2}dy = 0$$

On integrating, we get

$$\int\frac{2y}{1+y^2}dy - \int\frac{2x}{1+x^2}dx = \text{constant}$$

$$\Rightarrow \log(1+y^2) - \log(1+x^2) = \log C$$

$$\Rightarrow \log\frac{1+y^2}{1+x^2} = \log C$$

$$\Rightarrow 1 + y^2 = C(1 + x^2)$$

Putting $y = 1$ and $x = 0$, we get

$$1 + 1 = C(1 + 0)$$
$$\Rightarrow C = 2$$

$\therefore$ The required particular solution of equation (i) is

$$1 + y^2 = 2(1 + x^2)$$ **Ans.**

22. Find the vector and Cartesian equations of the line passing through the point (2, 1, 3) and

perpendicular to the lines $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{3}$ **and** $\frac{x}{-3}=\frac{y}{2}=\frac{z}{5}$. **[4]**

Solution : Let the equation of any line passing through (2, 1, 3) and perpendicular to the lines

$$\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{3}$$

and $$\frac{x}{-3}=\frac{y}{2}=\frac{z}{5}$$

be $$\frac{x-2}{l}=\frac{y-1}{m}=\frac{z-3}{n} \quad ...(i)$$

$$\therefore \quad l.1+m.2+n.3=0 \quad ...(ii)$$

$$l(-3)+m.2+n.5=0 \quad ...(iii)$$

Solving equations (ii) and (iii), we get

$$\frac{l}{10-6}=\frac{m}{-9-5}=\frac{n}{2+6}$$

$$\Rightarrow \quad \frac{l}{2}=\frac{m}{-7}=\frac{n}{4}$$

$\therefore$ The equation of the required line is

$$\frac{x-2}{2}=\frac{y-1}{-7}=\frac{z-3}{4}$$

Also its vector equation is

$$\vec{r}=(2\hat{i}+\hat{j}+3\hat{k})+\lambda(2\hat{i}-7\hat{j}+4\hat{k})$$

Ans.

SECTION — C

28. Evaluate : $\int\left(\sqrt{\cot x}+\sqrt{\tan x}\right)dx$. **[6]**

Solution : Let $I=\int\left(\sqrt{\cot x}+\sqrt{\tan x}\right)dx$

Putting $\sqrt{\tan x}=t$

$$\Rightarrow \quad \tan x=t^2$$

$$\Rightarrow \quad \sec^2 x\,dx=2t\,dt$$

$$\Rightarrow \quad dx=\frac{2t\,dt}{1+\tan^2 x}$$

$$(\because \sec^2 x=1+\tan^2 x)$$

$$=\frac{2t}{1+t^4}dt$$

$$\therefore \quad I=\int\left(\frac{1}{t}+t\right).\frac{2t}{1+t^4}dt$$

$$=2\int\frac{t^2+1}{t^4+1}dt$$

$$2\int\frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}}dt=2\int\frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+2}dt$$

Again putting, $t-\frac{1}{t}=y$

$$\Rightarrow \quad \left(1+\frac{1}{t^2}\right)dt=dy$$

$$\Rightarrow \quad I=2\int\frac{dy}{y^2+\left(\sqrt{2}\right)^2}+C$$

$$=2.\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{y}{\sqrt{2}}\right)+C$$

$$=\sqrt{2}\tan^{-1}\left(t-\frac{1}{t}\right)\Big/\sqrt{2}+C$$

$$=\sqrt{2}\tan^{-1}\left[\frac{\sqrt{\tan x}-\sqrt{\cot x}}{\sqrt{2}}\right]+C. \quad \textbf{Ans.}$$

29. Prove that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$**. Also find the maximum volume. [6]**

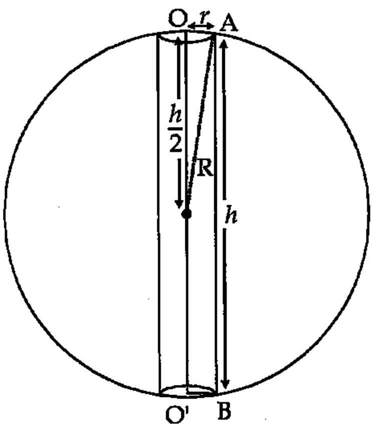
Solution : From the figure,

$$\left(\frac{h}{2}\right)^2+r^2=R^2$$

$$\Rightarrow \quad r^2=R^2-\frac{h^2}{4} \quad ...(i)$$

Now, V = Volume of the cylinder inscribed in a sphere

$$\therefore \quad V=\pi r^2 h$$



$$=\pi h\left(R^2-\frac{h^2}{4}\right) \quad \text{[using (i)]}$$

$$\therefore \quad V=\pi\left(R^2h-\frac{h^3}{4}\right)$$

Differentiating w. r. t. h, we get

$$\frac{dV}{dh}=\pi\left(R^2-\frac{3h^2}{4}\right) \quad ...(ii)$$

and $$\frac{d^2V}{dh^2}=\pi\left(0-\frac{3}{4}.2h\right) \quad ...(iii)$$

For maxima or minima

From (ii),
$$\frac{dV}{dh} = 0$$
$$R^2 - \frac{3}{4}h^2 = 0$$
$$\Rightarrow \quad h^2 = \frac{4}{3}R^2$$
$$\Rightarrow \quad h = \frac{2R}{\sqrt{3}}$$
For the value of h, from (iii),
$$\frac{d^2V}{dh^2} = -\frac{3}{2}\pi\frac{2R}{\sqrt{3}} = -\sqrt{3}.\pi R < 0(-ve)$$
$\Rightarrow$ V is maximum.

Also maximum value of V
$$= \pi.\frac{2R}{\sqrt{3}}\left(R^2 - \frac{1}{4}.\frac{4}{3}R^2\right)$$
$$= \pi.\frac{2R}{\sqrt{3}}.\frac{2}{3}R^2 = \frac{4\pi}{3\sqrt{3}}R^3$$
$$= \frac{4\sqrt{3}}{9}\pi R^3 \text{cu.units}$$ **Ans.**

Mathematics 2014 (Outside Delhi) SET III

Time allowed : 3 hours **Maximum marks : 100**

Note : Except for the following questions, all the remaining questions have been asked in previous sets.

SECTION — A

9. If $\int_0^a \frac{1}{4+x^2}dx = \frac{\pi}{8}$, find the value of a. [1]

Solution : Given,
$$\int_0^a \frac{1}{4+x^2}dx = \frac{\pi}{8}$$
$$\Rightarrow \quad \int_0^a \frac{1}{x^2+2^2}dx = \frac{\pi}{8}$$
$$\Rightarrow \quad \left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^a = \frac{\pi}{8}$$
$$\left(\because \int\frac{1}{x^2+a^2} = \frac{1}{2}\tan^{-1}\frac{x}{a}\right)$$
$$\Rightarrow \quad \frac{1}{2}\tan^{-1}\frac{a}{2} = \frac{\pi}{8}$$
$$\Rightarrow \quad \tan^{-1}\frac{a}{2} = \frac{\pi}{4}$$
$$\Rightarrow \quad \frac{a}{2} = \tan\frac{\pi}{4}$$
$$\Rightarrow \quad \frac{a}{2} = 1,$$
$$\Rightarrow \quad a = 2$$ **Ans.**

10. If $\vec{a}$ and $\vec{b}$ are perpendicular vectors, $|\vec{a}+\vec{b}| = 13$ and $|\vec{a}| = 5$, find the value of $|\vec{b}|$. [1]

Solution : Given, $|\vec{a}| = 5, |\vec{a}+\vec{b}| = 13$

Since $\vec{a}$ and $\vec{b}$ are perpendicular.
$$\therefore \quad \vec{a}.\vec{b} = 0$$
Now
$$|\vec{a}+\vec{b}| = 13$$
$$\Rightarrow \quad |\vec{a}+\vec{b}|^2 = 13^2$$
$$\Rightarrow \quad (\vec{a}+\vec{b}).(\vec{a}+\vec{b}) = 169$$
$$\Rightarrow \quad \vec{a}.\vec{a}+\vec{a}.\vec{b}+\vec{b}.\vec{a}+\vec{b}.\vec{b} = 169$$
$$\Rightarrow \quad |\vec{a}|^2+0+0+|\vec{b}|^2 = 169$$
$$(\because \vec{a}.\vec{b} = \vec{b}.\vec{a} = 0)$$
$$|\vec{b}|^2 = 169 - |\vec{a}|^2$$
$$= 169 - 5^2$$
$$= 169 - 25 = 144$$
$$\Rightarrow \quad |\vec{b}| = 12.$$ **Ans.**

SECTION–B

19. Using properties of determinants, prove that :
$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc+bc+ca+ab$$ [4]

Solution : Taking L. H. S.
$$\text{Let } \Delta = \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$$

Applying $R_1 \to \frac{R_1}{a}, R_2 \to \frac{R_2}{b}$ and $\to \frac{R_3}{c}$, we get

$$\Delta = abc\begin{vmatrix} \frac{1+a}{a} & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & \frac{1+b}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1+c}{c} \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$ and take $1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ common from R_1, we get

$$\Delta = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we get

$$\Delta = (abc + bc + ca + ab)\begin{vmatrix} 1 & 0 & 0 \\ \frac{1}{b} & 1 & 0 \\ \frac{1}{c} & 0 & 1 \end{vmatrix}$$

Expanding along R_1, we get

$$= (abc + bc + ca + ab)1.\,(1-0)$$
$$= abc + bc + ca + ab = \text{R. H. S.}$$

Hence Proved.

20. If $x = \cos t\,(3 - 2\cos^2 t)$ and $y = \sin t\,(3 - 2\sin^2 t)$, find the value of $\frac{dy}{dx}$ at $t = \frac{\pi}{4}$. [4]

Solution : Here, $x = \cos t\,(3 - 2\cos^2 t)$ ...(i)

$y = \sin t\,(3 - 2\sin^2 t)$ ...(ii)

Differentiating (i) w.r.t. t, we get

$$\frac{dx}{dt} = -\sin t\,(3 - 2\cos^2 t) + \cos t\,[2.2\cos t \sin t]$$
$$= -3\sin t + 6\cos^2 t \sin t$$

Differentiating (ii) w.r.t. t, we get

$$\frac{dy}{dt} = \cos t\,(3 - 2\sin^2 t) + \sin t\,(-2.2\sin t\cos t)$$
$$= 3\cos t - 6\sin^2 t\cos t$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$
$$= \frac{3\cos t - 6\sin^2 t\cos t}{-3\sin t + 6\cos^2 t\sin t}$$
$$= \frac{3\cos t(1-2\sin^2 t)}{3\sin t(2\cos^2 t - 1)}$$
$$= \frac{\cos t.\cos 2t}{\sin t.\cos 2t} = \cot t$$

$$(\because 1 - 2\sin^2 t = 2\cos^2 t - 1 = \cos 2t)$$

Put $t = \frac{\pi}{4}$

$\therefore \frac{dy}{dx} = \cot\frac{\pi}{4} = 1$ **Ans.**

21. Find the particular solution of the differential equation $\log\left(\frac{dy}{dx}\right) = 3x + 4y$, given that $y = 0$ when $x = 0$. [4]

Solution : The given differential equation is

$$\log\left(\frac{dy}{dx}\right) = 3x + 4y$$
$$\frac{dy}{dx} = e^{3x+4y} = e^{3x}.e^{4y}$$
$$\Rightarrow \quad e^{-4y}dy = e^{3x}dx$$

On integrating, we get

$$\int e^{3x}dx - \int e^{-4y}dy = C$$
$$\Rightarrow \quad \frac{e^{3x}}{3} - \frac{e^{-4y}}{-4} = C$$
$$\Rightarrow \quad \frac{e^{3x}}{3} + \frac{e^{-4y}}{4} = C$$

Putting, $y = 0$ when $x = 0$

$$\therefore \quad \frac{1}{3} + \frac{1}{4} = C$$
$$\Rightarrow \quad C = \frac{7}{12}$$

Hence, $\frac{e^{3x}}{3} + \frac{e^{-4y}}{4} = \frac{7}{12}$

$\therefore$ The required particular solution of the given differential equation is $4e^{3x} + 3e^{-4y} = 7$ **Ans.**

22. Find the value of p, so that the lines $l_1 : \frac{1-x}{3} = \frac{7y-14}{p} = \frac{z-3}{2}$ and $l_2 : \frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5}$ are perpendicular to each other. Also find the equations of a line passing through a point (3, 2, – 4) and parallel to line l_1. [4]

Solution : The given lines are

$$l_1 : \frac{1-x}{3} = \frac{7y-14}{p} = \frac{z-3}{2}$$
$$\Rightarrow \quad l_1 : \frac{x-1}{-3} = \frac{y-2}{p/7} = \frac{z-3}{2} \quad ...(i)$$
$$\text{and} \quad l_2 : \frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5}$$

$\Rightarrow \quad l_2: \dfrac{x-1}{-3p/7} = \dfrac{y-5}{1} = \dfrac{z-6}{-5}$ ...(ii)

Since l_1 and l_2 are perpendicular

$\therefore \quad (-3).\left(-\dfrac{3p}{7}\right)+\left(\dfrac{p}{7}\right).1+2.(-5) = 0$

$\Rightarrow \quad \dfrac{10p}{7}-10 = 0$

$\Rightarrow \quad p = 7$

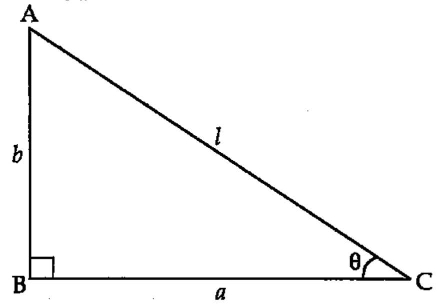
$\therefore$ Equation of the line passes through $(3, 2, -4)$ and parallel to l_1 is

$$\frac{x-3}{-3}=\frac{y-2}{1}=\frac{z+4}{2} \qquad \textbf{Ans.}$$

SECTION–C

28. If the sum of the lengths of the hypotenuse and a side of a right triangle is given, show that the area of the triangle is maximum, when the angle of between them is 60°. [6]

Solution : Let Δ ABC be right angled with side a and hypotenuse l be



$a+l = p$ (given)

$\Rightarrow \quad l = p-a$

Let θ be the angle between them.

Now, $\quad A = \text{Area of } \Delta \text{ ABC}$

$= \dfrac{1}{2}a.b = \dfrac{1}{2}a\sqrt{l^2-a^2}$

$(\because b^2 = l^2-a^2)$

$= \dfrac{1}{2}a\sqrt{(p-a)^2-a^2}$

$\therefore \quad A = \dfrac{1}{2}a\sqrt{p^2-2pa}$

Let $\quad z = A^2 = \dfrac{1}{4}a^2(p^2-2pa)$

Diff. w. r. t. a, $\quad \dfrac{dz}{da} = \dfrac{1}{4}(2ap^2-6pa^2)$ ...(i)

$= \dfrac{1}{4}.2ap(p-3a)$

For max. or min

$\dfrac{dz}{da} = 0$

$\Rightarrow \quad p-3a = 0$

$\Rightarrow \quad a = \dfrac{p}{3}$

Again differentiate eq. (i) w. r. t. a,

$\dfrac{d^2z}{da^2}\ a=\dfrac{p}{3} = \dfrac{1}{4}(2p^2-12pa)$

$\dfrac{d^2z}{da^2} = \dfrac{1}{4}\left(2p^2-12p.\dfrac{p}{3}\right)$

$= \dfrac{1}{4}(2p^2-4p^2)=\dfrac{-2p^2}{4}=\dfrac{-p^2}{2}<0$

$\Rightarrow$ Area, A is maximum.

Now from the figure,

$\cos\theta = \dfrac{a}{l}=\dfrac{a}{p-a}=\dfrac{a}{3a-a}=\dfrac{1}{2}$

$\Rightarrow \quad \theta = 60°$

Thus area A is maximum, when angle between the hypotenuse and a side is 60°. **Hence Proved.**

29. Evaluate :

$$\int\frac{1}{\sin^4 x+\sin^2 x\cos^2 x+\cos^4 x}dx. \qquad [6]$$

Solution : Let

$$I = \int\frac{1}{\sin^4 x+\sin^2 x\cos^2 x+\cos^4 x}dx.$$

Dividing by $\cos^4 x$ in Nr and Dr, we get

$= \int\dfrac{\sec^4 x\,dx}{\tan^4 x+\tan^2 x+1}$

$= \int\dfrac{\sec^2 x.\sec^2 x}{\tan^4 x+\tan^2 x+1}dx$

$= \int\dfrac{(1+\tan^2 x)\sec^2 x\,dx}{\tan^4 x+\tan^2 x+1}$

$(\because \sec^2 x = 1+\tan^2 x)$

Putting $\tan x = t$

$\Rightarrow \quad \sec^2 x\,dx = dt$

$\therefore \quad I = \int\dfrac{1+t^2}{t^4+t^2+1}dt$

$= \int\dfrac{1+\frac{1}{t^2}}{t^2+1+\frac{1}{t^2}}dt = \int\dfrac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+3}dt$

Again putting $t-\dfrac{1}{t} = y$

$\Rightarrow \quad \left(1+\dfrac{1}{t^2}\right)dt = dy$

Thus, $\quad I = \int\dfrac{dy}{y^2+(\sqrt{3})^2}$

$= \dfrac{1}{\sqrt{3}}\tan^{-1}\left(\dfrac{y}{\sqrt{3}}\right)+C$

$$\left(\because \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C\right)$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(t - \frac{1}{t}\right)/\sqrt{3} + C$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\tan x - \cot x}{\sqrt{3}}\right) + C \quad \textbf{Ans.}$$

Mathematics 2014 (Delhi) SET I

Time allowed : 3 hours **Maximum marks : 100**

SECTION — A

1. Let * be binary operation, on the set of all non-zero real numbers, given by $a * b = \frac{ab}{5}$ for all $a, b \in R - \{0\}$. Find the value of x, given that $2 * (x * 5) = 10$.** **[1]**

2. If $\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1}x\right) = 1$, then find the value of x. **[1]**

Solution : Given,

$$\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1}x\right) = 1$$

$$\Rightarrow \quad \sin^{-1}\frac{1}{5} + \cos^{-1}x = \sin^{-1} \quad (1)$$

$$\Rightarrow \quad \sin^{-1}\frac{1}{5} + \cos^{-1}x = \frac{\pi}{2}$$

$$\left(\because \sin^{-1}(1) = \frac{\pi}{2}\right)$$

$$\Rightarrow \quad \sin^{-1}\frac{1}{5} = \frac{\pi}{2} - \cos^{-1}x$$

$$\Rightarrow \quad \sin^{-1}\frac{1}{5} = \sin^{-1}x$$

$$\left(\because \cos^{-1}x + \sin^{-1}x = \frac{\pi}{2}\right)$$

$$\therefore \quad x = \frac{1}{5} \quad \textbf{Ans.}$$

3. If $2\begin{bmatrix}3 & 4\\5 & x\end{bmatrix} + \begin{bmatrix}1 & y\\0 & 1\end{bmatrix} = \begin{bmatrix}7 & 0\\10 & 5\end{bmatrix}$, find $(x - y)$. **[1]**

Solution : Given,

$$2\begin{bmatrix}3 & 4\\5 & x\end{bmatrix} + \begin{bmatrix}1 & y\\0 & 1\end{bmatrix} = \begin{bmatrix}7 & 0\\10 & 5\end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix}6 & 8\\10 & 2x\end{bmatrix} + \begin{bmatrix}1 & y\\0 & 1\end{bmatrix} = \begin{bmatrix}7 & 0\\10 & 5\end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix}6+1 & 8+y\\10+0 & 2x+1\end{bmatrix} = \begin{bmatrix}7 & 0\\10 & 5\end{bmatrix}$$

Equating the corresponding elements, we get

$$\therefore \quad 8 + y = 0$$

$$\Rightarrow \quad y = -8$$

$$\text{and} \quad 2x + 1 = 5$$

$$\Rightarrow \quad 2x = 4$$

$$\Rightarrow \quad x = 2$$

$$\therefore \quad x - y = 2 - (-8)$$

$$= 2 + 8 = 10 \quad \textbf{Ans.}$$

4. Solve the following matrix equation for $x : [x \quad 1]\begin{bmatrix}1 & 0\\-2 & 0\end{bmatrix} = 0.$ **[1]**

Solution : Given,

$$[x \quad 1]\begin{bmatrix}1 & 0\\-2 & 0\end{bmatrix} = 0$$

$$\Rightarrow \quad [x-2 \quad 0+0] = [0 \; 0]$$

$$\Rightarrow \quad x - 2 = 0$$

$$\therefore \quad x = 2 \quad \textbf{Ans.}$$

5. If $\begin{vmatrix}2x & 5\\8 & x\end{vmatrix} = \begin{vmatrix}6 & -2\\7 & 3\end{vmatrix}$, write the value of x. **[1]**

Solution : Given,

$$\begin{vmatrix}2x & 5\\8 & x\end{vmatrix} = \begin{vmatrix}6 & -2\\7 & 3\end{vmatrix}$$

$$\Rightarrow \quad 2x^2 - 40 = 18 - (-14)$$

$$\Rightarrow \quad 2x^2 = 18 + 14 + 40$$

$$\Rightarrow \quad 2x^2 = 72$$

$$\Rightarrow \quad x^2 = 36$$

$$\therefore \quad x = \pm 6 \quad \textbf{Ans.}$$

6. Write the antiderivative of $\left(3\sqrt{x} + \frac{1}{\sqrt{x}}\right)$. **[1]**

Solution : The antiderivative of $\left(3\sqrt{x} + \frac{1}{\sqrt{x}}\right)$

****Answer is not given due to the change in present syllabus**

$$= \int\left(3\sqrt{x}+\frac{1}{\sqrt{x}}\right)dx$$

(Antiderivative = Integral)

$$= 3\int x^{1/2}dx+\int x^{-1/2}dx$$

$$= 3.\frac{x^{3/2}}{3/2}+\frac{x^{1/2}}{1/2}+C$$

$$= 2x^{3/2}+2x^{1/2}+C$$

$$= 2x\sqrt{x}+2\sqrt{x}+C$$

$$= 2\sqrt{x}(x+1)+C.$$ **Ans.**

7. Evaluate : $\int_0^3 \frac{dx}{9+x^2}$. **[1]**

Solution : $\int_0^3 \frac{dx}{9+x^2} = \int_0^3 \frac{dx}{x^2+3^2}$

$$= \left[\frac{1}{3}\tan^{-1}\frac{x}{3}\right]_0^3$$

$$\left(\because \int\frac{dx}{x^2+a^2}=\frac{1}{a}\tan^{-1}\frac{x}{a}\right)$$

$$= \frac{1}{3}\left[\tan^{-1}1-\tan^{-1}0\right]$$

$$= \frac{1}{3}\left[\frac{\pi}{4}-0\right]=\frac{\pi}{12}.$$ **Ans.**

8. Find the projection of the vector $\hat{i}+3\hat{j}+7\hat{k}$ on the vector $2\hat{i}-3\hat{j}+6\hat{k}$. **[1]**

Solution: Let $\vec{a} = \hat{i}+3\hat{j}+7\hat{k}$ and $\vec{b} = 2\hat{i}-3\hat{j}+6\hat{k}$

$\therefore$ Projection of $\vec{a}$ on $\vec{b}$ $= \frac{\vec{a}.\vec{b}}{|\vec{b}|}$

$$= \frac{(\hat{i}+3\hat{j}+7\hat{k}).(2\hat{i}-3\hat{j}+6\hat{k})}{\sqrt{2^2+(-3)^2+6^2}}$$

$$= \frac{1\times2+3\times(-3)+7\times6}{\sqrt{4+9+36}}$$

$$= \frac{2-9+42}{7}=\frac{35}{7}=5.$$ **Ans.**

9. If $\vec{a}$ and $\vec{b}$ are two unit vectors such that $\vec{a}+\vec{b}$ is also a unit vector, then find the angle between $\vec{a}$ and $\vec{b}$. **[1]**

Solution : Given, $|\vec{a}|=1, |\vec{b}|=1$

and $|\vec{a}+\vec{b}| = 1$

$\Rightarrow |\vec{a}+\vec{b}|^2 = 1$

$\Rightarrow (\vec{a}+\vec{b}).(\vec{a}+\vec{b}) = 1$

$\Rightarrow \vec{a}.\vec{a}+\vec{a}.\vec{b}+\vec{b}.\vec{a}+\vec{b}.\vec{b} = 1$

$\Rightarrow |\vec{a}|^2+2\vec{a}.\vec{b}+|\vec{b}|^2 = 1$

$\Rightarrow 1+2\vec{a}.\vec{b}+1 = 1$

$\Rightarrow 2\vec{a}.\vec{b} = 1$

$\Rightarrow 2|\vec{a}|.|\vec{b}|\cos\theta = -1$

$\Rightarrow 2.1.1\cos\theta = -1$

$\Rightarrow \cos\theta = -\frac{1}{2}$

$\Rightarrow \cos\theta = \cos 120°$

$\therefore \theta = 120°$ **Ans.**

10. Write the vector equation of the plane, passing through the point (*a*, *b*, *c*) and parallel to the plane $\vec{r}.(\hat{i}+\hat{j}+\hat{k})=2$. **[1]**

Solution : The given plane is

$\vec{r}.(\hat{i}+\hat{j}+\hat{k}) = 2$ ...(i)

$\Rightarrow (x\hat{i}+y\hat{j}+z\hat{k}).(\hat{i}+\hat{j}+\hat{k}) = 2$

$\Rightarrow x+y+z = 2$ ...(ii)

$\therefore$ Equation of a plane parallel to (ii), is

$x+y+z = \lambda$ ...(iii)

The plane is passing through (*a*, *b*, *c*)

$\therefore \lambda = a+b+c$

$\therefore$ The vector equation of the required plane is

$\vec{r}.(\hat{i}+\hat{j}+\hat{k}) = a+b+c$. **Ans.**

SECTION — B

11. Let A = {1, 2 ,3, ..., 9} and R be relation in A × A defined by (*a*, *b*) R (*c*, *d*) if *a* + *d* = *b* + *c* for (*a*,*b*), (*c*,*d*) in A × A. Prove that R is an equivalence relation. Also obtain the equivalence class [(2, 5)]. **[4]**

Solution : Here, A = {1, 2, 3, ..., 9} and R is a relation on A × A defined by

$(a, b) R (c, d) \Leftrightarrow a+d=b+c \forall a, b, c, d \in A$

(i $\forall (a,b) \in A\times A$

$a+b = b+a$

$\Rightarrow (a,b)R(a,b) \ \forall (a,b)\in A\times A$

$\Rightarrow$ R is reflexive on A.

(ii) Let (*a*, *b*) R (*c*, *d*)

$\Rightarrow a+d=b+c$

$\Rightarrow b+c=a+d$

$\Rightarrow c+b=d+a$

$\Rightarrow \quad (c, d)\ R\ (a, b)$

$\Rightarrow$ R is symmetric on A.

(iii) Let $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f)$

$\Rightarrow \quad a + d = b + c$ and $c + f = d + e$

$\Rightarrow (a + d) + (c + f) = (b + c) + (d + e)$

$\Rightarrow \quad a + f = b + e$

$\Rightarrow \quad (a, b)\ R\ (e, f)$

$\Rightarrow$ R is transitive on A.

Hence R is an equivalence relation on A.

Also equivalence class [(2, 5)]

$$= \{(a, b) \in A \times A \mid (2,5)\ R\ (a, b)\}$$
$$= \{(a, b) \in A \times A \mid 2 + b = 5 + a\}$$
$$= \{(a, b) \in A \times A \mid b = a + 3\}$$
$$= \{(a, a + 3) \mid a \in A\}. \quad \textbf{Ans.}$$

12. Prove that

$$\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) = \frac{x}{2};\ x \in \left(0, \frac{\pi}{4}\right)$$

[4]

Solution : L. H. S.

$$= \cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right);\ x \in \left(0, \frac{\pi}{4}\right)$$

$$= \cot^{-1}\left(\frac{(\sqrt{1+\sin x}+\sqrt{1-\sin x})}{(\sqrt{1+\sin x}-\sqrt{1-\sin x})}\right.$$
$$\left.\frac{(\sqrt{1+\sin x}+\sqrt{1-\sin x})}{(\sqrt{1+\sin x}+\sqrt{1-\sin x})}\right)$$

$$= \cot^{-1}\left(\frac{1+\sin x+1-\sin x+2\sqrt{(1+\sin x)(1-\sin x)}}{(1+\sin x)-(1-\sin x)}\right)$$

$$= \cot^{-1}\left(\frac{2+2\sqrt{1-\sin^2 x}}{1+\sin x-1+\sin x}\right)$$

$$= \cot^{-1}\left(\frac{2(1+\cos x)}{2\sin x}\right)$$

$$= \cot^{-1}\left(\frac{4\cos^2 x/2}{4\sin x/2.\cos x/2}\right)$$

$$\left(\because 1+\cos x = 2\cos^2\frac{x}{2} \text{ and } \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}\right)$$

$$= \cot^{-1}\left(\frac{\cos x/2}{\sin x/2}\right)$$

$= \cot^{-1}(\cot x/2) = x/2 =$ R.H.S. **Hence Proved.**

OR

Prove that :

$$2\tan^{-1}\left(\frac{1}{5}\right)+\sec^{-1}\left(\frac{5\sqrt{2}}{7}\right)+2\tan^{-1}\left(\frac{1}{8}\right)=\frac{\pi}{4}.$$

Solution : L. H. S.

$$= 2\tan^{-1}\left(\frac{1}{5}\right)+\sec^{-1}\left(\frac{5\sqrt{2}}{7}\right)+2\tan^{-1}\left(\frac{1}{8}\right)$$

$$= 2\left[\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(\frac{1}{8}\right)\right]+\sec^{-1}\left(\frac{5\sqrt{2}}{7}\right)$$

$$= 2\tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{8}}{1-\frac{1}{5}\times\frac{1}{8}}\right)+\tan^{-1}\left(\sqrt{\left(\frac{5\sqrt{2}}{7}\right)^2-1}\right)$$

$$\left[\begin{array}{l}\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)\\ \because \sec^{-1}x=\tan^{-1}\sqrt{x^2-1}\end{array}\right]$$

$$= 2\tan^{-1}\left(\frac{13/40}{39/40}\right)+\tan^{-1}\left(\sqrt{\frac{1}{49}}\right)$$

$$= 2\tan^{-1}\left(\frac{13}{39}\right)+\tan^{-1}\left(\frac{1}{7}\right)$$

$$= 2\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left[\frac{2\times\frac{1}{3}}{1-\left(\frac{1}{3}\right)^2}\right]+\tan^{-1}\left(\frac{1}{7}\right)$$

$$\left[\because 2\tan^{-1}x=\tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]$$

$$= \tan^{-1}\left(\frac{\frac{2}{3}}{\frac{8}{9}}\right)+\tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left(\frac{3}{4}\right)+\tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left(\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\cdot\frac{1}{7}}\right)$$

$$= \tan^{-1}\left(\frac{\frac{25}{28}}{\frac{25}{28}}\right)$$

$$= \tan^{-1}(1)$$

$$= \frac{\pi}{4} = \text{R. H. S.}$$ **Hence Proved.**

13. Using properties of determinants, prove that

$$\begin{vmatrix} 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \\ x-y-z & 2x & 2x \end{vmatrix} = (x+y+z)^3$$ **[4]**

Solution : Taking L. H. S.

Let $$\Delta = \begin{vmatrix} 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \\ x-y-z & 2x & 2x \end{vmatrix}$$

Applying $R_1 \rightarrow R_1 + R_2 + R_3$, we get

$$\Delta = \begin{vmatrix} x+y+z & x+y+z & x+y+z \\ 2z & 2z & z-x-y \\ x-y-z & 2x & 2x \end{vmatrix}$$

Taking $(x + y + z)$ common from R_1,

$$= (x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ 2z & 2z & z-x-y \\ x-y-z & 2x & 2x \end{vmatrix}$$

Applying $C_2 \rightarrow C_2 - C_1$, $C_3 \rightarrow C_3 - C_2$, we get

$$\Delta = (x+y+z)\begin{vmatrix} 1 & 0 & 0 \\ 2z & 0 & -(x+y+z) \\ x-y-z & x+y+z & 0 \end{vmatrix}$$

By expanding along R_1, we get

$$(x+y+z).1.(x+y+z)^2$$

$$= (x+y+z)^3 = \text{R. H. S.}$$ **Hence Proved.**

14. Differentiate $\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$ with respect to $\cos^{-1}(2x\sqrt{1-x^2})$, when $x \neq 0$. **[4]**

Solution : Let $$y = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$$

Putting $x = \cos\theta$, we get

$$y = \tan^{-1}\left(\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}\right)$$

$$= \tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right) \quad (\because 1-\cos^2\theta = \sin^2\theta)$$

$$= \tan^{-1}(\tan\theta)$$

$$\Rightarrow \quad y = \theta$$

Differentiating w.r. t. θ, we get

$$\frac{dy}{d\theta} = 1 \quad \text{...(i)}$$

and let $t = \cos^{-1}(2x\sqrt{1-x^2})$ $(x \neq 0)$

Put $x = \cos\theta$, we get

$$t = \cos^{-1}(2\cos\theta\sqrt{1-\cos^2\theta})$$

$$= \cos^{-1}(2\cos\theta\sin\theta)$$

$$= \cos^{-1}(\sin 2\theta)$$

$$= \cos^{-1}\left[\cos\left(\frac{\pi}{2}-2\theta\right)\right]$$

$$\therefore \quad t = \frac{\pi}{2}-2\theta$$

Differentiating w. r. t. θ, we get

$$\frac{dt}{d\theta} = 0-2 = -2 \quad \text{...(ii)}$$

Now, $$\frac{dy}{dt} = \frac{dy/d\theta}{dt/d\theta} \quad \text{[Using (i) and (ii)]}$$

$$= -\frac{1}{2}$$ **Ans.**

15. If $y = x^x$, prove that $\frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x} = 0$. **[4]**

Solution : Given, $y = x^x$

Taking log on both sides, we get

$$\log y = x \log x$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{y}.\frac{dy}{dx} = x.\frac{1}{x} + 1.\log x = 1 + \log x$$

$$\Rightarrow \quad \frac{dy}{dx} = y(1+\log x)$$

Again differentiating, w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{dy}{dx}.(1+\log x) + y.\frac{1}{x}$$

$$= \frac{dy}{dx}\left[\frac{dy}{dx}.\frac{1}{y}\right] + \frac{y}{x} \quad \text{[Using (i)]}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x} = 0.$$ **Hence Proved.**

16. Find the intervals in which the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is **[4]**

(a) strictly increasing

(b) strictly decreasing

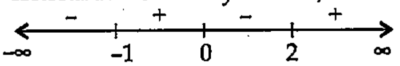
Solution : Here

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 5$$

$$\Rightarrow \quad f'(x) = 12x^3 - 12x^2 - 24x$$

$$= 12x(x^2 - x - 2)$$
$$= 12x[x^2 - 2x + x - 2]$$
$$= 12x[x(x-2) + 1(x-2)]$$
$$= 12x(x+1)(x-2)$$

The critical lvalues for f are -1, 0 and 2



Intervals	Test value	sign of $f'(x)$ $=12x(x+1)(x-2)$	Nature of function $f(x)$
$(-\infty,-1)$	$x=-1.5$	$(-)(-)(-)=-<0$	Strictly decreasing
$(-1,0)$	$x=-0.5$	$(-)(+)(-)=+>0$	Strictly increasing
$(0,2)$	$x=1$	$(+)(+)(-)=-<0$	Strictly decreasing
$(2,\infty)$	$x=3$	$(+)(+)(+)=+>0$	Strictly increasing

$\therefore f$ is strictly increasing in $(-1, 0) \cup (2, \infty)$ and strictly decreasing in $(-\infty, -1) \cup (0, 2)$. **Ans.**

OR

Find the equations of the tangent and normal to the curve $x = a\sin^3\theta$ and $y = a\cos^3\theta$ at $\theta = \frac{\pi}{4}$.

Solution : The given curve is $x = a\sin^3\theta$; $y = a\cos^3\theta$ ...(i)

At, $\theta = \frac{\pi}{4}$

$$x = a\sin^3\frac{\pi}{4}$$
$$= a\left(\frac{1}{\sqrt{2}}\right)^3$$
$$= \frac{a}{2\sqrt{2}}$$

and $y = a\cos^3\frac{\pi}{4}$

$$= a\left(\frac{1}{\sqrt{2}}\right)^3$$
$$= \frac{a}{2\sqrt{2}}$$

$\Rightarrow P\left(\frac{a}{2\sqrt{2}}, \frac{a}{2\sqrt{2}}\right)$ is a point on (i) corresponding to

$$\theta = \frac{\pi}{4}.$$

Differentiating (i) w.r.t. θ, we get

$$\frac{dx}{d\theta} = 3a\sin^2\theta\cos\theta$$

and $\frac{dy}{d\theta} = -3a\cos^2\theta\sin\theta$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$
$$= \frac{-3a\cos^2\theta\sin\theta}{3a\sin^2\theta\cos\theta} = -\cot\theta$$

$\therefore$ Slope of the tangent at

$$\theta = \frac{\pi}{4} \text{ is } -1$$

and slope of the normal at

$$\theta = \frac{\pi}{4} \text{ is } 1.$$

Hence equation of the tangent at P is

$$y - \frac{a}{2\sqrt{2}} = -1.\left(x - \frac{a}{2\sqrt{2}}\right)$$
$$\therefore \quad x + y = \frac{a}{\sqrt{2}}$$

and equation of the normal at P is

$$y - \frac{a}{2\sqrt{2}} = 1.\left(x - \frac{a}{2\sqrt{2}}\right)$$
$$\therefore \quad y = x$$ **Ans.**

17. Evaluate : $\int \frac{\sin^6 x + \cos^6 x}{\sin^2 x.\cos^2 x}dx.$ **[4]**

Solution : Let, $I = \int \frac{\sin^6 x + \cos^6 x}{\sin^2 x\cos^2 x}dx$

Here Numerator $= \sin^6 x + \cos^6 x$

$$= (\sin^2 x + \cos^2 x)(\sin^4 x + \cos^4 x - \sin^2 x\cos^2 x)$$
$$[\because a^3 + b^3 = (a+b)(a^2 + b^2 - ab)]$$
$$= 1.[(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x - \sin^2 x\cos^2 x]$$
$$= 1 - 3\sin^2 x\cos^2 x$$
$$\therefore I = \int \frac{1 - 3\sin^2 x\cos^2 x}{\sin^2 x\cos^2 x}dx$$
$$= \int \frac{1}{\sin^2 x\cos^2 x}dx - 3\int 1.dx$$

Multiplying and dividing denominator by $\cos^2 x$ in first Integral

$$= \int \frac{\sec^4 x}{\tan^2 x}dx - 3x + C$$
$$= \int \frac{\sec^2 x.\sec^2 x}{\tan^2 x}dx - 3x + C$$
$$= \int \frac{(1+\tan^2 x)\sec^2 x}{\tan^2 x}dx - 3x + C$$

Putting $\tan x = t$

$\Rightarrow \sec^2 x\, dx = dt$

$$\Rightarrow \quad I = \int \frac{(t^2+1)}{t^2}dt - 3x + C$$

$$= \int\left[1+\frac{1}{t^2}\right]dt - 3x + C$$

$$= t - \frac{1}{t} - 3x + C$$

$$= \tan x - \cot x - 3x + C.$$

OR

Evaluate : $\int (x-3)\sqrt{x^2+3x-18}\,dx.$

Solution : Let $I = \int (x-3)\sqrt{x^2+3x-18}\,dx$

$$= \int\left[\frac{1}{2}(2x+3) - \frac{9}{2}\right]\sqrt{x^2+3x-18}\,dx$$

$$= \frac{1}{2}\int(2x+3)\sqrt{x^2+3x-18}\,dx - \frac{9}{2}\int\sqrt{x^2+3x-18}\,dx$$

Putting, $x^2 + 3x - 18 = t$ in the first integral

$\Rightarrow$ $(2x + 3)\,dx = dt$

$$\therefore\ I = \frac{1}{2}\int\sqrt{t}\,dt - \frac{9}{2}\int\sqrt{\left(x+\frac{3}{2}\right)^2 - \left(\frac{9}{2}\right)^2}\,dx$$

$$= \frac{1}{2}\frac{t^{3/2}}{3/2} - \frac{9}{2}\left[\frac{1}{2}\left(x+\frac{3}{2}\right)\sqrt{\left(x+\frac{3}{2}\right)^2 - \left(\frac{9}{2}\right)^2}\right.$$

$$\left. - \frac{(9/2)^2}{2}\log\left|\left(x+\frac{3}{2}\right) + \sqrt{\left(x+\frac{3}{2}\right)^2 - \left(\frac{9}{2}\right)^2}\right|\right] + C$$

$$\left[\because \int\sqrt{x^2-a^2}\,dx = \frac{1}{2}x\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + C\right]$$

$$= \frac{1}{3}(x^2+3x-18)^{3/2} - \frac{9}{4}\left[\left(x+\frac{3}{2}\right)\sqrt{x^2+3x-18} - \frac{81}{4}\log\left|x+\frac{3}{2}+\sqrt{x^2+3x-18}\right|\right] + C \quad \textbf{Ans.}$$

18. Find the particular solution of the differential equation $e^x\sqrt{1-y^2}\,dx + \frac{y}{x}dy = 0$, given that $y = 1$ when $x = 0$. **[4]**

Solution : The given differential equation is

$$e^x\sqrt{1-y^2}\,dx + \frac{y}{x}dy = 0$$

Separating variables :

$$\int xe^x dx + \int\frac{y}{\sqrt{1-y^2}}dy = 0$$

On integrating, we get

$$xe^x dx + \frac{y}{\sqrt{1-y^2}}dy = C$$

Putting, $1 - y^2 = t$

$\Rightarrow$ $-2y.dy = dt$

$\Rightarrow$ $y.dy = -\frac{dt}{2}$

$$\Rightarrow\ x\int e^x dx - \int 1.e^x dx - \frac{1}{2}\int\frac{1}{\sqrt{t}}dt = C$$

$$\Rightarrow\ xe^x - e^x - \frac{1}{2}\left[\frac{t^{1/2}}{\left(\frac{1}{2}\right)}\right] = C$$

$$\Rightarrow\ e^x(x-1) - t^{1/2} = C$$

$$\Rightarrow\ e^x(x-1) - \sqrt{1-y^2} = C$$

$[\because t = 1 - y^2]$

Putting $y = 1$

and $x = 0$

$$\Rightarrow\ e^0(0-1) - \sqrt{1-1} = C$$

$$\Rightarrow\ C = -1$$

$$\therefore\ e^x(x-1) - \sqrt{1-y^2} = -1 \quad \textbf{Ans.}$$

19. Solve the following differential equation :

$$(x^2-1)\frac{dy}{dx} + 2xy = \frac{2}{x^2-1}$$ **[4]**

Solution : The given differential equation is

$$(x^2-1)\frac{dy}{dx} + 2xy = \frac{2}{x^2-1} \quad ...(i)$$

$$\Rightarrow\ \frac{dy}{dx} + \frac{2x}{x^2-1}.y = \frac{2}{(x^2-1)^2} \quad ...(ii)$$

This is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

Here, $P = \frac{2x}{x^2-1}$,

and $Q = \frac{2}{(x^2-1)^2}$

Now, $\int P\,dx = \int\frac{2x}{x^2-1}dx = \log(x^2-1)$

$$\therefore\ \text{I. F} = e^{\int P dx} = e^{\log(x^2-1)} = x^2 - 1$$

$\therefore$ The solution is

$$y.(\text{I.F.}) = \int Q(\text{I.F.})dx + C$$

$$\Rightarrow \quad y(x^2-1) = \int \frac{2}{(x^2-1)^2}\cdot(x^2-1)dx + C$$

$$= 2\int \frac{dx}{x^2-1} + C$$

$$= 2.\frac{1}{2}\log\left|\frac{x-1}{x+1}\right| + C$$

$\therefore$ The solution of the given differential equation is

$$y(x^2-1) = \log\left|\frac{x-1}{x+1}\right| + C.$$ **Ans.**

20. Prove that, for any three vectors : $\vec{a}, \vec{b}, \vec{c}$ $[\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}] = 2[\vec{a}, \vec{b}, \vec{c}]$ **[4]**

Solution : L. H. S. $= [\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}]$

$$= (\vec{a}+\vec{b}).[(\vec{b}+\vec{c})\times(\vec{c}+\vec{a})]$$

$$= (\vec{a}+\vec{b}).[\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{c}+\vec{c}\times\vec{a}]$$

$$= (\vec{a}+\vec{b}).[\vec{b}\times\vec{c}-\vec{a}\times\vec{b}+0+\vec{c}\times\vec{a}]$$

$(\because \vec{c}\times\vec{c} = 0)$

$$= \vec{a}.(\vec{b}\times\vec{c}) - \vec{a}.(\vec{a}\times\vec{b}) + \vec{a}.(\vec{c}\times\vec{a})$$
$$+ \vec{b}.(\vec{b}\times\vec{c}) - \vec{b}.(\vec{a}\times\vec{b}) + \vec{b}.(\vec{c}\times\vec{a})$$

$$= \vec{a}.(\vec{b}\times\vec{c}) - 0 + 0 + 0 - 0 + \vec{b}.(\vec{c}\times\vec{a})$$

$$= [\vec{a}, \vec{b}, \vec{c}] + [\vec{b}, \vec{c}, \vec{a}]$$

$$\left[\begin{array}{l}\because \vec{a}.(\vec{b}\times\vec{c}) = [\vec{a}, \vec{b}, \vec{c}] \\ \text{and } \vec{b}.(\vec{c}\times\vec{a}) = [\vec{b}, \vec{c}, \vec{a}]\end{array}\right]$$

$$= [\vec{a}, \vec{b}, \vec{c}] + [\vec{a}, \vec{b}, \vec{c}]$$

$= 2[\vec{a}, \vec{b}, \vec{c}] =$ R.H.S. **Hence Proved.**

OR

Vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are such that $\vec{a}+\vec{b}+\vec{c} = \vec{0}$ and $|\vec{a}| = 3, |\vec{b}| = 5$ and $|\vec{c}| = 7$. Find the angle between $\vec{a}$ and $\vec{b}$.

Solution : Given, $|\vec{a}| = 3, |\vec{b}| = 5, |\vec{c}| = 7$

and $\vec{a}+\vec{b}+\vec{c} = \vec{0}$

$$\Rightarrow \quad \vec{a}+\vec{b} = -\vec{c}$$

$$\Rightarrow \quad |\vec{a}+\vec{b}|^2 = |-\vec{c}|^2 = 7^2$$

$$\Rightarrow \quad (\vec{a}+\vec{b}).(\vec{a}+\vec{b}) = 49$$

$$\Rightarrow \quad \vec{a}.\vec{a}+\vec{a}.\vec{b}+\vec{b}.\vec{a}+\vec{b}.\vec{b} = 49$$

$$\Rightarrow \quad |\vec{a}|^2 + 2\vec{a}.\vec{b} + |\vec{b}|^2 = 49$$

$$\Rightarrow |\vec{a}|^2 + 2|\vec{a}||\vec{b}|\cos\theta + |\vec{b}|^2 = 49,$$

where θ is the angle between $\vec{a}$ and $\vec{b}$

$$\Rightarrow \quad 3^2 + 2\times3\times5\cos\theta + 5^2 = 49$$

$$\Rightarrow \quad 30\cos\theta = 15$$

$$\Rightarrow \quad \cos\theta = \frac{15}{30} = \frac{1}{2}$$

$$\Rightarrow \quad \theta = \cos^{-1}(1/2)$$

$$\therefore \quad \theta = 60^\circ$$ **Ans.**

21. Show that the lines $\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7}$ and $\frac{x-2}{1} = \frac{y-4}{3} = \frac{z-6}{5}$ intersect. Also find their point of intersection. **[4]**

Solution : Let $\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7} = r$ ...(i)

$\therefore$ Point (i) is $(3r-1, 5r-3, 7r-5)$.

And let $\frac{x-2}{1} = \frac{y-4}{3} = \frac{z-6}{5} = k$ ...(ii)

$\therefore$ Point (ii) is $(k+2, 3k+4, 5k+6)$

For lines (i) and (ii) to intersect, we get

$$3r-1 = k+2 \Rightarrow 3r-k = 3$$
$$5r-3 = 3k+4 \Rightarrow 5r-3k = 7$$
$$7r-5 = 5k+6 \Rightarrow 7r-5k = 11$$

Solving these equations, we get

$$r = \frac{1}{2};\ k = -\frac{3}{2}$$

$\therefore$ Lines (i) and (ii) intersect and their point of intersection is $\left(\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}\right)$. **Ans.**

22. Assume that each born child is equally likely to be a boy or a girl. If a family has two children, what is the conditional probability that both are girls ? Given that

(i) the youngest is a girl.

(ii) atleast one is a girl. **[4]**

Solution : The sample space

$$S = \{B_1B_2, B_1G_2, G_1G_2, G_1B_2\}$$

$$\Rightarrow \quad n(S) = 4$$

Let A be the event that both children are girls, B be the event that the youngest child is a girl and C be the event that atleast one of the children is a girl. Then

$$A = \{G_1G_2\}$$

$$\Rightarrow \quad n(A) = 1,$$

$$B = \{G_1G_2, B_1G_2\}$$

$$\Rightarrow \quad n(B) = 2,$$

and $$C = \{B_1G_2, G_1G_2, G_1B_2\}$$

$$\Rightarrow \quad n(C) = 3$$

$$\Rightarrow \quad A \cap B = \{G_1 G_2\}$$

$$\Rightarrow \quad n(A \cap B) = 1$$

and $$(A \cap C) = \{G_1 G_2\}$$

$$\Rightarrow \quad n(A \cap C) = 1$$

(i) The required probability

$$= P(A/B)$$

$$= \frac{P(A \cap B)}{P(B)} = \frac{\frac{n(A \cap B)}{n(S)}}{\frac{n(B)}{n(S)}}$$

$$= \frac{1/4}{2/4} = \frac{1}{2}.$$

(ii) The required probability

$$= P(A/C)$$

$$= \frac{P(A \cap C)}{P(C)} = \frac{\frac{n(A \cap C)}{n(S)}}{\frac{n(C)}{n(S)}}$$

$$= \frac{1/4}{3/4} = \frac{1}{3}. \qquad \textbf{Ans.}$$

SECTION — C

23. Two schools P and Q want to award their selected students on the values of Discipline, Politeness and Punctuality. The school P wants to award ₹ x each, ₹ y each and ₹ z each for the three respective values to its 3, 2 and 1 students with a total award money of ₹ 1,000. School Q wants to spend ₹ 1,500 to award its 4, 1 and 3 students on the respective values (by giving the same award money for the three values as before). If the total amount of awards for one prize on each value is ₹ 600, using matrices, find the award money for each value.

Apart from the above three values, suggest one more value for awards. [6]

Solution : The awards for Discipline, Politeness and Punctuality is ₹ x, ₹ y and ₹ z respectively.

According to question,

$$3x + 2y + z = 1{,}000$$
$$4x + y + 3z = 1{,}500$$
$$x + y + z = 600$$

The given equation can be written in matrix form,

$$AX = B \qquad \text{...(i)}$$

$$\begin{bmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1{,}000 \\ 1{,}500 \\ 600 \end{bmatrix}$$

Here $$A = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

And $$B = \begin{bmatrix} 1{,}000 \\ 1{,}500 \\ 600 \end{bmatrix}$$

Now, $$|A| = \begin{vmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 3(1-3) - 2(4-3) + 1(4-1)$$
$$= -6 - 2 + 3$$
$$= -5 \neq 0$$

$\Rightarrow$ A^{-1} exists.

For adj A,

$$A_{11} = (1-3) = -2,$$
$$A_{12} = -(4-3) = -1,$$
$$A_{13} = (4-1) = 3$$
$$A_{21} = -(2-1) = -1,$$
$$A_{22} = (3-1) = 2,$$
$$A_{23} = -(3-2) = -1$$
$$A_{31} = (6-1) = 5,$$
$$A_{32} = -(9-4) = -5,$$
$$A_{33} = (3-8) = -5$$

$$\therefore \text{adj } A = \begin{bmatrix} -2 & -1 & 3 \\ -1 & 2 & -1 \\ 5 & -5 & -5 \end{bmatrix}^T = \begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{-1}{5}\begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & 5 \end{bmatrix}$$

From (1), $X = A^{-1}B$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}\begin{bmatrix} 1{,}000 \\ 1{,}500 \\ 600 \end{bmatrix}$$

$$= -\frac{1}{5}\begin{bmatrix} -2,00 & -1,500 & +3,000 \\ -1,000 & +3,000 & -3,000 \\ 3,000 & -1,500 & -3,000 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} -500 \\ -1,000 \\ -1,500 \end{bmatrix} = \begin{bmatrix} 100 \\ 200 \\ 300 \end{bmatrix}$$

$\Rightarrow \quad x = ₹\,100;$

$y = ₹\,200$

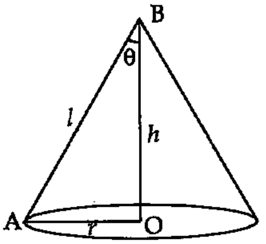
and $\quad z = ₹\,300.$

A part from the three values, Discipline, Politeness and Punctuality, another value for award, should be Hard Work. **Ans.**

24. Show that the semi-vertical angle of the cone of the maximum volume and of given slant height is $\cos^{-1}\frac{1}{\sqrt{3}}$. **[6]**

Solution : Let θ be the semi-vertical angle of a cone, h its height, r base radius and slant height:

$$l = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \quad \text{(Given)}$$



Then from ΔOAB,

$$r = l\sin\theta,\ h = l\cos\theta$$

Let V be the volume of the cone.

Now, $\quad V = \frac{1}{3}\pi r^2 h$

$$= \frac{1}{3}\pi l^2 \sin^2\theta . l\cos\theta$$

$\therefore \quad V = \frac{1}{3}\pi l^3 \sin^2\theta . \cos\theta$

Differentiating w.r. t. θ, we get

$$\frac{dV}{d\theta} = \frac{1}{3}\pi l^3\,(2\sin\theta.\cos\theta.\cos\theta - \sin^2\theta.\sin\theta)$$

$$= \frac{1}{3}\pi l^3 \sin\theta(2\cos^2\theta - \sin^2\theta)$$

And again differentiating, we get

$$\frac{d^2V}{d\theta^2} = \frac{1}{3}\pi l^3\ [\cos\theta\,(2\cos^2\theta - \sin^2\theta) + \sin\theta\,(-4\cos\theta\sin\theta - 2\sin\theta\cos\theta)]$$

For maxima or minima,

$$\frac{dV}{d\theta} = 0$$

$\Rightarrow \quad \sin\theta\,(2\cos^2\theta - \sin^2\theta) = 0$

$\Rightarrow \quad \sin\theta = 0 \text{ or } 2\cos^2\theta - \sin^2\theta = 0$

$[\because \theta = 0 \text{ not possible}]$

$\Rightarrow \quad 2\cos^2\theta - (1-\cos^2\theta) = 0$

$\Rightarrow \quad 2\cos^2\theta - 1 + \cos^2\theta = 0$

$\Rightarrow \quad \cos^2\theta = \frac{1}{3}$

$\Rightarrow \quad \cos^2\theta = \frac{1}{\sqrt{3}}$

$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

For $\quad \cos\theta = \frac{1}{\sqrt{3}}$

$\Rightarrow \quad \sin\theta = \frac{\sqrt{2}}{\sqrt{3}}$

$$\therefore \quad \frac{d^2V}{d\theta^2} = \frac{1}{3}\pi l^3\left[\frac{1}{\sqrt{3}}\left(2.\frac{1}{3} - \frac{2}{3}\right) + \frac{\sqrt{2}}{\sqrt{3}}\left(-\frac{4\times 1}{\sqrt{3}} \times \frac{\sqrt{2}}{\sqrt{3}} - \frac{2\sqrt{2}}{\sqrt{3}}\,\frac{1}{\sqrt{3}}\right)\right]$$

$$= \frac{1}{3}\pi l^3\left[\frac{1}{\sqrt{3}}(0) + \frac{\sqrt{2}}{\sqrt{3}}\left(-\frac{4\sqrt{2}}{3} - \frac{2\sqrt{2}}{3}\right)\right]$$

$$= \frac{1}{3}\pi l^3\left[\frac{\sqrt{2}}{\sqrt{3}} \times \left(-\frac{6\sqrt{2}}{3}\right)\right]$$

$$= \frac{1}{3}\pi l^3\left(-\frac{4}{\sqrt{3}}\right) < 0$$

$\therefore$ V is maximum for $\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

Hence Proved.

25. Evaluate : $\int_{\pi/6}^{\pi/3}\frac{dx}{1+\sqrt{\cot x}}$. **[6]**

Solution :

Let $\quad I = \int_{\pi/6}^{\pi/3}\frac{dx}{1+\sqrt{\cot x}}$

$$= \int_{\pi/6}^{\pi/3}\frac{dx}{1+\frac{\sqrt{\cos x}}{\sqrt{\sin x}}}$$

$$= \int_{\pi/6}^{\pi/3}\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx \quad \text{...(i)}$$

$$I = \int_{\pi/6}^{\pi/3}\frac{\sqrt{\sin(\pi/2-x)}}{\sqrt{\sin(\pi/2-x)}+\sqrt{\cos(\pi/2-x)}}dx$$

$$\left[\because \int_a^b f(x)\,dx = \int_a^b f(a+b-x)dx \text{ and } a+b = \frac{\pi}{6}+\frac{\pi}{3} = \frac{\pi}{2}\right]$$

$$\therefore \quad I = \int_{\pi/6}^{\pi/3}\frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx \quad \text{...(ii)}$$

Adding (i) and (ii), we get

$$\therefore \quad I + I = \int_{\pi/6}^{\pi/3} \left[\frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx \right]$$

$$\Rightarrow \quad 2I = \int_{\pi/6}^{\pi/3} 1.dx$$

$$2I = [x]_{\pi/6}^{\pi/3} = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$$

$$\therefore \quad I = \frac{\pi}{12}. \qquad \textbf{Ans.}$$

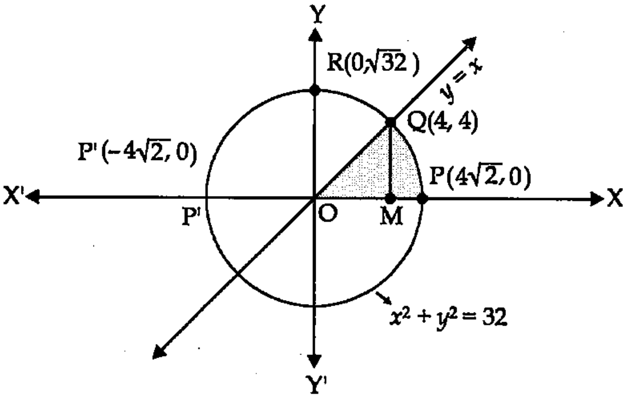
26. Find the area of the region in the first quadrant enclosed by the x-axis, the line $y = x$ and circle $x^2 + y^2 = 32$. [6]

Solution : Equation of the line and the circle are:

$$y = x \qquad ...(i)$$

and $$x^2 + y^2 = 32 \qquad ...(ii)$$

$$\Rightarrow \quad y = \sqrt{32 - x^2}$$



Circle (ii) meets x-axis at

$P(4\sqrt{2}, 0)$ and $P'(-4\sqrt{2}, 0)$.

Also (ii) meets (i) at Q in the first quadrant and Q has co-ordinates (4, 4).

The required area in first quadrant = Area OPQ

= Area under line + Area under circle

$$= \int_0^4 y_1 dx + \int_4^{4\sqrt{2}} y_2 dx$$

$$= \int_0^4 x\, dx + \int_4^{4\sqrt{2}} \sqrt{32 - x^2}\, dx$$

$$= \left[\frac{x^2}{2} \right]_0^4 + \left[\frac{x\sqrt{32 - x^2}}{2} + \frac{32}{2} \sin^{-1}\left(\frac{x}{4\sqrt{2}} \right) \right]_4^{4\sqrt{2}}$$

$$= 8 + \left[0 + 16 \sin^{-1}(1) - \frac{4.4}{2} - 16 \sin^{-1}\left(\frac{1}{\sqrt{2}} \right) \right]$$

$$= 16\left(\frac{\pi}{2} - \frac{\pi}{4} \right) = 16 \times \frac{\pi}{4} = 4\pi \text{ sq. units.} \qquad \textbf{Ans.}$$

27. Find the distance between the point (7, 2, 4) and the plane determined by the points A (2, 5, – 3), B (– 2, – 3, 5) and C (5, 3, – 3). [6]

Solution : The plane passing through A (2, 5, – 3) is

$$a(x - 2) + b(y - 5) + c(z + 3) = 0 \qquad ...(i)$$

It passes through B (– 2, – 3, 5) and C (5, 3, – 3);

So $$-4a - 8b + 8c = 0 \qquad ...(ii)$$

$$3a - 2b + 0c = 0 \qquad ...(iii)$$

Solving equation (ii) and (iii), we get

$$\frac{a}{0+16} = \frac{b}{24-0} = \frac{c}{8+24}$$

$$\Rightarrow \quad \frac{a}{2} = \frac{b}{3} = \frac{c}{4} \qquad ...(iv)$$

From (i) and (iv), the equation of the required plane is

$$2(x - 2) + 3(y - 5) + 4(z + 3) = 0$$

$$\Rightarrow \quad 2x + 3y + 4z - 7 = 0$$

$\therefore$ Distance of the point (7, 2, 4) from it

$$= \left| \frac{2 \times 7 + 3 \times 2 + 4 \times 4 - 7}{\sqrt{2^2 + 3^2 + 4^2}} \right|$$

$$= \left| \frac{14 + 6 + 16 - 7}{\sqrt{4 + 9 + 16}} \right|$$

$$= \frac{29}{\sqrt{29}} = \sqrt{29}. \qquad \textbf{Ans.}$$

OR

Find the distance of the point (– 1, – 5, – 10) from the point of intersection of the line $\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$ and the plane $\vec{r}.(\hat{i} - \hat{j} + \hat{k}) = 5$.

Solution : The given plane is

$$\vec{r}.(\hat{i} - \hat{j} + \hat{k}) = 5$$

$$\Rightarrow \quad x - y + z = 5 \qquad ...(i)$$

The given line is

$$\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$$

$$\Rightarrow \quad \frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{2} = \lambda \qquad ...(ii)$$

$\therefore$ Point (ii) is $(3\lambda + 2, 4\lambda - 1, 2\lambda + 2)$.

Let it lie on (i), so

$$3\lambda + 2 - (4\lambda - 1) + 2\lambda + 2 = 5$$

$$\Rightarrow \quad \lambda = 0$$

$\therefore$ The point of intersection of (i) and (ii) is (2, – 1, 2). Its distance from (– 1, – 5, – 10)

$$= \sqrt{(2+1)^2 + (-1+5)^2 + (2+10)^2}$$

$$= \sqrt{9 + 16 + 144} = \sqrt{169} = 13. \qquad \textbf{Ans.}$$

28. A dealer in rural area wishes to purchase a number of sewing machines. He has only ₹ 5,760 to invest and has space for at most 20 items for storage. An electronic sewing machine cost him ₹ 360 and a manually operated sewing machine ₹ 240. He can sell an electronic sewing machine at a profit of ₹ 22 and a manually operated sewing machine at a profit of ₹ 18. Assuming that he can sell all the items that he can buy, how should he invest his money in order to maximize his profit ? Make it as a LPP and solve it graphically. [6]

Solution : Let the dealer buy x electronic and y manually operated sewing machines. The LPP is Maximize

$$Z = 22x + 18y$$

Subject to constraints :

$$x + y \leq 20$$

$$360x + 240y \leq 5,760$$

$$\Rightarrow \quad 3x + 2y \leq 48$$

and $\quad x \geq 0, y \geq 0$

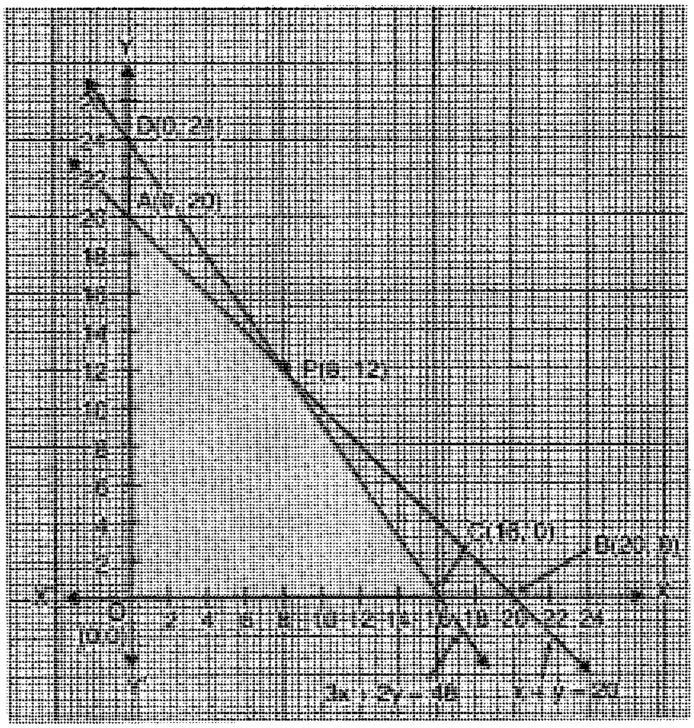
First we draw the lines AB and CD whose equations are

$$x + y = 20$$

	A	B
x	20	0
y	0	20

and $\quad 3x + 2y = 48$

	C	D
x	16	0
y	0	24



∴ The feasible region is OCPAO which is shaded in the figure.

The vertices of the feasible region are O (0, 0), C (16, 0), A (0, 20).

P is the point of intersection of the lines :

$$x + y = 20 \text{ and } 3x + 2y = 48.$$

Solving these equations, we get point P (8, 12).

The value of objective function $Z = 22x + 18y$ at these vertices are as follows :

Corner points	$Z = 22x + 18y$
At O (0, 0)	Z = 0
At C (16, 0)	Z = 352
At P (8, 12)	Z = 392 maximum
At A (0, 20)	Z = 360

∴ The maximum profit is ₹ 392 when 8 electronic and 12 manually operated machines are purchased.

29. A card from a pack of 52 playing cards is lost. From the remaining cards of the pack three cards are drawn at random (without replacement) and are found to be all spades. Find the probability of the lost card being a spade. [6]

Solution : Let E_1, E_2, E_3, E_4 and A be the events defined as below :

E_1 = the missing card is a heart card

E_2 = the missing card is a spade card

E_3 = the missing card is a club card

E_4 = the missing card is a diamond card

A = drawing three spades cards from the remaining cards.

$$P(E_1) = \frac{13}{52} = \frac{1}{4}$$

$$P(E_2) = \frac{13}{52} = \frac{1}{4}$$

$$P(E_3) = \frac{13}{52} = \frac{1}{4}$$

$$P(E_4) = \frac{13}{52} = \frac{1}{4}$$

$$P(A/E_1) = \frac{^{13}C_3}{^{51}C_3}$$

$$P(A/E_2) = \frac{^{12}C_3}{^{51}C_3}$$

$$P(A/E_3) = \frac{^{13}C_3}{^{51}C_3}$$

$$P(A/E_4) = \frac{^{13}C_3}{^{51}C_3}$$

By Bayes' theorem,

Required Probability $= P(E_2/A)$

$$= \frac{P(E_2)P(A/E_2)}{P(A/E_1).P(E_1)+P(A/E_2).P(E_2)+P(A/E_3).P(E_3)+P(A/E_4).P(E_4)}$$

$$= \frac{\frac{1}{4}\times\frac{^{12}C_3}{^{51}C_3}}{\frac{^{13}C_3}{^{51}C_3}\times\frac{1}{4}+\frac{^{12}C_3}{^{51}C_{13}}\times\frac{1}{4}+\frac{^{13}C_3}{^{51}C_3}\times\frac{1}{4}+\frac{^{13}C_3}{^{51}C_3}\times\frac{1}{4}}$$

$$= \frac{^{12}C_3}{^{12}C_3+3\times{}^{13}C_3}$$

$$= \frac{12\cdot 11\cdot 10}{12\cdot 11\cdot 10+3\cdot 13\cdot 12\cdot 11}$$

$$= \frac{10}{10+39} = \frac{10}{49}$$ **Ans.**

OR

From a lot of 15 bulbs which include 5 defectives, a sample of 4 bulbs is drawn one by one with replacement. Find the probability distribution of number of defective bulbs. Hence find the mean of the distribution.

Solution : Let D be the event of drawing a defective bulb and X denote the variable showing the number of defective bulbs in 4 draws. Then

$$P(D) = \frac{5}{15} = \frac{1}{3}$$

$$\Rightarrow \quad P(\overline{D}) = 1-\frac{1}{3} = \frac{2}{3}$$

The drawn bulb is replaced.

Hence X takes values 0, 1, 2, 3 and 4.

$\therefore$ P(X = 0) = P (Getting no defective bulb)

$$= {}^4C_0\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^4 = \left(\frac{2}{3}\right)^4 = \frac{16}{81}$$

$$P(X=1) = {}^4C_1\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^3 = \frac{32}{81}$$

$$P(X=2) = {}^4C_2\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^2 = \frac{24}{81}$$

$$P(X=3) = {}^4C_3\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right) = \frac{8}{81}$$

$$P(X=4) = {}^4C_4\left(\frac{1}{3}\right)^4 = \frac{1}{81}$$

$\therefore$ The probability distribution is

X	0	1	2	3	4
$P(X=x)$	$\frac{16}{81}$	$\frac{32}{81}$	$\frac{24}{81}$	$\frac{8}{81}$	$\frac{1}{81}$

Mean of the distribution $= \Sigma[P_i X_i]$

$$= \frac{16}{81}\times 0+\frac{32}{81}\times 1+\frac{24}{81}\times 2+\frac{8}{81}\times 3+\frac{1}{81}\times 4$$

$$= \frac{1}{81}(0+32+48+24+4) = \frac{108}{81} = \frac{4}{3}$$ **Ans.**

Mathematics 2014 (Delhi) SET II

Time allowed : 3 hours **Maximum marks : 100**

Note : Except for the following questions, all the remaining questions have been asked in previous set.

SECTION — A

9. Evaluate : $\int \cos^{-1}(\sin x)\,dx$. [1]

Solution : $\int \cos^{-1}(\sin x)\,dx$

$$= \int \cos^{-1}\left[\cos\left(\frac{\pi}{2}-x\right)\right]dx$$

$$\left(\because \cos\left(\frac{\pi}{2}-x\right)=\sin x\right)$$

$$= \int\left(\frac{\pi}{2}-x\right)dx$$

$$= \frac{\pi}{2}x-\frac{x^2}{2}+C. \quad \textbf{Ans.}$$

10. If vectors $\vec{a}$ and $\vec{b}$ are such that, $|\vec{a}|=3$, $|\vec{b}|=\frac{2}{3}$ and $\vec{a}\times\vec{b}$ is a unit vector, then write the angle between $\vec{a}$ and $\vec{b}$. [1]

Solution : Given, $|\vec{a}| = 3,\ |\vec{b}| = \frac{2}{3}$

and $|\vec{a}\times\vec{b}| = 1$

We know that $|\vec{a}\times\vec{b}| = |\vec{a}|\,|\vec{b}|\sin\theta$

$\Rightarrow \quad |\vec{a}|\,|\vec{b}|\sin\theta = 1$

$\Rightarrow \quad 3.\frac{2}{3}\sin\theta = 1$

$\Rightarrow \quad \sin\theta = \frac{1}{2}$

$\Rightarrow \quad \theta = \sin^{-1}\left(\frac{1}{2}\right)$

$\therefore \quad \theta = \frac{\pi}{6}$

Hence the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{6}$. **Ans.**

SECTION — B

19. Prove the following using properties of determinants :

$$\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3$$ [4]

Solution :

$$\text{L. H. S.} = \begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$= \begin{vmatrix} 2(a+b+c) & a & b \\ 2(a+b+c) & b+c+2a & b \\ 2(a+b+c) & a & c+a+2b \end{vmatrix}$$

Taking $2(a+b+c)$ common from C_1, we get

$$= 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$; $R_3 \to R_3 - R_1$, we get

$$= 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 0 & b+c+a & 0 \\ 0 & 0 & c+a+b \end{vmatrix}$$

Expanding along C_1, we get

$$2(a+b+c)\,[1.(b+c+a).(c+a+b)-0]$$

$$= 2(a+b+c)\,(a+b+c)^2$$

$$= 2(a+b+c)^3 = \text{R. H. S.} \quad \textbf{Hence Proved.}$$

20. Differentiate $\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$ with respect to $\sin^{-1}(2x\sqrt{1-x^2})$. [4]

Solution : Let $y = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

Put $x = \sin\theta$

$\therefore \quad y = \tan^{-1}\left(\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}\right)$

$= \tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right)$

$[\because 1-\sin^2\theta = \cos^2\theta]$

$= \tan^{-1}(\tan\theta)$

$\therefore \quad y = \theta$

Differentiating w.r.t. to θ, we get

$$\frac{dy}{d\theta} = 1 \quad \text{...(i)}$$

and let $t = \sin^{-1}(2x\sqrt{1-x^2})$

Put $x = \sin\theta$

$\therefore \quad t = \sin^{-1}(2\sin\theta\sqrt{1-\sin^2\theta})$

$= \sin^{-1}(2\sin\theta\cos\theta)$

$\therefore \quad t = 2\theta$

Differentiating w.r.t. to θ, we get

$$\frac{dt}{d\theta} = 2 \quad ...(ii)$$

Now, $$\frac{dy}{dt} = \frac{dy/d\theta}{dt/d\theta}$$

$$= \frac{1}{2} \text{ [Using (i) and (ii)]}$$ **Ans.**

21. Solve the following differential equation :

$$\text{cosec}\, x \log y \frac{dy}{dx} + x^2y^2 = 0.$$ **[4]**

Solution : The given differential equation is

$$\text{cosec}\, x \log y \frac{dy}{dx} + x^2y^2 = 0$$

Separating the variables, we get

$$\frac{\log y}{y^2}dy + \frac{x^2}{\text{cosec}\, x}dx = 0$$

On integrating, we get

$$\int \frac{\log y}{y^2}dy + \int x^2 \sin x\, dx = C$$

Put $\log y = t$

$$\Rightarrow \quad \frac{1}{y}dy = dt \text{ and } y = e^t$$

$$\Rightarrow \quad \int t.e^{-t}dt + \int x^2 \sin x\, dx = C$$

Integrate both by parts, we get

$$t.\frac{e^{-t}}{-1} - \int 1.\frac{e^{-t}}{-1}dt + x^2(-\cos x) - \int 2x(-\cos x)dx = C$$

$$\Rightarrow -te^{-t} - e^{-t} - x^2\cos x + 2x\sin x - 2\int 1.\sin x\, dx = C$$

$$\Rightarrow \quad \frac{-(1+t)}{e^t} - x^2\cos x + 2x\sin x - 2(-\cos x) = C$$

∴ The solution of the given differential equation is

$$-\left(\frac{1+\log y}{y}\right) - x^2\cos x + 2x\sin x + 2\cos x = C$$ **Ans.**

22. Show that the lines $\frac{5-x}{-4} = \frac{y-7}{4} = \frac{z+3}{-5}$ and $\frac{x-8}{7} = \frac{2y-8}{2} = \frac{z-5}{3}$ are coplanar. **[4]**

Solution : The given lines are

$$\frac{5-x}{-4} = \frac{y-7}{4} = \frac{z+3}{-5}$$

and $$\frac{x-8}{7} = \frac{2y-8}{2} = \frac{z-5}{3}$$

$$\Rightarrow \quad \frac{x-5}{4} = \frac{y-7}{4} = \frac{z+3}{-5}$$

and $$\frac{x-8}{7} = \frac{y-4}{1} = \frac{z-5}{3}$$

We know that the given lines are coplanar. If

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

Taking $$\text{L.H.S.} = \begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}$$

Here, $x_1 = 5, y_1 = 7, z_1 = -3, x_2 = 8, y_2 = 4, z_2 = 5, l_1 = 4, m_1 = 4, n_1 = -5, \quad l_2 = 7, m_2 = 1, n_2 = 3$

$$= \begin{vmatrix} 8-5 & 4-7 & 5-(-3) \\ 4 & 4 & -5 \\ 7 & 1 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & -3 & 8 \\ 4 & 4 & -5 \\ 7 & 1 & 3 \end{vmatrix}$$

$$= 3(12+5) + 3(12+35) + 8(4-28)$$

$$= 51 + 141 - 192 = 0 = \text{R.H.S.}$$

Hence, the given lines are coplanar.

Hence Proved.

SECTION — C

28. Evaluate : $\int_0^{\pi} \frac{x\tan x}{\sec x.\text{cosec}\, x}dx.$ **[6]**

Solution :

Let, $$I = \int_0^{\pi} \frac{x\tan x}{\sec x.\text{cosec}\, x}dx$$

$$= \int_0^{\pi} \frac{x\sin x}{\cos x}\cdot\cos x.\sin x\, dx$$

$$\therefore \quad I = \int_0^{\pi} x\sin^2 x\, dx \quad ...(i)$$

$$I = \int_0^{\pi} (\pi - x)\sin^2(\pi - x)dx$$

$$\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$\therefore \quad I = \int_0^\pi (\pi - x)\sin^2 x\,dx \qquad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^\pi [x\sin^2 x + (\pi - x)\sin^2 x\,dx]$$

$$= \int_0^\pi \sin^2 x(x + \pi - x)\,dx$$

$$= \pi\int_0^\pi \sin^2 x\,dx$$

$$= \pi\int_0^\pi \frac{1-\cos 2x}{2}\,dx$$

$$(\because \cos 2x = 1 - \sin^2 x)$$

$$= \frac{\pi}{2}\left[x - \frac{\sin 2x}{2}\right]_0^\pi$$

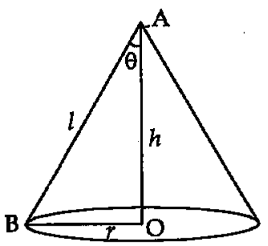
$$\Rightarrow \quad 2I = \frac{\pi}{2}[\pi - 0]$$

$$\Rightarrow \quad 2I = \frac{\pi^2}{2}$$

$$\therefore \quad I = \frac{\pi^2}{4}. \qquad \textbf{Ans.}$$

29. Prove that the semi-vertical angle of the right circular cone of given volume and least curved surface area is $\cot^{-1}\sqrt{2}$. **[6]**

Solution : Let r, h, l, V and S be the base radius, height, slant height, volume (given) and curved surface of the cone respectively. Then



$$\therefore \quad \text{Slant height } l^2 = r^2 + h^2 \qquad \text{...(i)}$$

$$V = \frac{1}{3}\pi r^2 h \text{ (given)}$$

$$\Rightarrow \quad r^2 = \frac{3V}{\pi h} \qquad \text{...(ii)}$$

and $$S = \pi r l = \pi r\sqrt{r^2 + h^2}$$

$$(\because l^2 = r^2 + h^2)$$

$$\Rightarrow \quad S^2 = \pi^2 r^2 (r^2 + h^2)$$

$$= \pi^2 . \frac{3V}{\pi h}\left(\frac{3V}{\pi h} + h^2\right)$$

[Using (ii)]

$$\therefore \quad S^2 = 3\pi V\left(\frac{3V}{\pi h^2} + h\right)$$

Fzor S to be least, S^2 is also least.

$$\therefore \quad \frac{dS^2}{dh} = 3\pi V.\left(\frac{-6V}{\pi h^3} + 1\right)$$

and $$\frac{d^2S^2}{dh^2} = 3\pi V\left(\frac{-6V}{\pi}\right)\frac{-3}{h^4}$$

$$= \frac{54V^2}{h^4}. \qquad \text{...(iii)}$$

For max. or min. S (and so S^2),

$$\frac{dS^2}{dh} = 0$$

$$\Rightarrow \quad 6V = \pi h^3$$

$$\Rightarrow \quad h = \left(\frac{6V}{\pi}\right)^{1/3} \qquad \text{...(iv)}$$

$\therefore$ From (iii),

$$\frac{d^2S^2}{dh^2} = \frac{54V^2}{h^4} > 0(+\text{ve})$$

$\Rightarrow S^2$ and therefore S is least.

In right angled Δ AOB,

$$\cot\theta = \frac{h}{r}$$

$$= \frac{h}{\sqrt{3V/\pi h}} = \sqrt{\frac{\pi}{3V}}.h^{3/2}$$

[From (ii)]

$$\Rightarrow \quad \cot\theta = \sqrt{\frac{\pi}{3V}}.\sqrt{\frac{6V}{\pi}} = \sqrt{2} \qquad \text{[From (iv)]}$$

$\therefore$ The semivertical angle,

$$\theta = \cot^{-1}(\sqrt{2}) \qquad \textbf{Ans.}$$

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Mathematics 2014 (Delhi) SET III

Time allowed : 3 hours **Maximum marks : 100**

Note : Except for the following questions, all the remaining questions have been asked in previous sets.

SECTION — A

9. Evaluate : $\int_0^{\pi/2} e^x(\sin x - \cos x)\,dx$. **[1]**

Solution :

$$\text{Let I} = \int_0^{\pi/2} e^x(\sin x - \cos x)dx$$

$$= \int_0^{\pi/2} e^x \sin x\,dx - \int_0^{\pi/2} e^x \cos x\,dx$$

On integrating IInd integral by parts, we get

$$\int_0^{\pi/2} e^x \sin x\,dx - \left[\cos x \int_0^{\pi/2} e^x dx - \int_0^{\pi/2}\left[\frac{d}{dx}\cos x \int e^x dx\right]dx\right]$$

$$= \int_0^{\pi/2} e^x \sin x\,dx - \left[\{e^x \cos x\}_0^{\pi/2} - \int_0^{\pi/2} e^x(-\sin x)dx\right]$$

$$= \int_0^{\pi/2} e^x \sin x\,dx - [e^{\pi/2}.0 - e^0.1] - \int_0^{\pi/2} e^x \sin x\,dx$$

$$= -(0-1) = 1.$$ **Ans.**

10. Write a unit vector in the direction of the sum of the vectors $\vec{a} = 2\hat{i}+2\hat{j}-5\hat{k}$ and $\vec{b} = 2\hat{i}+\hat{j}-7\hat{k}$. **[1]**

Solution :

Given, $\vec{a} = 2\hat{i}+2\hat{j}-5\hat{k}$

$\vec{b} = 2\hat{i}+\hat{j}-7\hat{k}$

$\Rightarrow \vec{a}+\vec{b} = (2\hat{i}+2\hat{j}-5\hat{k})+(2\hat{i}+\hat{j}-7\hat{k})$

$= 4\hat{i}+3\hat{j}-12\hat{k}$

and $|\vec{a}+\vec{b}| = \sqrt{4^2+3^2+(-12)^2}$

$= \sqrt{16+9+144} = \sqrt{169} = 13$

The unit vector in the direction of $\vec{a}+\vec{b}$ is

$$\frac{\vec{a}+\vec{b}}{|\vec{a}+\vec{b}|} = \frac{4\hat{i}+3\hat{j}-12\hat{k}}{13}$$

$$= \frac{4}{13}\hat{i}+\frac{3}{13}\hat{j}-\frac{12}{13}\hat{k}.$$ **Ans.**

SECTION — B

19. Using properties of determinants, prove the following :

$$\begin{vmatrix} x^2+1 & xy & xz \\ xy & y^2+1 & yz \\ xz & yz & z^2+1 \end{vmatrix} = 1+x^2+y^2+z^2$$ **[4]**

Solution : Taking L. H. S.

$$\text{Let } \Delta = \begin{vmatrix} x^2+1 & xy & xz \\ xy & y^2+1 & yz \\ xz & yz & z^2+1 \end{vmatrix}$$

Multiply C_1, C_2, C_3 by x, y, z respectively, we get

$$\Delta = \frac{1}{xyz}\begin{vmatrix} x(x^2+1) & xy^2 & xz^2 \\ x^2y & y(y^2+1) & yz^2 \\ x^2z & y^2z & (z^2+1)z \end{vmatrix}$$

Taking x, y, z common from R_1, R_2, R_3 respectively, we get

$$\Delta = \frac{1}{xyz}\cdot xyz\begin{vmatrix} x^2+1 & y^2 & z^2 \\ x^2 & y^2+1 & z^2 \\ x^2 & y^2 & z^2+1 \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} x^2+y^2+z^2+1 & y^2 & z^2 \\ x^2+y^2+z^2+1 & y^2+1 & z^2 \\ x^2+y^2+z^2+1 & y^2 & z^2+1 \end{vmatrix}$$

Taking $(x^2+y^2+z^2+1)$ common from C_1, we get

$$\Delta = (x^2+y^2+z^2+1)\begin{vmatrix} 1 & y^2 & z^2 \\ 1 & y^2+1 & z^2 \\ 1 & y^2 & z^2+1 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$; $R_3 \to R_3 - R_1$, we get

$$\Delta = (x^2+y^2+z^2+1)\begin{vmatrix} 1 & y^2 & z^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding along C_1, we get

$$(x^2+y^2+z^2+1).1.(1-0)$$
$$= x^2+y^2+z^2+1 = \text{R. H. S.}$$

Hence Proved.

20. Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ with respect to $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$, when $x \neq 0$. **[4]**

Solution : Let, $y = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

Putting $x = \tan\theta$,

$$\therefore \quad y = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$$
$$= \tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)$$
$$= \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$
$$= \tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$$
$$= \tan^{-1}\left(\tan\frac{\theta}{2}\right)$$
$$\therefore \quad y = \frac{\theta}{2}$$

Differentiating w.r. t. θ, we get

$$\frac{dy}{d\theta} = \frac{1}{2} \quad \text{...(i)}$$

and let $t = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Putting $x = \tan\theta$

$$t = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$
$$= \sin^{-1}\left(\frac{2\tan\theta}{\sec^2\theta}\right)$$
$$(\because 1+\tan^2\theta = \sec^2\theta)$$
$$= \sin^{-1}(2\sin\theta\cos\theta)$$
$$= \sin^{-1}(\sin 2\theta)$$
$$\therefore \quad t = 2\theta$$

Differentiating w.r. t. θ, we get

$$\frac{dt}{d\theta} = 2 \quad \text{...(ii)}$$

Using (i) and (ii), we get

$$\frac{dy}{dt} = \frac{dy/d\theta}{dt/d\theta} = \frac{\frac{1}{2}}{2} = \frac{1}{4}$$ **Ans.**

21. Find the particular solution of the differential equation $\frac{dy}{dx} = \frac{x(2\log x+1)}{\sin y + y\cos y}$ given that $y = \frac{\pi}{2}$ when $x = 1$. **[4]**

Solution : The given differential equation is

$$\frac{dy}{dx} = \frac{x(2\log x+1)}{\sin y+y\cos y} \quad \text{...(i)}$$

Separating the variables, we get

$$(\sin y + y\cos y)\,dy = x(2\log x+1)\,dx$$

On integrating both sides, we get

$$\int \sin y\,dy + \int y\cos y\,dy = \int x(2\log x+1)\,dx$$
$$\Rightarrow -\cos y + y.\sin y - \int 1.\sin y\,dy$$
$$= \frac{x^2}{2}(2\log x+1) - \int \frac{x^2}{2}\cdot\frac{2}{x}dx + C$$
$$\Rightarrow -\cos y + y\sin y - (-\cos y)$$
$$= \frac{x^2}{2}(2\log x+1) - \frac{x^2}{2} + C$$
$$\Rightarrow \quad y\sin y = x^2\log x + C$$

Putting $y = \frac{\pi}{2}$ and $x = 1$, we get

$$\frac{\pi}{2}\sin\frac{\pi}{2} = 1^2.\log 1 + C$$
$$\Rightarrow \quad \frac{\pi}{2}.1 = 0 + C \Rightarrow C = \frac{\pi}{2}$$

∴ The particular solution of the given differential equation is

$$y\sin y = \frac{\pi}{2} + x^2\log x$$ **Ans.**

22. Show that lines $\vec{r} = (\hat{i}+\hat{j}-\hat{k}) + \lambda(3\hat{i}-\hat{j})$ and $\vec{r} = (4\hat{i}-\hat{k}) + \mu(2\hat{i}+3\hat{k})$ interesect. Also find their point of intersection. **[4]**

olution : The equation of given lines are

$$\vec{r} = (\hat{i}+\hat{j}-\hat{k}) + \lambda(3\hat{i}-\hat{j}) \quad \text{...(i)}$$
$$\vec{r} = (4\hat{i}-\hat{k}) + \mu(2\hat{i}+3\hat{k}) \quad \text{...(ii)}$$

For lines (i) and (ii) to be intersecting

$$(\hat{i}+\hat{j}-\hat{k}) + \lambda(3\hat{i}-\hat{j}) = (4\hat{i}-\hat{k}) + \mu(2\hat{i}+3\hat{k})$$

for some values of λ and μ.

$$\Rightarrow (1+3\lambda)\hat{i} + (1-\lambda)\hat{j} - \hat{k} = (4+2\mu)\hat{i} + (3\mu-1)\hat{k}$$

On equating coefficients,

$$1+3\lambda = 4+2\mu \quad \text{...(iii)}$$
$$1-\lambda = 0$$
$$-1 = 3\mu - 1$$

$\Rightarrow \qquad \lambda = 1$ and $\mu = 0$.

These values of λ and μ satisfy the equation (iii).

$\therefore$ Lines (i) and (ii) intersect at the point whose position vector is $4\hat{i}-\hat{k}$.

Thus, the coordinates of the point of intersection are (4, 0, – 1). **Ans.**

SECTION — C

28. Evaluate : $\int_0^{\pi/2} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x} dx$ **[6]**

Solution :

Let $\quad I = \int_0^{\pi/2} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x} dx \qquad$ ...(i)

$$\therefore \quad I = \int_0^{\pi/2} \frac{\left(\frac{\pi}{2}-x\right)\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)}{\sin^4\left(\frac{\pi}{2}-x\right)+\cos^4\left(\frac{\pi}{2}-x\right)} dx$$

$$\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$= \int_0^{\pi/2} \frac{\left(\frac{\pi}{2}-x\right)\cos x \sin x}{\cos^4 x + \sin^4 x} dx$$

$$\left(\because \sin\left(\frac{\pi}{2}-x\right) = \cos x \text{ and } \cos\left(\frac{\pi}{2}-x\right) = \sin x\right)$$

$$\therefore \quad I = \int_0^{\pi/2} \frac{\left(\frac{\pi}{2}-x\right)\sin x \cos x}{\sin^4 x + \cos^4 x} dx \qquad ...(ii)$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{\frac{\pi}{2}\sin x \cos x}{\sin^4 x + \cos^4 x} dx$$

Dividing Nr and Dr by $\cos^4 x$, we get

$$I = \frac{\pi}{4}\int_0^{\pi/2} \frac{\tan x \sec^2 x}{\tan^4 x + 1} dx$$

Putting $\quad \tan^2 x = t$

$\Rightarrow 2 \tan x \sec^2 x\, dx = dt$

Also, $\quad x = 0 \Rightarrow t = 0$

and $\quad x = \frac{\pi}{2} \Rightarrow t = \infty$

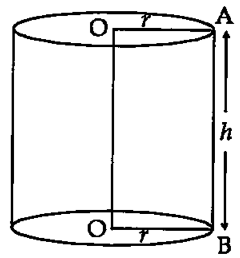
$$\therefore \quad I = \frac{\pi}{8}\int_0^{\infty} \frac{dt}{t^2+1} = \frac{\pi}{8}\left[\tan^{-1} t\right]_0^{\infty}$$

$$= \frac{\pi}{8}[\tan^{-1}\infty - \tan^{-1} 0]$$

$$= \frac{\pi}{8}\left(\frac{\pi}{2}-0\right) = \frac{\pi^2}{16}. \quad \textbf{Ans.}$$

29. Of all the closed right circular cylindrical cans of volume 128π cm^3, find the dimensions of the can which has minimum surface area. [6]

Solution : Let r cm be the base radius and h cm be the height of the closed cylindrical cans of given volume = 128π cm^3.



Then volume, $\quad V = \pi r^2 h = 128\pi$

$$\Rightarrow \quad h = \frac{128}{r^2} \qquad ...(i)$$

Also surface area, $\quad S = 2\pi r h + 2\pi r^2$

$$= 2\pi\left(r\cdot\frac{128}{r^2} + r^2\right)$$

$$\Rightarrow \quad S = 2\pi\left(r^2 + \frac{128}{r}\right)$$

Differentiating w.r.t. r, we get

$$\therefore \quad \frac{dS}{dr} = 2\pi\left(2r - \frac{128}{r^2}\right)$$

and $$\frac{d^2S}{dr^2} = 2\pi\left(2 + \frac{128\times 2}{r^3}\right)$$

For maxima or minima,

$$\frac{dS}{dr} = 0$$

$$\Rightarrow \quad 2\pi\left(2r - \frac{128}{r^2}\right) = 0$$

$\Rightarrow \quad 2r^3 = 128$

$\Rightarrow \quad r^3 = 64$

$\Rightarrow \quad r = 4$ cm

and $$\frac{d^2S}{dr^2} = 2\pi\left(2 + \frac{256}{64}\right) = 12\pi > 0$$

$\Rightarrow$ S is minimum.

The dimensions of such a can are

$r = 4$ cm

and $$h = \frac{128}{r^2} = \frac{128}{4^2} = \frac{128}{16} = 8 \text{ cm.} \qquad \textbf{Ans.}$$

Mathematics 2015 (Outside Delhi) SET I

Time allowed : 3 hours **Maximum marks : 100**

SECTION — A

1. Write the value of $\Delta = \begin{vmatrix} x+y & x+z & z+x \\ z & x & y \\ -3 & -3 & -3 \end{vmatrix}$ **[1]**

Solution : Given, $\Delta = \begin{vmatrix} x+y & x+z & z+x \\ z & x & y \\ -3 & -3 & -3 \end{vmatrix}$

Applying $R_1 \to R_1 + R_2$, we get

$$\Delta = \begin{vmatrix} x+y+z & x+y+z & x+y+z \\ z & x & y \\ -3 & -3 & -3 \end{vmatrix}$$

$$\Delta = -3(x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$$

[Taking out $(x+y+z)$ and (-3) common from R_1 and R_3 respectively]

$$\Delta = -3(x+y+z).\,0$$

[$\because R_1$ and R_3 are identical]

$\Rightarrow \quad \Delta = 0$ **Ans.**

2. Write the sum of the order and degree of the following differential equation :

$$\frac{d}{dx}\left\{\left(\frac{dy}{dx}\right)^3\right\} = 0 \qquad \textbf{[1]}$$

Solution : Given,

$$\frac{d}{dx}\left\{\left(\frac{dy}{dx}\right)^3\right\} = 0.$$

On differentiating w.r.t. x both sides, we get

$$3\left(\frac{dy}{dx}\right)^2 \times \frac{d^2y}{dx^2} = 0.$$

Since the order and degree of the differential equation is 2 and 1 respectively.

So, the sum of the order and degree is 3. **Ans.**

3. Write the integrating factor of the following differential equation :

$$(1+y^2) + (2xy - \cot y)\frac{dy}{dx} = 0. \qquad \textbf{[1]}$$

Solution : Given,

$$(1+y^2) + (2xy - \cot y)\,\frac{dy}{dx} = 0$$

$\Rightarrow \quad (1+y^2)\,dx + 2xy\,dy = \cot y\,dy$

$\Rightarrow \quad \dfrac{dx}{dy} + \left(\dfrac{2y}{1+y^2}\right)x = \dfrac{\cot y}{1+y^2}$

This is a linear differential equation of the form

$$\frac{dx}{dy} + Rx = S,$$

where $\quad R = \dfrac{2y}{1+y^2}$

and $\quad S = \dfrac{\cot y}{1+y^2}$

$\therefore$ Integrating factor $= e^{\int R\,dy} = e^{\int \frac{2y}{1+y^2}dy}$

$= e^{\log(1+y^2)}$

$= 1+y^2.$ **Ans.**

4. If $\hat{a}$, $\hat{b}$ and $\hat{c}$ are mutually perpendicular unit vectors, then find the value of $|2\hat{a} + \hat{b} + \hat{c}|$. [1]

Solution : Let $x = |2\hat{a} + \hat{b} + \hat{c}|$

$\therefore \quad x^2 = |2\hat{a} + \hat{b} + \hat{c}|^2$

$= 4|\hat{a}|^2 + |\hat{b}|^2 + |\hat{c}|^2 + 2(2\hat{a}.\hat{b} + \hat{b}.\hat{c} + 2\hat{c}.\hat{a})$

Since, $\hat{a}$, $\hat{b}$ and $\hat{c}$ are mutually perpendicular unit vectors.

$\therefore \quad \hat{a}.\hat{b} = \hat{b}.\hat{c} = \hat{c}.\hat{a} = 0$

So, $\quad x^2 = 4 \times 1 + 1 + 1 + 2\,(2 \times 0 + 0 + 2 \times 0)$

$\Rightarrow \quad x^2 = 6$

$\therefore \quad x = \sqrt{6}$

Hence, $|2\hat{a} + \hat{b} + \hat{c}| = \sqrt{6}.$ **Ans.**

5. Write a unit vector perpendicular to both the vectors $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + \hat{j}$. [1]

Solution : A vector perpendicular to both $\vec{a}$ and $\vec{b}$ is $\vec{a} \times \vec{b}$

$\therefore$ Unit vector $\perp$ to $\vec{a}$ and $\vec{b}$ $= \dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= \hat{i}(0-1) - \hat{j}(0-1) + \hat{k}(1-1)$$

$$= -\hat{i} + \hat{j}$$

$$|\vec{a} \times \vec{b}| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$$

$\therefore$ Required unit vector $= \dfrac{-\hat{i}+\hat{j}}{\sqrt{2}}$ **Ans.**

6. The equations of a line are $5x - 3 = 15y + 7 = 3 - 10z$. Write the direction cosines of the line. [1]

Solution : Given line is $5x-3=15y+7=3-10z$

Rewritting the eq. in standard form :

$$5\left(x-\frac{3}{5}\right) = 15\left(y+\frac{7}{15}\right) = -10\left(z-\frac{3}{10}\right)$$

i.e., $$\frac{x-\frac{3}{5}}{\frac{1}{5}} = \frac{y+\frac{7}{15}}{\frac{1}{15}} = \frac{z-\frac{3}{10}}{\frac{-1}{10}}$$

Thus, the direction ratios of the line are

$\frac{1}{5}, \frac{1}{15}, \frac{-1}{10}$ *i.e.,* $6, 2, -3$.

Hence, its direction cosines are

$$\pm\frac{6}{\sqrt{6^2+2^2+(-3)^2}}, \pm\frac{2}{\sqrt{6^2+2^2+(-3)^2}},$$

$$\pm\frac{-3}{\sqrt{6^2+2^2+(-3)^2}} \text{ i.e., } \pm\frac{6}{7}, \pm\frac{2}{7}, \mp\frac{3}{7}$$

i.e., $\frac{6}{7}, \frac{2}{7}, \frac{-3}{7}$ or $\frac{-6}{7}, \frac{-2}{7}, \frac{3}{7}$. **Ans.**

SECTION — B

7. To promote the making of toilets for women, an organisation tried to generate awareness through

(i) house calls (ii) letters, and (iii) announcements. The cost for each mode per attempt is given below :

(i) ₹ 50 (ii) ₹ 20 (iii) ₹ 40

The number of attempts made in three villages X, Y and Z are given below :

	(i)	(ii)	(iii)
X	400	300	100
Y	300	250	75
Z	500	400	150

Find the total cost incurred by the organisation for the three villages separately, using matrices.

Write one value generated by the organisation in the society. [4]

Solution : The number of attempts made in three villages X, Y and Z can be represented by the 3×3 matrix.

$$X = \begin{bmatrix} 400 & 300 & 100 \\ 300 & 250 & 75 \\ 500 & 400 & 150 \end{bmatrix}$$

and the cost for each mode per attempt can be represented by the 3×1 matrix.

$$Y = \begin{bmatrix} 50 \\ 20 \\ 40 \end{bmatrix}$$

$\therefore$ By matrix multiplication the cost incurred by the organisation for the three villages.

$$XY = \begin{bmatrix} 400 & 300 & 100 \\ 300 & 250 & 75 \\ 500 & 400 & 150 \end{bmatrix}\begin{bmatrix} 50 \\ 20 \\ 40 \end{bmatrix}$$

$$\Rightarrow \quad XY = \begin{bmatrix} 30,000 \\ 23,000 \\ 39,000 \end{bmatrix}$$

Hence the total cost incurred by the organisation for the three villages separately are ₹ 30,000, ₹ 23,000 and ₹ 39,000.

The organisation in the society generated the value of cleanliness for the women welfare. **Ans.**

8. Solve for x :

$\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}\frac{8}{31}$ [4]

Solution : Given, $\tan^{-1}(x+1) + \tan^{-1}(x-1)$

$$= \tan^{-1}\frac{8}{31}$$

$$\Rightarrow \quad \tan^{-1}\frac{x+1+x-1}{1-(x+1)(x-1)} = \tan^{-1}\frac{8}{31}$$

$$\left[\because \tan^{-1}A + \tan^{-1}B = \tan^{-1}\frac{A+B}{1-AB}\right]$$

$$\Rightarrow \quad \tan^{-1}\frac{2x}{2-x^2} = \tan^{-1}\frac{8}{31}$$

$$\Rightarrow \quad \frac{2x}{2-x^2} = \frac{8}{31}$$

$\Rightarrow \quad 62x = 16 - 8x^2$

$\Rightarrow \quad 4x^2 + 31x - 8 = 0$

Solving the quadratic equation, we get

$$(4x - 1)(x + 8) = 0$$

$\Rightarrow \quad x = \frac{1}{4}$ or -8.

Since, $x = -8$ doesn't satisfy the given equation. So neglecting it.

$\therefore \quad x = \frac{1}{4}$. **Ans.**

OR

Prove the following :

$$\cot^{-1}\left(\frac{xy+1}{x-y}\right)+\cot^{-1}\left(\frac{yz+1}{y-z}\right)+\cot^{-1}\left(\frac{zx+1}{z-x}\right)$$

$$= 0\ (0 < xy, yx, zx < 1).$$

Solution : L. H. S.

$$= \cot^{-1}\left(\frac{xy+1}{x-y}\right)+\cot^{-1}\left(\frac{yz+1}{y-z}\right)+\cot^{-1}\left(\frac{zx+1}{z-x}\right)$$

$$= \tan^{-1}\left(\frac{x-y}{xy+1}\right)+\tan^{-1}\left(\frac{y-z}{yz+1}\right)+\tan^{-1}\left(\frac{z-x}{zx+1}\right)$$

$$\left[\because \tan^{-1} x = \cot^{-1}\frac{1}{x}\right]$$

$= \tan^{-1}x - \tan^{-1}y + \tan^{-1}y - \tan^{-1}z + \tan^{-1}z - \tan^{-1}x = 0 =$ R. H. S. **Hence Proved.**

9. Using properties of determinants, prove the following:

$$\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2. \quad [4]$$

Solution :

$$\text{Let } \Delta = \begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix}$$

Taking a, b and c common from C_1, C_2 and C_3 respectively.

$$\Delta = abc\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$

$$\Delta = abc\begin{vmatrix} 2(a+b) & 2(b+c) & 2(a+c) \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$$

Taking 2 common from R_1,

$$\Delta = 2abc\begin{vmatrix} a+b & b+c & a+c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$$

Applying $R_1 \to R_1 - R_2$ and $R_3 \to R_3 - R_1$

$$\Delta = 2abc\begin{vmatrix} 0 & c & c \\ a+b & b & a \\ -a & 0 & -a \end{vmatrix}$$

Now, taking c and a common from R_1 and R_3

$$\Delta = 2a^2bc^2\begin{vmatrix} 0 & 1 & 1 \\ a+b & b & a \\ -1 & 0 & -1 \end{vmatrix}$$

Expanding along R_1,

$\Delta = 2a^2bc^2\,[0(-b-0) - 1\{-(a+b)+a\} + 1(0+b)]$

$= 2a^2bc^2\,[0 - (-a - b + a) + b]$

$= 2a^2bc^2\,[0 + b + b]$

$= 2a^2bc^2\,[2b]$

$= 4a^2b^2c^2 =$ R. H. S. **Hence Proved.**

10. Find the adjoint of the matrix :

$$A = \begin{pmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

and hence show that A. (adj A) = |A| I_3. **[4]**

Solution : We have,

$$A = \begin{pmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

$$A_{11} = \begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} = 1 - 4 = -3$$

$$A_{12} = -\begin{vmatrix} 2 & -2 \\ 2 & 1 \end{vmatrix} = -(2+4) = -6$$

$$A_{13} = \begin{vmatrix} 2 & 1 \\ 2 & -2 \end{vmatrix} = -4 - 2 = -6$$

$$A_{21} = -\begin{vmatrix} -2 & -2 \\ -2 & 1 \end{vmatrix} = -(-2-4) = 6$$

$$A_{22} = \begin{vmatrix} -1 & -2 \\ 2 & 1 \end{vmatrix} = -1 + 4 = 3$$

$$A_{23} = -\begin{vmatrix} -1 & -2 \\ 2 & -2 \end{vmatrix} = -(2+4) = -6$$

$$A_{31} = \begin{vmatrix} -2 & -2 \\ 1 & -2 \end{vmatrix} = +4+2 = +6$$

$$A_{32} = -\begin{vmatrix} -1 & -2 \\ 2 & -2 \end{vmatrix} = -(2+4) = -6$$

$$A_{33} = \begin{vmatrix} -1 & -2 \\ 2 & 1 \end{vmatrix} = -1+4 = +3$$

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

$$\text{adj } A = \begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{bmatrix}$$

$$|A| = \begin{vmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{vmatrix}$$

$$= -1(1-4)+2(2+4)-2(-4-2)$$

$$= -1(-3)+2(6)-2(-6)$$

$$= 3+12+12 = 27$$

Now L. H. S. = A (adj A)

$$= \begin{pmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}\begin{pmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 27 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 27 \end{pmatrix}$$

$$= 27\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$= |A|.I_3 =$ R. H. S. **Hence Proved.**

11. Show that the function $f(x) = |x-1| + |x+1|$, for all $x \in R$, is not differentiable at the points $x = -1$ and $x = 1$. **[4]**

Solution : Given,

$$f(x) = |x-1| + |x+1|$$

$$= \begin{cases} -(x-1)-(x+1) = -2x, & x < -1 \\ -(x-1)+(x+1) = 2, & -1 \le x < 1 \\ (x-1)+(x+1) = 2x, & x \ge 1 \end{cases}$$

Now differentiability at $x = -1$

$$\text{LHD} = \lim_{h\to 0} \frac{f(-1-h)-f(-1)}{-h}$$

$$= \lim_{h\to 0} \frac{-2(-1-h)-2}{-h}$$

$$= \lim_{h\to 0} \frac{2+2h-2}{-h}$$

$$= \lim_{h\to 0} \frac{2h}{-h}$$

$$= \lim_{h\to 0} -2$$

$$= -2$$

$$\text{RHD} = \lim_{h\to 0} \frac{f(-1+h)-f(-1)}{h}$$

$$= \lim_{h\to 0} \frac{2-2}{h} = \lim_{h\to 0} 0$$

$$= 0$$

Since (LHD) $\neq$ (RHD)

$\therefore f(x)$ is not differentiable at $x = -1$.

Now differentiability at $x = 1$

$$(\text{LHD at } x = 1) = \lim_{h\to 0} \frac{f(1-h)-f(1)}{-h}$$

$$= \lim_{h\to 0} \frac{2-2}{-h}$$

$$= \lim_{h\to 0} 0$$

$$= 0$$

$$(\text{RHD at } x = 1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0} \frac{2(1+h)-2}{h}$$

$$= \lim_{h\to 0} \frac{2+2h-2}{h}$$

$$= \lim_{h\to 0} \frac{2h}{h} = \lim_{h\to 0} 2$$

$$= 2$$

Since LHD $\neq$ RHD

$\therefore f(x)$ is not differentiable at $x = 1$, also.

Hence Proved.

12. If $y = e^{m\sin^{-1}x}$, then show that

$$\left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - m^2y = 0.$$ **[4]**

Solution : Given,

$$y = e^{m\sin^{-1}x}, \qquad \text{...(i)}$$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = e^{m\sin^{-1}x} \times \frac{m}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad \sqrt{1-x^2}\frac{dy}{dx} = m e^{m\sin^{-1}x}$$

Again differentiating, w.r.t. x, we get

$$\sqrt{1-x^2}\frac{d^2y}{dx^2}+\frac{dy}{dx}\times\left(\frac{-x}{\sqrt{1-x^2}}\right) = \frac{m^2 e^{m\sin^{-1}x}}{\sqrt{1-x^2}}$$

Multiplying both sides by $\sqrt{1-x^2}$, we get

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx} = m^2 e^{m\sin^{-1}x}$$

$$\Rightarrow \left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-m^2y = 0 \quad \text{[using (i)]}$$

Hence Proved.

13. If $f(x)=\sqrt{x^2+1}$; $g(x)=\frac{x+1}{x^2+1}$ and $h(x)=2x-3$, then find $f'[h'\{g'(x)\}]$. **[4]**

Solution : Given,

$$f(x) = \sqrt{x^2+1},\ g(x)=\frac{x+1}{x^2+1}$$

and $\quad h(x) = 2x-3$

Now, $\quad f'(x) = \frac{2x}{2\sqrt{x^2+1}}=\frac{x}{\sqrt{x^2+1}}$

$$g'(x) = \frac{\left(x^2+1\right)-(x+1)2x}{\left(x^2+1\right)^2}=\frac{1-2x-x^2}{\left(x^2+1\right)^2}$$

and $\quad h'(x) = 2.$

$$\therefore f'[h'\{g'(x)\}] = f'\left[h'\left\{\frac{1-2x-x^2}{\left(x^2+1\right)^2}\right\}\right]$$

$$= f'[2] \qquad [\because h'(x)=2]$$

$$= \frac{2}{\sqrt{2^2+1}}=\frac{2}{\sqrt{5}}.$$ **Ans.**

14. Evaluate : $\int(3-2x).\sqrt{2+x-x^2}\,dx.$ **[4]**

Solution :

Let, $\quad I = \int(3-2x).\sqrt{2+x-x^2}\,dx.$

Let $\quad 3-2x = \lambda\frac{d}{dx}\left(2+x-x^2\right)+\mu$

$\Rightarrow \quad 3-2x = \lambda(1-2x)+\mu \quad$ ...(i)

$\Rightarrow \quad 3-2x = (-2\lambda)x+\lambda+\mu$

Equating the coefficients of like terms, we get

$$-2\lambda = -2$$

$\Rightarrow \quad \lambda = 1$

and $\quad \lambda+\mu = 3$

$\Rightarrow \quad \mu = 3-1=2$

$\therefore$ Equation. (i) becomes $3-2x=(1-2x)+2$

$$I = \int\{(1-2x)+2\}\sqrt{2+x-x^2}\,dx$$

$$\Rightarrow I = \int(1-2x)\sqrt{2+x-x^2}\,dx+2\int\sqrt{2+x-x^2}\,dx$$

$\Rightarrow I = I_1+2I_2 \quad$ ...(ii)

$$\Rightarrow I_1 = \int(1-2x)\sqrt{2+x-x^2}\,dx$$

Let $\quad 2+x-x^2 = t$

$\Rightarrow \quad (1-2x)\,dx = dt$

$\therefore \quad I_1 = \int\sqrt{t}\,dt$

$$\Rightarrow \quad I_1 = \frac{t^{1/2+1}}{\frac{1}{2}+1}+C_1$$

$$\Rightarrow \quad I_1 = \frac{2}{3}t^{3/2}+C_1$$

$$\Rightarrow \quad I_1 = \frac{2}{3}\left(2+x-x^2\right)^{3/2}+C_1 \quad \text{...(iii)}$$

and $\quad I_2 = \int\sqrt{2+x-x^2}\,dx$

$$\Rightarrow \quad I_2 = \int\sqrt{2+\frac{1}{4}-\left(x^2-x+\frac{1}{4}\right)}dx$$

$$\Rightarrow \quad I_2 = \int\sqrt{\left(\frac{3}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}\,dx$$

$$\Rightarrow \quad I_2 = \frac{1}{2}\left(x-\frac{1}{2}\right)\sqrt{\left(\frac{3}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}+\frac{\frac{9}{4}}{2}\sin^{-1}\left(\frac{x-\frac{1}{2}}{\frac{1}{2}}\right)+C_2$$

$$\Rightarrow \quad I_2 = \frac{1}{2}\left(x-\frac{1}{2}\right)\sqrt{2+x-x^2}+\frac{9}{8}\sin^{-1}\left(\frac{2x-1}{3}\right)+C_2 \quad \text{...(iv)}$$

From (ii), (iii) and (iv), we get

$$I = \frac{2}{3}\left(2+x-x^2\right)^{3/2}+\left(x-\frac{1}{2}\right)\sqrt{2+x-x^2}+\frac{9}{4}\sin^{-1}\left(\frac{2x-1}{3}\right)+C,$$

where $\quad C = C_1+C_2.$ **Ans.**

OR

Evaluate : $\int\frac{x^2+x+1}{\left(x^2+1\right)(x+2)}dx.$

Solution : Let $\quad I = \int \frac{x^2+x+1}{(x^2+1)(x+2)}dx$

$$I = \int\left(\frac{x^2+1}{(x^2+1)(x+2)}+\frac{x}{(x^2+1)(x+2)}\right)dx$$

$$\Rightarrow \quad I = \int\frac{1}{x+2}dx+\int\frac{x}{(x^2+1)(x+2)}dx$$

$$\Rightarrow \quad I = \log|x+2|+\int\frac{x}{(x^2+1)(x+2)}dx \quad ...(i)$$

Let $\quad \frac{x}{(x^2+1)(x+2)} = \frac{A}{x+2}+\frac{Bx+C}{x^2+1}$

$$x = A(x^2+1)+(Bx+C)(x+2) \quad ...(ii)$$

Putting $x+2=0$

$\Rightarrow \quad x = -2$ in (ii), we get $-2 = 5A$

$\Rightarrow \quad A = \frac{-2}{5}$

Putting $\quad x = 0$ and $x = 1$ in (ii), we get

$0 = A + 2C$

$\Rightarrow \quad C = \frac{1}{5}$

$1 = 2A + 3B + 3C$

$\Rightarrow \quad 3B = \frac{6}{5}$

$\Rightarrow \quad B = \frac{2}{5}$

$\therefore \quad I = \log|x+2| - \frac{2}{5}\int\frac{dx}{x+2}+\int\frac{\frac{2}{5}x+\frac{1}{5}}{x^2+1}dx$

$\Rightarrow \quad I = \log|x+2| - \frac{2}{5}\int\frac{dx}{x+2}+\frac{2}{5}\int\frac{xdx}{x^2+1}+\frac{1}{5}\int\frac{1}{x^2+1}dx$

$\Rightarrow \quad I = \log|x+2| - \frac{2}{5}\log|x+2|+\frac{1}{5}\log|x^2+1|+\frac{1}{5}\tan^{-1}x+C$

$\therefore \quad I = \frac{3}{5}\log|x+2|+\frac{1}{5}\log|x^2+1|+\frac{1}{5}\tan^{-1}x+C.$ **Ans.**

15. Find $\left(\int_0^{\pi/4}\frac{dx}{\cos^3 x\sqrt{2\sin 2x}}.\right)$. **[4]**

Solution : Let $\quad I = \int_0^{\pi/4}\frac{dx}{\cos^3 x\sqrt{2\sin 2x}}$.

Multiplying and dividing denominator by cos x

$$\int_0^{\pi/4}\frac{dx}{\cos^4 x\frac{\sqrt{2\sin 2x}}{\cos x}} = \int_0^{\pi/4}\frac{\sec^4 x\,dx}{\sqrt{\frac{2\sin 2x}{\cos^2 x}}}$$

$$I = \int_0^{\pi/4}\frac{(1+\tan^2 x)\sec^2 x}{2\sqrt{\frac{\sin x\cos x}{\cos^2 x}}}dx$$

$[\because \sin 2x = 2\sin x\cos x]$

$$\Rightarrow \quad I = \int_0^{\pi/4}\frac{(1+\tan^2 x)\sec^2 x}{2\sqrt{\tan x}}dx$$

Putting $\quad \tan x = t$

$\Rightarrow \quad \sec^2 x\,dx = dt$, we get $\quad \left(\begin{matrix}\because \text{when } x=0, t=0\\ \text{and } x=\frac{\pi}{4}, t=1\end{matrix}\right)$

$$\therefore \quad I = \int_0^1\frac{(1+t^2)dt}{2\sqrt{t}}$$

$$\Rightarrow \quad I = \frac{1}{2}\int_0^1\left(\frac{1}{\sqrt{t}}+t^{\frac{3}{2}}\right)dt$$

$$\Rightarrow \quad I = \frac{1}{2}\left[2\sqrt{t}+\frac{2}{5}t^{\frac{5}{2}}\right]_0^1$$

$$\Rightarrow \quad I = \left[\sqrt{t}+\frac{1}{5}t^{\frac{5}{2}}\right]_0^1$$

$$\Rightarrow \quad I = \left[\sqrt{1}+\frac{1}{5}1^{\frac{5}{2}}\right]-\left[\sqrt{0}+\frac{1}{5}0^{\frac{5}{2}}\right]$$

$\therefore \quad I = \frac{6}{5}.$ **Ans.**

16. Find $\left(\int\frac{\log x}{(x+1)^2}dx.\right)$. **[4]**

Solution : Let $\quad I = \int\frac{\log x}{(x+1)^2}dx$

$$= \int \underset{I}{\log x}\cdot\underset{II}{\frac{1}{(x+1)^2}}dx$$

Integrating by parts

$$I = \log x\int\frac{dx}{(x+1)^2}-\int\left(\frac{d}{dx}\log x\int\frac{dx}{(x+1)^2}\right)dx$$

$\Rightarrow I = -\frac{\log x}{(x+1)}+\int\frac{1}{x(x+1)}dx \quad \left[\because \frac{d}{dx}\frac{1}{x^2}=-\frac{1}{x}\right]$

$$= -\frac{\log x}{(x+1)} + \int \frac{(x+1)-x}{x(x+1)} dx$$

[Add and subtract x]

$$\Rightarrow \quad I = -\frac{\log x}{(x+1)} + \int \left(\frac{1}{x} - \frac{1}{x+1}\right) dx$$

$$\Rightarrow \quad I = -\frac{\log x}{(x+1)} + \log|x| - \log|x+1| + C$$

$$\therefore \quad I = \log\left|\frac{x}{x+1}\right| - \frac{\log x}{(x+1)} + C. \qquad \textbf{Ans.}$$

17. If $\vec{a} = \hat{i} + 2\hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} + \hat{j}$ and $\vec{c} = 3\hat{i} - 4\hat{j} - 5\hat{k}$ then find a unit vector perpendicular to both of the vectors $(\vec{a} - \vec{b})$ and $(\vec{c} - \vec{b})$. [4]

Solution : Here,

$$(\vec{a} - \vec{b}) = \hat{i} + 2\hat{j} + \hat{k} - 2\hat{i} - \hat{j}$$
$$= -\hat{i} + \hat{j} + \hat{k}$$

and
$$(\vec{c} - \vec{b}) = 3\hat{i} - 4\hat{j} - 5\hat{k} - 2\hat{i} - \hat{j}$$
$$= \hat{i} - 5\hat{j} - 5\hat{k}$$

So, required unit vector $\bar{r} = \dfrac{(\vec{a}-\vec{b}) \times (\vec{c}-\vec{b})}{|(\vec{a}-\vec{b}) \times (\vec{c}-\vec{b})|}$,

where

$$\vec{r} = (\hat{a} - \hat{b}) \times (\vec{c} - \vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 1 \\ 1 & -5 & -5 \end{vmatrix}$$

$$= \hat{i}(-5+5) - \hat{j}(5-1) + \hat{k}\ (5-1)$$
$$= -4\hat{j} + 4\hat{k}$$

$$\text{Hence, } \hat{r} = \frac{4\hat{k} - 4\hat{j}}{\sqrt{4^2 + 4^2}} = \left(\frac{\hat{k} - \hat{j}}{\sqrt{2}}\right). \qquad \textbf{Ans.}$$

18. Find the equation of a line passing through the point (1, 2, – 4) and perpendicular to two lines

$$\vec{r} = (8\hat{i} - 19\hat{j} + 10\hat{k}) + \lambda(3\hat{i} - 16\hat{j} + 7\hat{k})$$

and $\vec{r} = (15\hat{i} + 29\hat{j} + 5\hat{k}) + \mu\ (3\hat{i} + 8\hat{j} - 5\hat{k})$.[4]

Solution : Let the direction ratios of required line be a, b, c, since, the line is perpendicular to

$$\vec{r} = (8\hat{i} - 19\hat{j} + 10\hat{k}) + \lambda(3\hat{i} - 16\hat{j} + 7\hat{k})$$

and
$$\vec{r} = (15\hat{i} + 29\hat{j} + 5\hat{k}) + \mu\ (3\hat{i} + 8\hat{j} - 5\hat{k})$$

$$\therefore \quad 3a - 16b + 7c = 0$$

and
$$3a + 8b - 5c = 0.$$

Solving by cross multiplication, we get

$$\frac{a}{80-56} = \frac{b}{21+15} = \frac{c}{24+48}$$

$$\Rightarrow \quad \frac{a}{24} = \frac{b}{36} = \frac{c}{72}$$

$\therefore$ the direction ratios of line : 2, 3, 6.

Hence, required line through the point (1, 2, – 4) is

$$\vec{r} = \vec{a} + \lambda \vec{b}$$

$$\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda\ (2\hat{i} + 3\hat{j} + 6\hat{k}) \ \textbf{Ans.}$$

OR

Find the equation of the plane passing through the points (–1, 2, 0), (2, 2, –1) and parallel to the line $\dfrac{x-1}{1} = \dfrac{2y+1}{2} = \dfrac{z+1}{-1}$.

Solution : The equation of a plane passing through (–1, 2, 0) is

$$a\ (x+1) + b\ (y-2) + c\ (z-0) = 0 \qquad \text{...(i)}$$

It passes through (2, 2, –1)

$$\therefore \quad a(2+1) + b(2-2) + c(-1-0) = 0$$
$$3a + 0b - c = 0 \qquad \text{...(ii)}$$

The given line is

$$\frac{x-1}{1} = \frac{2y+1}{2} = \frac{z+1}{-1}$$

i. e.
$$\frac{x-1}{1} = \frac{y+\frac{1}{2}}{1} = \frac{z+1}{-1}$$

$\therefore$ *d. r'. s* of line are 1, 1, –1

The plane (i) is parallel to the given line

$$a + b - c = 0 \qquad \text{...(iii)}$$

Solving (ii) and (iii) by cross multiplication, we get

$$\frac{a}{1} = \frac{b}{2} = \frac{c}{3}$$

i.e., direction ratios of normal to the plane are 1, 2, 3.

$$\therefore \quad 1(x+1) + 2(y-2) + 3(z-0) = 0$$

i.e.,
$$x + 2y + 3z = 3. \qquad \textbf{Ans.}$$

19. Three cards are drawn successively with replacement from a well shuffled pack of 52 cards. Find the probability distribution of the number of spades. Hence find the mean of the distribution. [4]

Solution : Let X denote the number of spades when three cards are drawn, then, X is a random variable that can take values 0, 1, 2, 3.

Let E be the event when spade card is drawn,

$$p = P(E) = \frac{13}{52} = \frac{1}{4}$$

$$q = 1 - p = 1 - \frac{1}{4} = \frac{3}{4}$$

P(X = 0) = Probability of getting no spade

$$= {}^3C_0\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^3 = \frac{27}{64}$$

P(X = 1) = Probability of getting one spade

$$= {}^3C_1\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^{3-1} = \frac{27}{64}$$

P(X = 2) = Probability of getting two spades

$$= {}^3C_2\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^{3-2} = \frac{9}{64}$$

P(X = 3) = Probability of getting three spades

$$= {}^3C_3\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^{3-3} = \frac{1}{64}$$

Thus, the probability distribution of random variable X is given by

X	0	1	2	3
P(X)	$\frac{27}{64}$	$\frac{27}{64}$	$\frac{9}{64}$	$\frac{1}{64}$

$\therefore$ Mean $(\overline{X}) = \Sigma XP(X)$

$$= 0\times\frac{27}{64} + 1\times\frac{27}{64} + 2\times\frac{9}{64} + 3\times\frac{1}{64}$$

$$= \frac{48}{64} = \frac{3}{4}.$$ **Ans.**

OR

For 6 trials of an experiment, let X be a binomial variate which satisfies the relation 9P(X = 4) = P(X = 2). Find the probability of success.

Solution : Let p denote the probability of getting success and q be the probability of failure.

Since, $P(x = r) = {}^nC_r\, p^r\, q^{n-r}$

$\therefore$ $P(x = 4) = {}^6C_4\, p^4\, q^{6-4}$

and $P(x = 2) = {}^6C_2\, p^2\, q^{6-2}$

We have $9P(X = 4) = P(X = 2)$

$\Rightarrow$ $9\,{}^6C_4\, p^4\, q^{6-4} = {}^6C_2 p^2 q^4$ $[\because {}^6C_4 = {}^6C_2]$

$\Rightarrow$ $9p^2 = q^2$

$\Rightarrow$ $9p^2 = (1-p)^2$ $[\because p + q = 1]$

$\Rightarrow$ $9p^2 = 1^2 + p^2 - 2p$

$\Rightarrow$ $9p^2 - p^2 + 2p - 1 = 0$

$\Rightarrow$ $8p^2 + 2p - 1 = 0$

$\Rightarrow$ $8p^2 + 4p - 2p - 1 = 0$

$\Rightarrow$ $(4p-1)(2p+1) = 0$

$\therefore$ $p = \frac{1}{4}, \frac{-1}{2}$

Since probability can not be – ve

$\therefore$ $p = \frac{1}{4}$

Hence, the probability of success $= \frac{1}{4}$. **Ans.**

SECTION — C

20. Consider $f : R_+ \to [-9, \infty)$ given by $f(x) = 5x^2 + 6x - 9$. Prove that f is invertible with

$$f^{-1}(y) = \left(\frac{\sqrt{54+5y}-3}{5}\right).$$ **[6]**

Solution : To prove f is invertible we have to prove that f is one-one and onto.

For one-one

Let $x_1, x_2 \in R_+$, then

$$f(x_1) = f(x_2)$$

$\Rightarrow$ $5x_1^2 + 6x_1 - 9 = 5x_2^2 + 6x_2 - 9$

$\Rightarrow$ $5(x_1^2 - x_2^2) + 6(x_1 - x_2) = 0$

$\Rightarrow$ $(x_1 - x_2)(5x_1 + 5x_2 + 6) = 0$

$\Rightarrow$ $x_1 - x_2 = 0$ as $5x_1 + 5x_2 + 6 \neq 0$

$\Rightarrow$ $x_1 = x_2$

i.e., f is one-one function.

For onto

Let $f(x) = y$

$\because$ $y = 5x^2 + 6x - 9$

$\therefore$ $5x^2 + 6x - (9 + y) = 0$

$$x = \frac{-6 \pm \sqrt{36 + 4\times 5(9+y)}}{10}$$

$$= \frac{-6 \pm \sqrt{216 + 20y}}{10}$$

$$= \frac{\pm\sqrt{54+5y} - 3}{5}$$

$\Rightarrow$ $x = \frac{\sqrt{54+5y}-3}{5}$ $(\because x \in R_+)$

Clearly $\forall\, y \in [-9, \infty]$, the value of $x \in R_+$

$\Rightarrow f$ is onto function.

Hence f is one-one onto function.

$\Rightarrow f$ is invertible function with

$$f^{-1}(y) = \frac{\sqrt{54+5y}-3}{5}$$ **Hence Proved.**

OR

A binary operation * is defined on the set $x = R - \{-1\}$ by $x * y = x + y + xy, \forall x, y \in X$. Check whether * is commutative and associative. Find its identity element and also find the inverse of each element of X.**

21. Find the value of p for which the curves $x^2 = 9p(9-y)$ and $x^2 = p(y+1)$ cut each other at right angles. **[6]**

Solution : Given,

$$x^2 = 9p(9-y) \quad ...(i)$$

and $$x^2 = p(y+1) \quad ...(ii)$$

From (i) and (ii), we get

$$9p(9-y) = p(y+1)$$
$$\Rightarrow \quad 81p - 9py = py + p$$
$$\Rightarrow \quad 10py = 80p$$
$$\Rightarrow \quad y = 8$$
$$\therefore \quad x^2 = 9p$$

Now, differentiating (i) and (ii) w.r.t. x, we get

$$2x = -9p\frac{dy}{dx}$$
$$\Rightarrow \quad \frac{dy}{dx} = -\frac{2x}{9p}$$

and $$2x = p\frac{dy}{dx}$$
$$\Rightarrow \quad \frac{dy}{dx} = \frac{2x}{p}.$$

Since curves cut each other at right angles

$$\therefore \quad \left(\frac{2x}{p}\right)\left(-\frac{2x}{9p}\right) = -1 \quad [\because m_1 m_2 = -1]$$
$$\Rightarrow \quad \frac{-4x^2}{9p^2} = -1$$
$$\Rightarrow \quad \frac{4}{9p^2}(9p) = 1 \quad (\because x^2 = 9p)$$
$$\therefore \quad p = 4.$$ **Ans.**

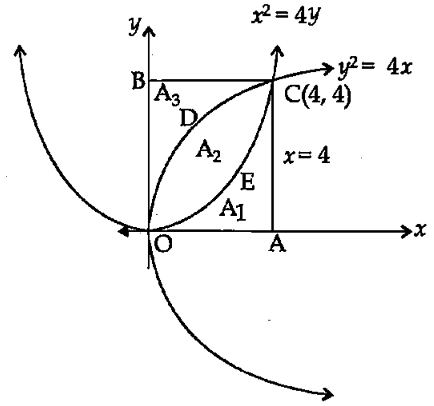
22. Using integration, prove that the curves $y^2 = 4x$ and $x^2 = 4y$ divide the area of the square bounded by $x = 0$, $x = 4$, $y = 0$ and $y = 4$ into three equal parts. **[6]**

Solution : Given,

$$y^2 = 4x \quad ...(i)$$

and $$x^2 = 4y \quad ...(ii)$$

Solving (i) and (ii), we get the point of intersection (0, 0) and (4, 4).



The area of the region OECDO bounded by the given curve A_2 = Area under $(y^2 = 4x)$ – Area under $(x^2 = 4y)$

$$= \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right)dx$$
$$= \left[\frac{4}{3}x^{3/2} - \frac{x^3}{12}\right]_0^4 = \left(\frac{32}{3} - \frac{16}{3}\right)$$
$$= \frac{16}{3} \text{ sq. units} \quad ...(iii)$$

and the area of the region OACEO A_1 = Area under $x^2 = 4y$

$$= \int_0^4 \left(\frac{x^2}{4}\right)dx = \left[\frac{x^3}{12}\right]_0^4$$
$$= \frac{16}{3} \text{ sq. units} \quad ...(iv)$$

Similarly, the area of the region ODCBO

$$A_3 = \text{Area of square} - (A_1 + A_2)$$
$$= 16 - \left(\frac{16}{3} + \frac{16}{3}\right)$$
$$= 16 - \frac{32}{3}$$
$$= \frac{16}{3} \text{ sq. units} \quad ...(v)$$

From (iii), (iv) and (v), it can be concluded that the given curves divide the area of the square bounded by $x = 0$, $x = 4$, $y = 0$ into three equal parts. **Hence Proved.**

23. Show that the differential equation $\frac{dy}{dx} = \frac{y^2}{xy - x^2}$ is homogeneous and also solve it. **[6]**

Solution : We have

**Answer is not given due to the change in present syllabus

$$\frac{dy}{dx} = \frac{y^2}{xy-x^2} \quad ...(i)$$

Let $$f(x, y) = \frac{y^2}{xy-x^2}$$

$$\therefore \quad f(\lambda x, \lambda y) = \frac{\lambda^2 y^2}{\lambda^2 xy - \lambda^2 x^2} = \frac{y^2}{xy-x^2}$$

$$= \lambda^0 f(x, y).$$

Hence, the differential equation is homogeneous.

Putting $y = vx$ so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (i), we get

$$v + \frac{xdv}{dx} = \frac{v^2x^2}{vx^2 - x^2}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v}{v-1}$$

$$\Rightarrow \quad \frac{v-1}{v}dv = \frac{dx}{x}$$

Integrating both sides, we get

$$\Rightarrow \quad \int\frac{v-1}{v}dv = \int\frac{dx}{x}$$

$$\Rightarrow \quad \int\left(1-\frac{1}{v}\right)dv = \int\frac{dx}{x}$$

$$v - \log|v| = \log|x| + C$$

$$\Rightarrow \frac{y}{x} - \log|y| + \log|x| = \log|x| + C$$

Hence, $x = \frac{y}{\log|y|+C}$ is the required solution of the differential equation. **Hence Proved.**

OR

Find the particular solution of the differential equation $(\tan^{-1}y - x)\,dy = (1 + y^2)\,dx$, given that $x = 1$, when $y = 0$.

Solution : Given,

$$(\tan^{-1} y - x)dy = (1 + y^2)dx$$

$$\Rightarrow \quad \frac{\tan^{-1} y - x}{1+y^2}dy = dx \quad ...(i)$$

Putting $\tan^{-1} y = t$ and $\frac{1}{1+y^2} = \frac{dt}{dy}$ in (i), we get

$(t - x)\,dt = dx$

$$\Rightarrow \quad \frac{dx}{dt} = t - x$$

$$\Rightarrow \frac{dx}{dt} + x = t \quad ...(ii)$$

Here, I.F. $= e^{\int 1.dt} = e^t$

Hence, the solution of the differential equation (ii) is given by

$$x(\text{I.F.}) = \int(\text{I.F.})\, t\, dt$$

$xe^t = \int e^t t\, dt + C$, where C is arbitrary constant

$$\Rightarrow \quad xe^t = te^t - \int e^t dt + C$$

$$\Rightarrow \quad xe^t = te^t - e^t + C$$

$$\Rightarrow xe^{\tan^{-1}y} = e^{\tan^{-1}y}(\tan^{-1}y - 1) + C \quad ...(iii)$$

$$[\because t = \tan^{-1} y]$$

It is given that $x = 1$ when $y = 0$

So, $e^0 = e^0(0 - 1) + C$

$\Rightarrow \quad C = 2$

Putting C = 2 in (iii) we get

$$xe^{\tan^{-1}y} = e^{\tan^{-1}y}(\tan^{-1} y - 1) + 2$$

$$\therefore \quad x = \tan^{-1}y + 2\,e^{\tan y} - 1$$

is the particular solution the given differential equation. **Ans.**

24. Find the distance of the point P(3, 4, 4) from the point, where the line joining the points A(3, – 4, – 5) and B(2, –3, 1) intersects the plane $2x + y + z = 7$. **[6]**

Solution : Equation of the line joining the points A(3, – 4, – 5) and B(2, – 3, 1) is

$$\Rightarrow \quad \frac{x-3}{2-3} = \frac{y+4}{-3+4} = \frac{z+5}{1+5}$$

$$\Rightarrow \quad \frac{x-3}{-1} = \frac{y+4}{1} = \frac{z+5}{6} = k.$$

Coordinates of any point on the line are

$Q\,(3 - k, k - 4, 6k - 5)$

The given line intersects the plane $2x + y + z = 7$.

So, $\quad 2(3 - k) + k - 4 + 6k - 5 = 7$

$\Rightarrow \quad 6 - 2k + k - 4 + 6k - 5 = 7$

$\Rightarrow \quad 5k = 10$

$\Rightarrow \quad k = 2$

$\therefore$ Coordinates of a point where line Intersect plane are $(3 - k, k - 4, 6k - 5) = (3 - 2, 2 - 4, 12 - 5) = (1, - 2, 7)$

Now, the distance between the point P(3, 4, 4) and A(1, – 2, 7) is given by

$$PQ = \sqrt{(3-1)^2 + (4+2)^2 + (4-7)^2}$$

$$\Rightarrow \quad PQ = \sqrt{(2)^2 + (6)^2 + (-3)^2}$$

$$\Rightarrow \quad PQ = \sqrt{4+36+9} = \sqrt{49}$$

$\therefore \quad PQ = 7$ units. **Ans.**

25. A company manufactures three kinds of calculators : A, B and C in its two factories I and II. The company has got an order for

manufacturing at least 6400 calculators of kind A, 4000 of kind B and 4800 of kind C. The daily output of factory I is of 50 calculators of kind A, 50 calculators of kind B, and 30 calculators of kind C. The daily output of factory II is of 40 calculators of kind A, 20 of kind B and 40 of kind C. The cost per day to run factory I is ₹12,000 and of factory II is ₹15,000. How many days do the two factories have to be in operation to produce the order with the minimum cost? Formulate this problem as an LPP and solve it graphically. **[6]**

Solution : Let the factories I and II work for x and y number of days respectively.

Thus, the given linear programming problem is Minimize Z = ₹ $(12000x + 15000y)$

Subject to the constraints

$$50x + 40y \geq 6400$$
$$50x + 20y \geq 4000$$
$$30x + 40y \geq 4800$$
$$x \geq 0$$

and $y \geq 0$

i.e.
$$5x + 4y \geq 640$$
$$5x + 2y \geq 400$$
$$3x + 4y \geq 480$$
$$x \geq 0, y \geq 0.$$

To solve this L. P. P,

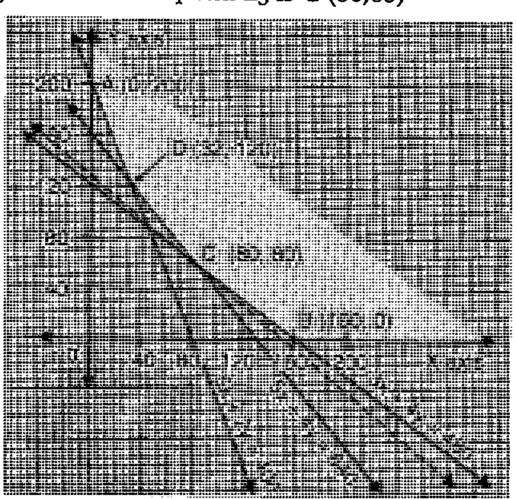
Let us consider the equations

$L_1 : 5x + 4y = 640$ ...(i)

$L_2 : 5x + 2y = 400$ ...(ii)

$L_3 : 3x + 4y = 480$ ...(iii)

The point of intersection of L_1 and L_2 is D(32, 120) and that of L_1 and L_3 is C (80,60)



The shaded region is the solution region of the given L. P. P.

Corner Points	Values of the objective function $Z=12000x+15000y$
A (0, 200)	$12000 \times 0 + 15000 \times 200 = 30,00,000$
B (160,0)	$12000 \times 160 + 15000 \times 0 = 19,20,000$
C (80, 60)	$12000 \times 80 + 15000 \times 60 = 18,60,000$
D (32,120)	$12000 \times 32 + 15000 \times 120 = 21,84,000$

Out of these values of Z, the minimum value of Z is 18,60,000 at $x = 80$ and $y = 60$.

Since the feasible region is unbounded so we draw the graph of inequality

$$12000\,x + 15000\,y < 1860000$$
$$i.e., 4x + 5y \leq 620$$

$L : 4x + 5y = 620$

x	0	155
y	124	0

We observe that open half one represented by L have no point common with feasible region.

$$Z = 12000 \times 80 + 15000 \times 60$$
$$= ₹\,18,60,000.$$

Hence, the factories I and II work for 80 and 60 number of days respectively. **Ans.**

26. In a factory which manufactures bolts, machines A, B and C manufacture respectively 30%, 50% and 20% of the bolts. Of their outputs 3, 4 and 1 percent respectively are defective bolts. A bolt is drawn at random from the product and is found to be defective. Find the probability that this is not manufactured by machine B. **[6]**

Solution : Let E_1, E_2, E_3 and A be the events defined as below

E_1 = the bolt is manufactured by machine A.

E_2 = the bolt is manufactured by machine B.

E_3 = the bolt is manufactured by machine C.

A = the bolt is defective.

then, $P(E_1)$ = Probability that the bolt drawn is manufactured by machine A = $\frac{30}{100}$

$P(E_2)$ = Probability that the bolt drawn is manufactured by machine B = $\frac{50}{100}$

$P(E_3)$ = Probability that the bolt drawn is manufactured by machine C = $\frac{20}{100}$

$P(A/E_1)$ = Probability that the bolt drawn is defective given that it is manufactured by machine A.

$$P(A/E_1) = \frac{3}{100}$$

Similarly, we have, $$P(A/E_2) = \frac{4}{100}$$

and $$P(A/E_3) = \frac{1}{100}$$

Now, using Bayes' theorem

$P(E_2/A)$ = Probability that the bolt is manufactured by machine B given that the bolt drawn is defective.

$$= \frac{P(E_2)\,P(A/E_2)}{P(E_1)P(A/E_1)+P(E_2)P(A/E_2)+P(E_3)P(A/E_3)}$$

$$= \frac{\frac{50}{100}\times\frac{4}{100}}{\frac{30}{100}\times\frac{3}{100}+\frac{50}{100}\times\frac{4}{100}+\frac{20}{100}\times\frac{1}{100}}$$

$$= \frac{200}{90+200+20} = \frac{200}{310} = \frac{20}{31}.$$

Hence, the probability that this is not manufactured by Machine B = $1 - P(E_2/A)$

$$= 1 - \frac{20}{31} = \frac{11}{31}$$ **Ans.**

●●

All questions are same in Outside Delhi Set II and Set III

Mathematics 2015 (Delhi) SET I

Time allowed : 3 hours **Maximum marks : 100**

SECTION — A

1. If $\vec{a} = 7\hat{i} + \hat{j} - 4\hat{k}$ and $\vec{b} = 2\hat{i} + 6\hat{j} + 3\hat{k}$, then find the projection of $\vec{a}$ on $\vec{b}$. **[1]**

Solution : Given,

$$\vec{a} = 7\hat{i}+\hat{j}-4\hat{k}$$

$$\vec{b} = 2\hat{i}+6\hat{j}+3\hat{k}$$

$\therefore$ The projection of $\vec{a}$ on $\vec{b}$ $= \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

$$= \frac{(7\times2)+(1\times6)+(-4\times3)}{\sqrt{(2)^2+(6)^2+(3)^2}} = \frac{8}{7}$$

Ans.

2. Find λ, if the vectors $\vec{a} = \hat{i}+3\hat{j}+\hat{k}$, $\vec{b} = 2\hat{i}-\hat{j}-\hat{k}$ and $\vec{c} = \lambda\hat{j}+3\hat{k}$ are coplanar. **[1]**

Solution : Since, the given vectors are coplanar.

$$[\vec{a}\ \vec{b}\ \vec{c}] = 0$$

Here, $$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 1 & 3 & 1 \\ 2 & -1 & -1 \\ 0 & \lambda & 3 \end{vmatrix} = 0$$

Expanding along R_3, we get

$$0.(-3+1) - \lambda(-1-2) + 3(-1-6) = 0$$

$$\Rightarrow \quad 3\lambda = 21$$

$$\therefore \quad \lambda = 7$$ **Ans.**

3. If a line makes angles 90°, 60° and θ with x, y and z–axis respectively, where θ is acute, then find θ. **[1]**

Solution : Given,

$$\alpha = 90°$$
$$\beta = 60°$$
$$\lambda = \theta$$

Let l, m, n be the direction cosines of the given vector.

Then, $l = \cos\alpha$,

$m = \cos\beta$

and $n = \cos\lambda$

Now, $$l^2+m^2+n^2 = 1$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\lambda = 1$$

$$\Rightarrow \cos^2(90°) + \cos^2(60°) + \cos^2\theta = 1$$

$$\Rightarrow 0^2 + \left(\frac{1}{2}\right)^2 + \cos^2\theta = 1$$

$$\Rightarrow \cos^2\theta = 1 - \frac{1}{4} = \frac{3}{4}$$

$$\Rightarrow \cos\theta = \frac{\sqrt{3}}{2} \quad (\because \theta \text{ is acute})$$

$$\therefore \theta = 30°$$ **Ans.**

4. Write the element a_{23} of a 3×3 matrix $A = (a_{ij})$ whose elements a_{ij} are given be $a_{ij} = \dfrac{|i-j|}{2}$. **[1]**

Solution : Given,

$$a_{ij} = \frac{|i-j|}{2}$$

$\therefore \quad a_{23} = \frac{|2-3|}{2} = \frac{|-1|}{2} = \frac{1}{2}$ **Ans.**

5. Find the differential equation representing the family of curves $v = \frac{A}{r} + B$, where A and B are arbitrary constants. [1]

Solution : We have,

$$v = \frac{A}{r} + B \quad ...(i)$$

Since, the given equation contains two arbitrary constants, we shall differentiate it two times. Now, differentiating (i) w.r.t. r, we get

$$\frac{dv}{dr} = -\frac{A}{r^2} + 0$$

$$\Rightarrow \quad r^2 \frac{dv}{dr} = -A \quad ...(ii)$$

Again, differentiating (ii) w.r.t. r, we get

$$r^2 \times \frac{d^2v}{dr^2} + 2r \times \frac{dv}{dr} = 0$$

$$\Rightarrow \quad r\frac{d^2v}{dr^2} + 2\frac{dv}{dr} = 0$$

This is the required differential equation representing the family of the given curve. **Ans.**

6. Find the integrating factor of the differential equation $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1.$ [1]

Solution : We have,

$$\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{y}{\sqrt{x}} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}},$$

which is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q,$$

where $P = \frac{1}{\sqrt{x}}$

and $Q = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$

$$\therefore \quad \text{I. F.} = e^{\int P dx}$$

$$= e^{\int \frac{1}{\sqrt{x}} dx} = e^{2\sqrt{x}}$$ **Ans.**

SECTION — B

7. If $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$, find $A^2 - 5A + 4I$ and hence find a matrix X such that $A^2 - 5A + 4I + X = 0$.[4]

Solution : We have,

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$$

$$\Rightarrow \quad A^2 = A.A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix}$$

$$\Rightarrow \quad -5A = \begin{bmatrix} (-5).2 & (-5).0 & (-5).1 \\ (-5).2 & (-5).1 & (-5).3 \\ (-5).1 & (-5)(-1) & (-5).0 \end{bmatrix}$$

$$= \begin{bmatrix} -10 & 0 & -5 \\ -10 & -5 & -15 \\ -5 & 5 & 0 \end{bmatrix}$$

$$\Rightarrow \quad 4I_3 = 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$\therefore \quad A^2 - 5A + 4I =$

$$\begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} + \begin{bmatrix} -10 & 0 & -5 \\ -10 & -5 & -15 \\ -5 & 5 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} -1 & -1 & -3 \\ -1 & -3 & -10 \\ -5 & 4 & 2 \end{bmatrix}$$

Now, $A^2 - 5A + 4I + X = 0$

$$\Rightarrow \quad X = -(A^2 - 5A + 4I)$$

$$\Rightarrow \quad X = (-1)\begin{bmatrix} -1 & -1 & -3 \\ -1 & -3 & -10 \\ -5 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & 10 \\ 5 & -4 & -2 \end{bmatrix}$$

Ans.

OR

If $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{bmatrix}$, find $(A')^{-1}$.

Solution : Given,

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{bmatrix}$$

Now, $$A' = \begin{bmatrix} 1 & 0 & -2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{bmatrix}$$

$$|A'| = \begin{vmatrix} 1 & 0 & -2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{vmatrix}$$

$$= 1(-1-8) - 0(-2-6) - 2(-8+3)$$
$$= -9 + 10 = 1 \neq 0$$

So, A′ is invertible.

$$A'_{11} = -9, A'_{12} = 8, A'_{13} = -5$$
$$A'_{21} = -8, A'_{22} = 7, A'_{23} = -4$$
$$A'_{31} = -2, A'_{32} = 2, A'_{33} = -1$$

$$\text{adj } A' = \begin{bmatrix} A'_{11} & A'_{21} & A'_{31} \\ A'_{12} & A'_{22} & A'_{32} \\ A'_{13} & A'_{23} & A'_{33} \end{bmatrix}$$

$$= \begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}$$

$$\therefore \quad (A')^{-1} = \frac{1}{|A'|} \text{adj } A'$$

$$= \text{adj } A' \qquad [\because |A'| = 1]$$

$$\Rightarrow \quad (A')^{-1} = \begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}$$ **Ans.**

8. If $f(x) = \begin{vmatrix} a & -1 & 0 \\ ax & a & -1 \\ ax^2 & ax & a \end{vmatrix}$, **using properties of determinants find the value of** $f(2x) - f(x)$. **[4]**

Solution : Given, $$f(x) = \begin{bmatrix} a & -1 & 0 \\ ax & a & -1 \\ ax^2 & ax & a \end{bmatrix}$$

Taking a common from C_1

$$\Rightarrow \quad f(x) = a\begin{bmatrix} 1 & -1 & 0 \\ x & a & -1 \\ x^2 & ax & a \end{bmatrix}$$

Applying $C_2 \to C_2 + C_1$, we get

$$f(x) = a\begin{bmatrix} 1 & 0 & 0 \\ x & x+a & -1 \\ x^2 & x^2+ax & a \end{bmatrix}$$

On expanding along R_1,

$$f(x) = a(a^2 + ax + ax + x^2)$$
$$\Rightarrow \quad f(x) = a(a^2 + 2ax + x^2)$$

Also, $$f(2x) = \begin{vmatrix} a & -1 & 0 \\ 2ax & a & -1 \\ 4ax^2 & 2ax & a \end{vmatrix}$$

$$\Rightarrow \quad f(2x) = a\begin{vmatrix} 1 & -1 & 0 \\ 2x & a & -1 \\ 4x^2 & 2ax & a \end{vmatrix}$$

Applying $C_2 \to C_2 + C_1$, we get

$$f(2x) = a\begin{vmatrix} 1 & 0 & 0 \\ 2x & 2x+a & -1 \\ 4x^2 & 4x^2+2ax & a \end{vmatrix}$$

On Expanding along R_1

$$\Rightarrow \quad f(2x) = a\{a(2x+a) + 4x^2 + 2ax\}$$
$$\Rightarrow \quad f(2x) = a\{4x^2 + a^2 + 4ax\}$$
$$\therefore \quad f(2x) - f(x) = a(4x^2 + a^2 + 4ax - a^2 - 2ax - x^2)$$
$$\Rightarrow \quad = a(3x^2 + 2ax)$$
$$\Rightarrow \quad = ax(3x + 2a).$$ **Ans.**

9. Find : $\int \dfrac{dx}{\sin x + \sin 2x}$. **[4]**

Solution : Let $$I = \int \frac{1}{\sin x + \sin 2x} dx$$

$$\Rightarrow \quad I = \int \frac{1}{\sin x + 2\sin x \cos x} dx$$

$$\Rightarrow \quad I = \int \frac{1}{\sin x(1 + 2\cos x)} dx$$

$$\Rightarrow \quad I = \int \frac{\sin x}{\sin^2 x(1 + 2\cos x)} dx$$

$$= \int \frac{\sin x \, dx}{(1 - \cos^2 x)(1 + 2\cos x)}$$

Putting $\cos x = t, -\sin x \, dx = dt$

$$\Rightarrow \quad \sin x \, dx = -dt$$

$$I = \int \frac{-dt}{(1-t^2)(1+2t)}$$

$$= \int \frac{-1}{(1-t)(1+t)(1+2t)} dt$$

Let $\frac{-1}{(1+t)(1-t)(1+2t)} = \frac{A}{(1+t)} + \frac{B}{(1-t)} + \frac{C}{(1+2t)}$ ...(i)

$-1 = A(1-t)(1+2t) + B(1+t)(1+2t) + C(1+t)(1-t)$

Putting $1-t = 0$

or $t = 1$ in (i), we get

$-1 = 6B$

$\Rightarrow \quad B = \frac{-1}{6}$

Putting $1+t = 0$

or $t = -1$ in (i), we get

$-1 = -2A$

$\Rightarrow \quad A = \frac{1}{2}$

Putting $1+2t = 0$

or $t = -\frac{1}{2}$ in (i), we get

$-1 = \frac{3}{4}C$

$\Rightarrow \quad C = -\frac{4}{3}$

$$\frac{-1}{(1+t)(1-t)(1+2t)} = \frac{1}{2(1+t)} - \frac{1}{6(1-t)} - \frac{4}{3(1+2t)}$$

$$\Rightarrow I = \int \frac{-dt}{(1+t)(1-t)(1+2t)}$$

$$\Rightarrow I = \frac{1}{2}\int \frac{1}{1+t}dt - \frac{1}{6}\int \frac{1}{1-t}dt - \frac{4}{3}\int \frac{1}{1+2t}dt$$

$$\Rightarrow I = \frac{1}{2}\log|1+t| + \frac{1}{6}\log|1-t| - \frac{4}{3\times 2}\log|1+2t| + C$$

$$\Rightarrow I = \frac{1}{2}\log|1+\cos x| + \frac{1}{6}\log|1-\cos x| - \frac{2}{3}\log|1+2\cos x| + C$$

Ans.

OR

Integrate the following w.r.t. x : $\frac{x^2-3x+1}{\sqrt{1-x^2}}$. **[1]**

Solution : Given,

$$\frac{x^2-3x+1-1+1}{\sqrt{1-x^2}} = -\left[\frac{-x^2+3x-1+1-1}{\sqrt{1-x^2}}\right]$$

$$= -\left[\frac{1-x^2+3x-2}{\sqrt{1-x^2}}\right]$$

$$= \frac{1-x^2}{\sqrt{1-x^2}} - \frac{3x-2}{\sqrt{1-x^2}}$$

$$= -\sqrt{1-x^2} - \frac{3x-2}{\sqrt{1-x^2}}$$

$$\Rightarrow \int \frac{x^2-3x+1}{\sqrt{1-x^2}}dx = \int\left(-\sqrt{1-x^2} - \frac{3x-2}{\sqrt{1-x^2}}\right)dx$$

$$= -\int\sqrt{1-x^2}dx - \int\frac{3x-2}{\sqrt{1-x^2}}dx$$

$$= -\int\sqrt{1-x^2}dx - 3\int\frac{x}{\sqrt{1-x^2}}dx + 2\int\frac{1}{\sqrt{1-x^2}}dx$$

$$= -\int\sqrt{1-x^2}dx + \frac{3}{2}\int\frac{1}{\sqrt{t}}dt + 2\int\frac{1}{\sqrt{1-x^2}}dx$$

(Putting $1-x^2 = t \Rightarrow -2x\,dx = dt$)

$$= \frac{-x}{2}\sqrt{1-x^2} - \frac{1}{2}\sin^{-1}x + 3\sqrt{1-x^2} + 2\sin^{-1}x + C$$

$$= \frac{-x}{2}\sqrt{1-x^2} + \frac{3}{2}\sin^{-1}x + 3\sqrt{1-x^2} + C$$ **Ans.**

10. Evaluate : $\int_{-\pi}^{\pi}(\cos ax - \sin bx)^2 dx$. **[4]**

Solution :

Let $I = \int_{-\pi}^{\pi}(\cos ax - \sin bx)^2 dx$

$I = \int_{-\pi}^{\pi}(\cos^2 ax + \sin^2 bx - 2\cos ax \sin bx)dx$

$I = \int_{-\pi}^{\pi}(\cos^2 ax\,dx + \int_{-\pi}^{\pi}\sin^2 bx\,dx - 2\int_{-\pi}^{\pi}\cos ax \sin bx\,dx$

$\Rightarrow I = 2\int_0^{\pi}\cos^2 ax\,dx + 2\int_0^{\pi}\sin^2 bx\,dx - 0$

[Since $\cos^2 ax$ and $\sin^2 bx$ are even functions and $\cos ax \sin bx$ is an odd function]

$$\Rightarrow I = 2\int_0^{\pi}\left(\frac{1+\cos 2ax}{2}\right)dx + 2\int_0^{\pi}\left(\frac{1-\cos 2bx}{2}\right)dx$$

$$\Rightarrow I = \int_0^{\pi}(1+\cos 2ax)dx + \int_0^{\pi}(1-\cos 2bx)\,dx$$

$$\Rightarrow I = \int_0^{\pi}(1+\cos 2ax + 1 - \cos 2bx)dx$$

$$\Rightarrow I = \int_0^{\pi}(2+\cos 2ax - \cos 2bx)dx$$

$$\Rightarrow I = 2[x]_0^{\pi} + \left[\frac{\sin 2ax}{2a}\right]_0^{\pi} - \left[\frac{\sin 2bx}{2b}\right]_0^{\pi}$$

$$\Rightarrow I = 2\pi + \frac{\sin 2a\pi}{2a} - \frac{\sin 2b\pi}{2b}$$

$\Rightarrow I = 2\pi$ if $a, b \in Z$ **Ans.**

11. A bag 'A' contains 4 black and 6 red balls and bag 'B' contains 7 black and 3 red balls. A die is thrown. If 1 or 2 appears on it, then bag A is chosen, otherwise bag B. If two balls are drawn at random (without replacement) from the selected bag, find the probability of one of them being red and another black. **[4]**

Solution : Consider the following events :

E_1 = Getting 1 or 2 on die.

E_2 = Getting 3, 4, 5 or 6 on die.

E = One of the ball drawn is red and another is black.

$$P(E_1) = \frac{2}{6} = \frac{1}{3}$$

and $$P(E_2) = \frac{4}{6} = \frac{2}{3}$$

$P(E/E_1)$ = Probability of drawing a red and a black ball when bag A has been chosen.

$$= P(RB) + P(BR)$$

$$P(E/E_1) = \frac{6}{10}\times\frac{4}{9}+\frac{4}{10}\times\frac{6}{9} = \frac{48}{90} = \frac{8}{15}$$

$P(E/E_2)$ = Probability of drawing a red and a black ball when bag B has been chosen.

$$= \frac{3}{10}\times\frac{7}{9}+\frac{7}{10}\times\frac{3}{9} = \frac{42}{90} = \frac{7}{15}$$

Using the law of total probability, we have

$$P(E) = P(E_1)\,P(E/E_1) + P(E_2)\,P(E/E_2)$$

$$= \frac{1}{3}\times\frac{8}{15}+\frac{2}{3}\times\frac{7}{15}$$

$$= \frac{22}{45}$$ **Ans.**

OR

An unbiased coin is tossed 4 times. Find the mean and variance of the number of heads obtained.

Solution : Let X denote the number of heads in the four tosses of the coin, then X is a random variable that can have values 0, 1, 2, 3, 4.

$P(X = 0)$ = Probability of getting no head (TTTT)

$$= \frac{1}{16}$$

$P(X = 1)$ = Probability of getting one head (HTTT, THTT, TTHT, TTTH)

$$= 4\times\frac{1}{16} = \frac{1}{4}$$

$P(X = 2)$ = Probability of getting two head (HHTT, HTHT, HTTH, THHT, THTH, TTHH)

$$= 6\times\frac{1}{16} = \frac{3}{8}$$

$P(X = 3)$ = Probability of getting three head (HHHT, HHTH, HTHH, THHH)

$$= 4\times\frac{1}{16} = \frac{1}{4}$$

$P(X = 4)$ = Probability of getting four head (HHHH)

$$= \frac{1}{16}$$

Thus, the probability distribution of random variable X is given by

X	0	1	2	3	4
P(X)	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$

x_i	$P_i = P(X = x_i)$	$P_i x_i$	$P_i x_i^2$
0	$\frac{1}{16}$	0	0
1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
2	$\frac{3}{8}$	$\frac{3}{4}$	$\frac{3}{2}$
3	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{9}{4}$
4	$\frac{1}{16}$	$\frac{1}{4}$	1
		$\Sigma P_i x_i = 2$	$\Sigma P_i x_i^2 = 5$

$\therefore$ Mean = $\bar{X} = \Sigma P_i x_i = 2$

and Var $(x) = \Sigma P_i x_i^2 - (\Sigma P_i x_i)^2 = 5 - 4 = 1$

Hence, Mean = 2 and Variance = 1. **Ans.**

12. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, find $(\vec{r}\times\hat{i}).(\vec{r}\times\hat{j}) + xy$. **[4]**

Solution : Given,

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k},$$

Now, $(\vec{r}\times\hat{i}).(\vec{r}\times\hat{j}) + xy$

$$= [(x\hat{i}+y\hat{j}+z\hat{k})\times\hat{i}].[(x\hat{i}+y\hat{j}+z\hat{k})\times\hat{j}] + xy$$

$$[\because \vec{r} = x\hat{i}+y\hat{j}+z\hat{k}]$$

$$= [(x\hat{i}\times\hat{i}+y\hat{j}\times\hat{i}+z\hat{k}\times\hat{i})].[(x\hat{i}\times\hat{j}+y\hat{j}\times\hat{j}$$

$+ z\hat{k} \times \hat{j})] + xy$

$$\left[\begin{array}{l} \because \hat{i} \times \hat{j} = \hat{k},\ \hat{j} \times \hat{k} = \hat{i}, \\ \hat{k} \times \hat{i} = \hat{j},\ \hat{i} \times \hat{i} = 0 \\ \hat{j} \times \hat{j} = 0,\ \hat{k} \times \hat{k} = 0 \end{array}\right]$$

$= (0 - y\hat{k} + z\hat{j}).(x\hat{k} + 0 - z\hat{i}) + xy$

$= (0z - xy + 0z) + xy$

$= - xy + xy = 0.$ **Ans.**

13. Find the distance between the point (– 1, – 5, – 10) and the point of intersection of the line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ and the plane $x - y + z = 5$. **[4]**

Solution : Let $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12} = k$

$$\Rightarrow \quad x = 3k + 2, \quad y = 4k - 1, \quad z = 12k + 2$$

Coordinates of any point on the line are

$(3k + 2, 4k - 1, 12k + 2)$.

The point of intersection of the line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ and the plane $x - y + z = 5$ will also be in the form $(3k + 2, 4k - 1, 12k + 2)$ and it will satisfy the equation of plane.

Now, putting $x = 3k + 2$, $y = 4k - 1$

and $z = 12k + 2$ in $x - y + z = 5$,

we get

$$3k + 2 - (4k - 1) + 12k + 2 = 5$$

$$\Rightarrow \quad 11k + 5 = 5$$

$$\Rightarrow \quad 11k = 0$$

$$\Rightarrow \quad k = 0$$

$$\therefore \quad x = 2, \quad y = -1, \quad z = 2$$

Hence, the point of intersection of $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ and the plane $x - y + z = 5$ is $(2, -1, 2)$.

$\therefore$ Distance between the point $(-1, -5, -10)$ and $(2, -1, 2)$

Using distance formula

$$= \sqrt{(2+1)^2 + (-1+5)^2 + (2+10)^2}$$

$$= \sqrt{3^2 + 4^2 + 12^2}$$

$$= \sqrt{169} = 13 \text{ units}$$

Hence, the distance between the point (–1, –5, – 10) and the point of intersection of the line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ and the plane $x - y + z = 5$ is 13 units. **Ans.**

14. If $\sin[\cot^{-1}(x+1)] = \cos(\tan^{-1} x)$, then find x. **[4]**

Solution : Given, $\sin[\cot^{-1}(x+1)] = \cos(\tan^{-1} x)$

$$\sin\left\{\sin^{-1}\frac{1}{\sqrt{1+(x+1)^2}}\right\} = \cos\left\{\cos^{-1}\frac{1}{\sqrt{1+x^2}}\right\}$$

$$\left[\because \cot^{-1} x = \sin^{-1}\frac{1}{\sqrt{1+x^2}} \text{ and } \tan^{-1} x = \cos^{-1}\frac{1}{\sqrt{1+x^2}}\right]$$

$$\Rightarrow \quad \frac{1}{\sqrt{1+(x+1)^2}} = \frac{1}{\sqrt{1+x^2}}$$

$$\Rightarrow \quad \frac{1}{\sqrt{1+x^2+1+2x}} = \frac{1}{\sqrt{1+x^2}}$$

$$\Rightarrow \quad \frac{1}{\sqrt{x^2+2x+2}} = \frac{1}{\sqrt{1+x^2}}$$

$$\Rightarrow \quad \sqrt{1+x^2} = \sqrt{x^2+2x+2}$$

On squaring both sides, we get

$$1 + x^2 = x^2 + 2x + 2$$

$$\Rightarrow \quad 2x + 2 = 1$$

$$\Rightarrow \quad x = \frac{-1}{2}$$ **Ans.**

OR

If $(\tan^{-1} x)^2 + (\cot^{-1} x)^2 = \frac{5\pi^2}{8}$, then find x.

Solution : Given, $(\tan^{-1} x)^2 + (\cot^{-1} x)^2 = \frac{5\pi^2}{8}$

$$\Rightarrow \quad (\tan^{-1} x + \cot^{-1} x)^2 - 2\tan^{-1} x \cot^{-1} x = \frac{5\pi^2}{8}$$

$$\Rightarrow \quad \left(\frac{\pi}{2}\right)^2 - 2\tan^{-1} x\left(\frac{\pi}{2} - \tan^{-1} x\right) = \frac{5\pi^2}{8}$$

$$\Rightarrow \quad \frac{\pi^2}{4} - \pi\tan^{-1} x + 2(\tan^{-1} x)^2 = \frac{5\pi^2}{8}$$

$$\Rightarrow \quad 2(\tan^{-1} x)^2 - \pi\tan^{-1} x + \frac{\pi^2}{4} - \frac{5\pi^2}{8} = 0$$

$$\Rightarrow \quad 2(\tan^{-1} x)^2 - \pi \tan^{-1} x - \frac{3\pi^2}{8} = 0$$

Solving the quadratic equation, we get

$$\tan^{-1} x = \frac{\pi \pm \sqrt{\pi^2 + 4 \times 2 \times \frac{3\pi^2}{8}}}{2 \times 2}$$

$$\Rightarrow \quad \tan^{-1} x = \frac{\pi \pm 2\pi}{4}$$

$$\Rightarrow \quad \tan^{-1} x = \frac{3\pi}{4} \text{ or } \tan^{-1} x = \frac{-\pi}{4}$$

$$\Rightarrow \quad x = \tan\frac{3\pi}{4} \text{ or } x = \tan\left(\frac{-\pi}{4}\right)$$

$$\Rightarrow \quad x = -1. \qquad \textbf{Ans.}$$

15. If $y = \tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)$, $x^2 \le 1$, then find $\frac{dy}{dx}$. **[4]**

Solution : Given,

$$y = \tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)$$

Putting $x^2 = \cos 2\theta$ we have

$$y = \tan^{-1}\left(\frac{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}\right)$$

$$\Rightarrow \quad y = \tan^{-1}\left(\frac{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}\right)$$

$$\Rightarrow \quad y = \tan^{-1}\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)$$

Dividing the numerator and denominator by $\cos\theta$, we get

$$y = \tan^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right)$$

$$\Rightarrow \quad y = \tan^{-1}\left(\frac{\tan\frac{\pi}{4}+\tan\theta}{1-\tan\frac{\pi}{4}.\tan\theta}\right)$$

$$\Rightarrow \quad y = \tan^{-1}\left[\tan\left(\frac{\pi}{4}+\theta\right)\right]$$

$$\Rightarrow \quad y = \frac{\pi}{4}+\theta$$

$$\therefore \quad y = \frac{\pi}{4}+\frac{1}{2}\cos^{-1} x^2 \quad (\because x^2 = \cos 2\theta) \qquad \text{...(i)}$$

Now, differentiating (i), w.r.t. x, we get

$$\frac{dy}{dx} = 0+\frac{1}{2}\times\left(\frac{-1}{\sqrt{1-(x^2)^2}}\right)\times 2x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-x}{\sqrt{1-x^4}} \qquad \textbf{Ans.}$$

16. If $x = a\cos\theta + b\sin\theta$, $y = a\sin\theta - b\cos\theta$, show that $\frac{y^2 d^2y}{dx^2} - \frac{xdy}{dx} + y = 0$. **[4]**

Solution : We have,

$$x = a\cos\theta + b\sin\theta \qquad \text{...(i)}$$

$$y = a\sin\theta - b\cos\theta \qquad \text{...(ii)}$$

On squaring and adding (i) and (ii), we get

$$x^2 + y^2 = (a\cos\theta + b\sin\theta)^2 + (a\sin\theta - b\cos\theta)^2$$

$$= a^2\cos^2\theta + b^2\sin^2\theta + 2ab\cos\theta\sin\theta + a^2\sin^2\theta + b^2\cos^2\theta - 2ab\cos\theta\sin\theta$$

$$= a^2(\cos^2\theta + \sin^2\theta) + b^2(\sin^2\theta + \cos^2\theta)$$

$$\Rightarrow x^2 + y^2 = a^2 + b^2 \qquad \text{...(iii)}$$

Now, differentiating both sides of equation (iii) w.r.t. x, we get

$$2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad 2y\frac{dy}{dx} = -2x$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{x}{y} \qquad \text{...(iv)}$$

Again, differentiating both sides of (iv) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = -\left(\frac{y \times 1 - x \times \frac{dy}{dx}}{y^2}\right)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -\left[\frac{y - x\left(-\frac{x}{y}\right)}{y^2}\right] \quad \text{[from (iv)]}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -\left[\frac{y^2+x^2}{y^3}\right] \qquad \text{...(v)}$$

Now, we have

$$\text{L. H. S.} = y^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y$$

$$= y^2\left(\frac{-y^2-x^2}{y^3}\right) - x\left(-\frac{x}{y}\right) + y$$

$$= \frac{-y^2-x^2}{y} + \frac{x^2}{y} + y$$

$$= \frac{-y^2 - x^2 + x^2 + y^2}{y}$$

$= 0 =$ R. H. S. **Hence Proved.**

17. The side of an equilateral triangle is increasing at the rate of 2 cm/s. At what rate is its area increasing when the side of the triangle is 20 cm ? [4]

Solution : We know that, Area of an equilateral triangle,

$$A = \frac{\sqrt{3}}{4}a^2,$$

where a = side of an equilateral triangle.

Given $\quad \frac{da}{dt} = 2 \text{ cm/s}$

Now, $\quad \frac{dA}{dt} = \frac{d}{dt}\left(\frac{\sqrt{3}}{4}a^2\right)$

$$= \frac{\sqrt{3}}{4} \times 2 \times a \times \frac{da}{dt}$$

$$= \left(\frac{\sqrt{3}}{2}a.\frac{da}{dt}\right)$$

$$= \frac{\sqrt{3}a}{2}.2 = \sqrt{3}a \text{ cm}^2/\text{s}$$

when the side of the triangle is 20 cm.

$$\therefore \quad \left[\frac{dA}{dt}\right]_{a=20} = 20\sqrt{3} \text{ cm}^2/\text{s}$$

Hence, the area is increasing at the rate of $20\sqrt{3} \text{ cm}^2/\text{s}$ when the side of the triangle is 20 cm. **Ans.**

18. Find $\int (x+3)\sqrt{3-4x-x^2}\,dx$. [4]

Solution :

Let $\quad I = \int (x+3)\sqrt{3-4x-x^2}\,dx$

Let $\quad x+3 = \lambda\frac{d}{dx}(3-4x-x^2) + \mu$

$\Rightarrow \quad x+3 = \lambda(-4-2x) + \mu$

$\Rightarrow \quad x+3 = (-2\lambda)x - 4\lambda + \mu$

Equating the coefficients of like terms, we get

$$-2\lambda = 1$$

$\Rightarrow \quad \lambda = \frac{-1}{2}$

and $\quad -4\lambda + \mu = 3$

$\Rightarrow -4\left(\frac{-1}{2}\right) + \mu = 3$

$\Rightarrow \quad \mu = 3 - 2 = 1$

$\therefore \quad I = \int \left[\frac{1}{2}(4+2x)+1\right]\sqrt{3-4x-x^2}\,dx$

$$= \int \frac{1}{2}(4+2x)\sqrt{3-4x-x^2}\,dx + \int \sqrt{3-4x-x^2}\,dx$$

$$= \frac{1}{2}I_1 + I_2 \quad \text{...(i)}$$

Now, $\quad I_1 = \int (4+2x)\sqrt{3-4x-x^2}\,dx$

Let $3 - 4x - x^2 = t$

$\Rightarrow -(4+2x)dx = dt$

$\therefore \quad I_1 = -\int \sqrt{t}\,dt$

$$= -\int t^{\frac{1}{2}}dt$$

$$= -\frac{2}{3}t^{\frac{3}{2}} + C_1$$

$$= -\frac{2}{3}(3-4x-x^2)^{\frac{3}{2}} + C_1 \quad \text{...(ii)}$$

$(\because t = 3 - 4x - x^2)$

and $\quad I_2 = \int \sqrt{3-4x-x^2}\,dx$

$$= \int \sqrt{3+4-(x^2+4x+4)}\,dx$$

$$= \int \sqrt{7-(x+2)^2}\,dx$$

$$= \int \sqrt{(\sqrt{7})^2-(x+2)^2}\,dx$$

$$= \frac{1}{2}(x+2)\sqrt{(\sqrt{7})^2-(x+2)^2} + \frac{(\sqrt{7})^2}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{7}}\right) + C_2$$

$$= \frac{1}{2}(x+2)\sqrt{3-4x-x^2} + \frac{7}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{7}}\right) + C_2 \quad \text{...(iii)}$$

From (i), (ii) and (iii), we get

$$I = -\frac{2}{3}(3-4x-x^2)^{\frac{3}{2}} + \frac{1}{2}(x+2)\sqrt{3-4x-x^2} + \frac{7}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{7}}\right) + C,$$

where $C = C_1 + C_2$ **Ans.**

19. Three schools A, B and C organised a mela for collecting funds for helping the rehabilitation of flood victims. They sold handmade fans, mats and plates from recycled material at a cost of ₹ 25, ₹ 100 and ₹ 50 each. The number of articles sold are given below :

Article \ School	A	B	C
Hand-fans	40	25	35
Mats	50	40	50
Plates	20	30	40

Find the fund collected by each school separately by selling the above articles. Also find the total funds collected for the purpose. Write one value generated by the above situation. **[4]**

Solution : The number of articles sold by each school can be represented by the 3 × 3 matrix

$$X = \begin{bmatrix} 40 & 25 & 35 \\ 50 & 40 & 50 \\ 20 & 30 & 40 \end{bmatrix}$$

and the cost of each article can be represented by the 1 × 3 matrix

$$Y = [2510050]$$

∴ Funds collected by each school separately is given by the matrix multiplication.

$$YX = [25 \;\; 100 \;\; 50] \begin{bmatrix} 40 & 25 & 35 \\ 50 & 40 & 50 \\ 20 & 30 & 40 \end{bmatrix}$$

$$YX = [7000 \;\; 6125 \;\; 7875]$$

Hence, the funds collected by schools A, B and C are ₹ 7,000, ₹ 6,125 and ₹ 7,875 respectively.

The total funds collected for flood victims

$$= ₹(7{,}000 + 6{,}125 + 7{,}875)$$
$$= ₹\,21{,}000$$

The above situation exhibits the helping nature of students. **Ans.**

SECTION – C

20. Let N denote the set of all natural numbers and R be the relation on N × N defined by (a, b) R (c, d) if $ad(b + c) = bc(a + d)$. Show that R is an equivalence relation. **[6]**

Solution : We know that relation R will be an equivalence relation, if we prove it as a reflexive, symmetric and transitive relation.

(i) Reflexivity :

Let (a, b), be an arbitrary element of N × N

then, $(a, b) \in N \times N$

$a, b \in N$

$\Rightarrow \quad ab(b + a) = ba(a + b)$

$\Rightarrow \quad (a, b) \text{ R } (a, b)$

$\therefore \quad (a, b) \text{ R } (a, b) \;\forall\; (a, b) \in N \times N.$

(ii) Symmetry :

Let (a, b), (c, d) be an arbitrary element of N × N such that (a, b) R (c, d)

$\Rightarrow \quad ad(b + c) = bc(a + d)$

$\Rightarrow \quad cb(d + a) = da(c + b)$

$\Rightarrow \quad (c, d) \text{ R } (a, b)$

$\therefore (a, b) \text{ R } (c, d) \Rightarrow (c, d) \text{ R } (a, b) \;\forall\; (a, b), (c, d) \in N \times N$

So, R is symmetric on N × N.

(iii) Transitivity :

Let (a, b), (c, d), (e, f) be an arbitrary element of N × N such that (a, b) R (c, d) and (c, d) R (e, f), then (a, b) R (c, d)

$\Rightarrow \quad ad(b + c) = bc(a + d)$

$\Rightarrow \quad \dfrac{b+c}{bc} = \dfrac{a+d}{ad}$

$\Rightarrow \quad \dfrac{1}{b} + \dfrac{1}{c} = \dfrac{1}{a} + \dfrac{1}{d}$ ...(i)

Also, (c, d) R (e, f)

$\Rightarrow \quad cf(d + e) = de(c + f)$

$\Rightarrow \quad \dfrac{d+e}{de} = \dfrac{c+f}{cf}$

$\Rightarrow \quad \dfrac{1}{d} + \dfrac{1}{e} = \dfrac{1}{c} + \dfrac{1}{f}$ ...(ii)

Adding (i) and (ii), we get

$$\left(\frac{1}{b} + \frac{1}{c}\right) + \left(\frac{1}{d} + \frac{1}{e}\right) = \left(\frac{1}{a} + \frac{1}{d}\right) + \left(\frac{1}{c} + \frac{1}{f}\right)$$

$\Rightarrow \quad \dfrac{1}{b} + \dfrac{1}{e} = \dfrac{1}{a} + \dfrac{1}{f}$

$\Rightarrow \quad \dfrac{b+e}{be} = \dfrac{a+f}{af}$

$\Rightarrow \quad af(b + e) = be(a + f)$

$\Rightarrow \quad (a, b) \text{ R } (e, f)$

Thus, (a, b) R (c, d) and (c, d) R $(e, f) \Rightarrow (a, b)$ R (e, f). $\forall\; (a, b)\, (c, d)\, (e, f) \in N \times N$

So, R is transitive on N × N.

Hence, R being reflexive, symmetric and transitive, is an equivalence relation on N × N.

Hence Proved.

21. Using integration find the area of the triangle formed by positive x-axis and tangent and normal to the circle $x^2 + y^2 = 4$ at $(1, \sqrt{3})$. **[6]**

Solution : The equation of the given circle is $x^2 + y^2 = 4$. The equation of the normal to the circle at $(1, \sqrt{3})$ is same as the line joining the points $(1, \sqrt{3})$ and $(0, 0)$ which is given by

$$y - 0 = \frac{\sqrt{3} - 0}{1 - 0}(x - 0)$$

$$\left[\because y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)\right]$$

$\Rightarrow \quad y = \sqrt{3}x \quad ...(i)$

So, the slope of the normal is $\sqrt{3}$.

We know that, slope of normal × slope of tangent $= -1$.

$\therefore$ the slope of tangent $= \frac{-1}{\sqrt{3}}$

Now, the equation of the tangent to the circle at $(1, \sqrt{3})$ is given by :

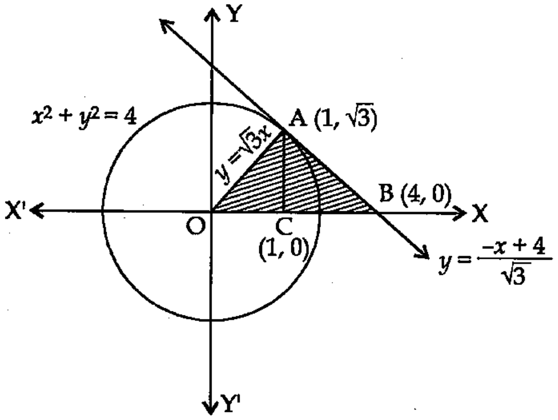
$$y - \sqrt{3} = \frac{-1}{\sqrt{3}}(x-1) \quad [\because y - y_1 = m(x - x_1)]$$

$$\Rightarrow \quad \sqrt{3}y - 3 = -x + 1$$

$$\Rightarrow \quad y = \frac{-x+4}{\sqrt{3}} \quad ...(ii)$$

Putting $y = 0$ in (ii), we get $x = 4$.

Thus, AOB is triangle formed by the tangent, normal and the positive x-axis.



Now, Area of Δ AOB = Area of Δ AOC + Area of Δ ACB.

$$= \int_0^1 y\,dx + \int_1^4 y\,dx$$

$$= \int_0^1 \sqrt{3}\,x\,dx + \int_1^4 \left(\frac{-x+4}{\sqrt{3}}\right)dx$$

$$= \sqrt{3}\int_0^1 x\,dx - \frac{1}{\sqrt{3}}\int_1^4 x\,dx + \frac{4}{\sqrt{3}}\int_1^4 1\,dx$$

$$= \sqrt{3}\left[\frac{x^2}{2}\right]_0^1 - \frac{1}{\sqrt{3}}\left[\frac{x^2}{2}\right]_1^4 + \frac{4}{\sqrt{3}}[x]_1^4$$

$$= \sqrt{3}\left(\frac{1}{2} - 0\right) - \frac{1}{\sqrt{3}}\left(\frac{16}{2} - \frac{1}{2}\right) + \frac{4}{\sqrt{3}}(4-1)$$

$$= \frac{\sqrt{3}}{2} - \frac{15}{2\sqrt{3}} + \frac{12}{\sqrt{3}}$$

$$= \frac{\sqrt{3}}{2} + \frac{9}{2\sqrt{3}} = \frac{\sqrt{3}}{2} + \frac{3\sqrt{3}}{2}$$

$$= 2\sqrt{3} \text{ sq. units.}$$

Hence, the area of the tringle so formed is $2\sqrt{3}$ square units. **Ans.**

OR

Evaluate $\int_1^3 (e^{2-3x} + x^2 + 1)\,dx$ as a limit of a sum.

Solution : We have, $\int_a^b f(x)dx$

$$= \lim_{h \to 0} h\,[f(a) + f(a+h) + f(a+2h) + ... + f(a + (n-1)h)]$$

where $\quad h = \frac{b-a}{n}$

Here, $\quad a = 1, b = 3$

and $\quad f(x) = e^{2-3x} + x^2 + 1$

$\therefore \quad h = \frac{2}{n}$

$\Rightarrow \quad hn = 2$

Now, $\quad f(a) = f(1)$

$$= e^{2-3\times 1} + 1^2 + 1$$

$$f(a+h) = f(1+h) = e^{2-3(1+h)} + (1+h)^2 + 1$$

$$f(a+2h) = f(1+2h) = e^{2-3(1+2h)} + (1+2h)^2 + 1$$

$$f(a+(n-1)h) = f(1+(n-1)h) = e^{2-3[1+(n-1)h]} + [1+(n-1)h]^2 + 1$$

Adding these equations, we get

$$f(a) + f(a+h) + f(a+2h) + ... + f(a+(n-1)h)$$
$$= [e^{2-3\times 1} + 1^2 + 1] + [e^{2-3\times(1+h)} + (1+h)^2 + 1] + [e^{2-3\times(1+2h)}] + (1+2h)^2 + 1] + ... + \{e^{2-3[1+(n-1)h]} + [1+(n-1)h]^2 + 1\}$$

$$\therefore \int_1^3 (e^{2-3x} + x^2 + 1)dx$$

$$= \lim_{h\to 0} h\,[e^2.(e^{-3} + e^{-3(1+h)} + e^{-3(1+2h)} + ... + e] + \lim_{h\to 0} h\,[1^2 + (1+h)^2 + (1+2h)^2 + ... + \{1 + (n-1)h\}^2] + \lim_{h\to 0} h(1 + 1 + ... n \text{ times})$$

$$= \lim_{h\to 0} h\left\{e^2 \times e^{-3}\frac{(1-e^{-3nh})}{1-e^{-3h}}\right\} + \lim_{h\to 0} h\{1 + 1 + ... n \text{ times}\} + 2h[1+2+3+...+(n-1)] + h^2(1^2+2^2+3^2+...+(n-1)^2)\}$$

$$+ \lim_{h\to 0} h(n) \quad \left[\because \text{sum to } n \text{ terms of G. P.} \quad S_n = \frac{a(1-r^n)}{1-r}\right]$$

$$= \lim_{h\to 0} h\left\{\frac{e^{-1}(1-e^{-6})}{1-e^{-3h}}\right\} + \lim_{h\to 0} h\left[n+2h\times\frac{(n-1)n}{2} + h^2\times\frac{(n-1)n(2n-1)}{6}\right] + \lim_{h\to 0} h(n)$$

$$= \lim_{h\to 0} h\left(\frac{e^{-1}(1-e^{-6})}{1-e^{-3h}}\right) + \lim_{h\to 0}\left[2nh+(nh-h)\,nh+\frac{(nh-h)nh(2nh-h)}{6}\right]$$

$$= \frac{1}{e}\left(1-\frac{1}{e^6}\right)\times\frac{\lim_{h\to 0} e^{3h}}{3\times\lim_{h\to 0}\left(\frac{e^{3h}-1}{3h}\right)} + \lim_{h\to 0}\left[2\times 2+(2-h)\times 2+\frac{(2-h)\times 2(2\times 2-h)}{6}\right]$$

$$= \frac{1}{e}\left(1-\frac{1}{e^6}\right)\times\frac{1}{3\times 1}+\left(4+4+\frac{8}{3}\right)$$

$$= \frac{1}{3e}\left(1-\frac{1}{e^6}\right)+\frac{32}{3}$$ **Ans.**

22. Solve the differential equation :

$$(\tan^{-1} y - x)\,dy = (1+y^2)\,dx.$$

[6]

Solution : Same as solution Q. 23 (OR) Set 1 (Outside Delhi) upto eq.

$$x = \tan^{-1} y - 1 + c\,e^{\tan^{-1} y}$$

Ans.

OR

Find the particular solution of the differential equation $\frac{dy}{dx} = \frac{xy}{x^2+y^2}$ given that $y = 1$, when $x = 0$.

Solution : Given, $\frac{dy}{dx} = \frac{xy}{x^2+y^2}$ ...(i)

which is a homogeneous differential equation.

Putting $y = vx$

and $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = \frac{vx^2}{x^2+v^2x^2}$$

$$x\frac{dv}{dx} = \frac{v}{1+v^2} - v$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{-v^3}{1+v^2}$$

$$\Rightarrow \quad \frac{1+v^2}{v^3}dv = \frac{-1}{x}dx$$

$$\Rightarrow \quad \frac{1}{v^3}dv + \frac{1}{v}dv = \frac{-1}{x}dx$$

Now, integrating both sides, we get

$$\frac{v^{-3+1}}{-3+1} + \log|v| = -\log|x| + C$$

$$\Rightarrow \quad \frac{-1}{2v^2} + \log|v| = -\log|x| + C$$

$$\Rightarrow \quad \frac{-1}{2v^2} + \log|vx| = C$$

$$\Rightarrow \quad \frac{-x^2}{2y^2} + \log|y| = C \qquad (\because y = vx)\text{...(ii)}$$

It is given that $y = 1$ when $x = 0$

Putting $x = 0, y = 1$ in (ii)

we get, $C = 0$

Putting $C = 0$ in (ii), we get

$$\log|y| = +\frac{x^2}{2y^2}$$

$$\Rightarrow \quad x^2 = +2y^2 \log|y|$$

Hence, $x^2 = 2y^2 \log|y|$ is the solution of the given equation. **Ans.**

23. If lines $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4}$ and $\frac{x-3}{1} = \frac{y-k}{2} = \frac{z}{1}$ intersect, then find the value of k and hence find the equation of the plane containing these lines. [6]

Solution : The coordinates of any point on first line are

$$\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4} = \lambda$$

i.e., $x = 2\lambda+1$, $y = 3\lambda-1$, $z = 4\lambda+1$

i.e., $(2\lambda+1, 3\lambda-1, 4\lambda+1)$

and the coordinates of any point on second line are

$$\frac{x-3}{1} = \frac{y-k}{2} = \frac{z}{1} = \mu$$

i.e., $x = \mu+3$, $y = 2\mu+k$, $z = \mu$

i.e., $(\mu+3, 2\mu + k, \mu)$

if these two lines intersect each other, then

$$2\lambda+1 = \mu+3,\ 3\lambda-1 = 2\mu+k,\ 4\lambda+1 = \mu$$

i.e. $2\lambda-\mu = 2,\ 3\lambda-2\mu = k+1,\ 4\lambda-\mu = -1$

solving, $2\lambda-\mu = 2$ and $4\lambda-\mu = -1$, we get

$$\lambda = \frac{-3}{2} \text{ and } \mu = -5$$

and substituting the values of λ and μ in $3\lambda - 2\mu = k + 1$, we get

$$k = \frac{9}{2}$$

Now, we have $\vec{b_1} = 2\hat{i} + 3\hat{j} + 4\hat{k}$ and $\vec{b_2} = \hat{i} + 2\hat{j} + \hat{k}$. So, the required plane contains both lines and it passes through a point $\vec{a}$ (1, – 1, 1) and perpendicular vector $\vec{n}$, given by, $\vec{n} = \vec{b_1} \times \vec{b_2}$

$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix}$$

$$= \hat{i}(3-8) - \hat{j}(2-4) + \hat{k}(4-3)$$

$$= -5\hat{i} + 2\hat{j} + \hat{k}$$

∴ The equation of plane passing through $\vec{a}$ and perpendicular to $\vec{n}$ is given by

$$(\vec{r} - \vec{a}).\vec{n} = 0$$

$$[\vec{r} - (\hat{i} - \hat{j} + \hat{k})].(-5\hat{i} + 2\hat{j} + \hat{k}) = 0$$

$$\Rightarrow \vec{r}(-5\hat{i} + 2\hat{j} + \hat{k}) = (\hat{i} - \hat{j} + \hat{k}).(-5\hat{i} + 2\hat{j} + \hat{k})$$

$$\Rightarrow \vec{r}(-5\hat{i} + 2\hat{j} + \hat{k}) = -6$$

Writing $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

or $5x - 2y - z - 6 = 0.$ **Ans.**

24. If A and B are two independent events such that $P(\bar{A} \cap B) = \frac{2}{15}$ and $P(A \cap \bar{B}) = \frac{1}{6}$, then find P(A) and P(B). **[6]**

Solution : Let $P(A) = x$ and $P(B) = y$

$$\therefore \quad P(\bar{A}) = 1 - P(A) = 1 - x, P(\bar{B}) = 1 - y$$

We have, $P(\bar{A} \cap B) = \frac{2}{15}$ and $(P \cap \bar{B}) = \frac{1}{6}$

Now, $P(\bar{A} \cap B) = \frac{2}{15}$

$$\Rightarrow \quad P(\bar{A})\,P(B) = \frac{2}{15}$$

$$\Rightarrow \quad (1-x)y = \frac{2}{15} \quad \text{...(i)}$$

Since, A and B are independent events so are $\bar{A}$ and B as well as A and $\bar{B}$.

Given, $P(A \cap \bar{B}) = \frac{1}{6}$

$$\Rightarrow \quad P(A)P(\bar{B}) = \frac{1}{6}$$

$$\Rightarrow \quad x(1-y) = \frac{1}{6} \quad \text{...(ii)}$$

$$\Rightarrow \quad x = \frac{1}{6-6y}$$

Putting the value of x in eq. (i), we get

$$\left(1 - \frac{1}{6-6y}\right)y = \frac{2}{15}$$

$$\Rightarrow \quad -90y^2 + 87y = 12$$

or $\quad 30y^2 - 29y + 4 = 0$

Solving the quadratic equation, we get

$$y = \frac{4}{5}, y = \frac{1}{6}$$

For $y = \frac{4}{5}$; using (ii), we get

$$x = \frac{1}{6 - 6 \times \frac{4}{5}} = \frac{5}{30-24} = \frac{5}{6}$$

For $y = \frac{1}{6}$ using (ii), we get

$$x = \frac{1}{6 - 6 \times \frac{1}{6}} = \frac{1}{5}$$

$$\therefore \quad P(A) = \frac{5}{6}, P(B) = \frac{4}{5}$$

or $\quad P(A) = \frac{1}{5}, P(B) = \frac{1}{6}.$ **Ans.**

25. Find the local maxima and local minima of the function $f(x) = \sin x - \cos x, 0 < x < 2\pi$. Also find the local maximum and local minimum values. **[6]**

Solution : We have, $f(x) = \sin x - \cos x, 0 < x < 2\pi$.

$$\Rightarrow \quad f'(x) = \cos x + \sin x$$

For local maximum or minimum, we have

$$\Rightarrow \quad f'(x) = 0$$

$$\Rightarrow \quad \cos x + \sin x = 0$$

$$\Rightarrow \quad \cos x = -\sin x$$

$$\Rightarrow \quad \tan x = -1$$

$$\Rightarrow \quad x = \frac{3\pi}{4} \text{ or } x = \frac{7\pi}{4}$$

Thus, $x = \frac{3\pi}{4}$ and $x = \frac{7\pi}{4}$ are possible points of local maximum or minimum.

We have, $f'(x) = \frac{d}{dx}(\cos x + \sin x)$

$= -\sin x + \cos x$

At $x = \frac{3\pi}{4}$, we have

$$f''\left(\frac{3\pi}{4}\right) = -\sin\frac{3\pi}{4} + \cos\frac{3\pi}{4}$$

$$= -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2} < 0$$

$$\Rightarrow \quad f''\left(\frac{3\pi}{4}\right) < 0$$

So, $x = \frac{3\pi}{4}$ is the point of local maximum.

Local maximum value

$$= f\left(\frac{3\pi}{4}\right)$$

$$= \sin\frac{3\pi}{4} - \cos\frac{3\pi}{4}$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$$

At $x = \frac{7\pi}{4}$, we have

$$f''\left(\frac{7\pi}{4}\right) = -\sin\frac{7\pi}{4} + \cos\frac{7\pi}{4}$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2} > 0$$

$$\Rightarrow \quad f''\left(\frac{7\pi}{4}\right) > 0$$

So, $x = \frac{7\pi}{4}$, is the point of local minimum.

Local minimum value

$$= f\left(\frac{7\pi}{4}\right)$$

$$= \sin\frac{7\pi}{4} - \cos\frac{7\pi}{4}$$

$$= -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2}.$$ **Ans.**

26. Find graphically, the maximum value of $z = 2x + 5y$, subject to constraints given below : [6]

$$2x + 4y \le 8$$
$$3x + y \le 6$$
$$x + y \le 4$$
$$x \ge 0,\ y \le 0$$

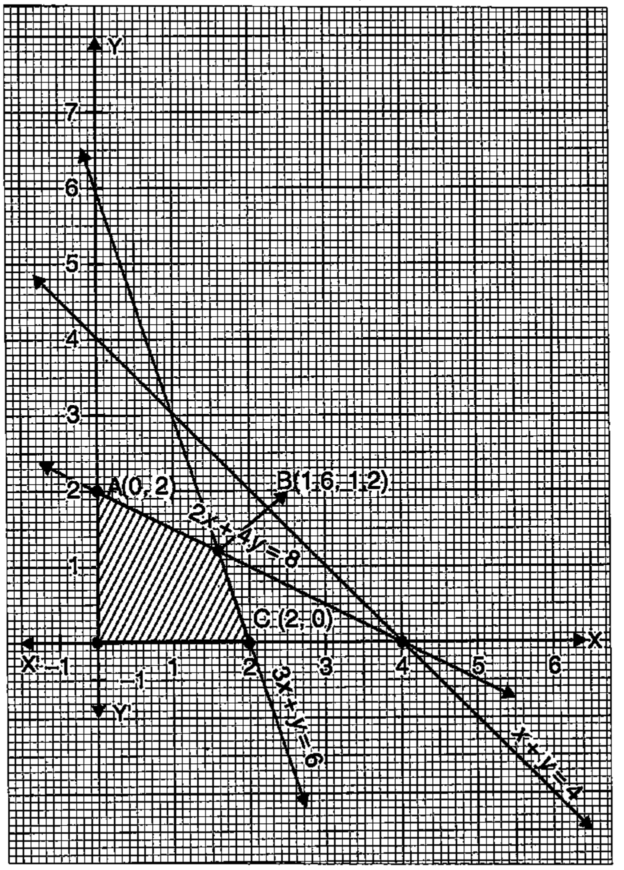
Solution : We first convert the inequalities into equations to obtain lines

$$2x + 4y = 8 \quad \text{...(i)}$$
$$3x + y = 6 \quad \text{...(ii)}$$
$$x + y = 4 \quad \text{...(iii)}$$
$$x = 0$$
and $$y = 0.$$

We need to maximize the objective function

$$z = 2x + 5y$$

These lines are drawn and the feasible region of the L.P.P. is the shaded region :



The point of intersection of (i) and (ii) is B (1.6, 1.2)

The coordinates of the corner points of the feasible region are O(0, 0), A(0, 2), B (1.6, 1.2) and C(2, 0).

The value of the objective function at these points are given in the following table :

Corner Points	Value of the objective function $z = 2x + 5y$
O(0, 0)	$2\times0+5\times0=0$
A(0, 2)	$2\times0+5\times2=10$ maximum
B(1.6, 1.2)	$2\times1.6+5\times1.2=9.2$
C(2, 0)	$2\times2+5\times0=4$

Out of these values of z, the maximum value of z is 10 which is attained at the point (0, 2). Thus the maximum value of z is 10. **Ans.**

●●

All questions are same in Delhi Set II and Set III

Mathematics 2016 (Outside Delhi) SET I

Time allowed : 3 hours **Maximum marks : 100**

SECTION — A

1. If $x \in$ N and $\begin{vmatrix} x+3 & -2 \\ -3x & 2x \end{vmatrix} = 8$, then find the value of x. **[1]**

Solution : We have, $\begin{vmatrix} x+3 & -2 \\ -3x & 2x \end{vmatrix} = 8$

$$\Rightarrow \quad (x+3) \times 2x - (-2) \times (-3x) = 8$$

$$\Rightarrow \quad 2x^2 + 6x - 6x = 8$$

$$\Rightarrow \quad 2x^2 = 8$$

$$\Rightarrow \quad x^2 = 4$$

But $x \neq -2$ as $x \in$ N

$$\Rightarrow \quad x = 2$$ **Ans.**

2. Use elementary column operation $C_2 \to C_2 + 2C_1$ in the following matrix equation :

$$\begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$ **[1]**

Solution : We have

$$\begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$P = A.B \text{ (say)}$$

Applying $C_2 \to C_2 + 2C_1$ on P, we get

$$\begin{bmatrix} 2 & 5 \\ 2 & 4 \end{bmatrix} = Q \text{ (say)}$$

Applying $C_2 \to C_2 + 2C_1$ on B, we get

$$\begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} = C \text{ (say)}$$

Now, $$AC = \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} = Q$$

$$\therefore \quad \begin{bmatrix} 2 & 5 \\ 2 & -4 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$$ **Ans.**

3. Write the number of all possible matrices of order 2 × 2 with each entry 1, 2 or 3. **[1]**

Solution : Total number of all possible matrices of order 2 × 2 with each entry 1, 2 or 3 are 3^4 *i.e.*, 81. **Ans.**

4. Write the position vector of the point which divides the join of points with position vector $3\vec{a} - 2\vec{b}$ and $2\vec{a} + 3\vec{b}$ in the ratio 2 : 1. **[1]**

Solution : Let A and B be the given points with position vectors $3\vec{a} - 2\vec{b}$ and $2\vec{a} + 3\vec{b}$ respectively.

Let P and Q be the points dividing AB in the ratio 2 : 1 internally and externally respectively. Then,

$$\text{Position vector of P} = \frac{1(3\vec{a} - 2\vec{b}) + 2(2\vec{a} + 3\vec{b})}{1+2}$$

$$= \frac{7\vec{a}}{3} + \frac{4\vec{b}}{3}$$

$$\text{Position vector of Q} = \frac{1(3\vec{a} - 2\vec{b}) - 2(2\vec{a} + 3\vec{b})}{1-2}$$

$$= \vec{a} + 8\vec{b}$$ **Ans.**

5. Write the number of vectors of unit length perpendicular to both the vectors $\vec{a} = 2\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = \hat{j} + \hat{k}$. **[1]**

Solution : We know that the unit vectors perpendicular to the plane of $\vec{a}$ and $\vec{b}$ are $\pm \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$

So, $$|\vec{a} \times \vec{b}| = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 2 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= (1-2)\hat{i} - (2-0)\hat{j} + (2-0)\hat{k}$$

$$= -\hat{i} - 2\hat{j} + 2\hat{k}$$

$$\Rightarrow \quad |\vec{a} \times \vec{b}| = \sqrt{(-1)^2 + (-2)^2 + (2)^2} = \sqrt{9} = 3$$

Hence, required vectors

$$= \pm \frac{1}{3}(-\hat{i} - 2\hat{j} + 2\hat{k})$$

and number of vectors are 2. **Ans.**

6. Find the vector equation of the plane with intercepts 3, – 4 and 2 on x, y and z-axis respectively. **[1]**

Solution : The equation of the required plane is,

$$\frac{x}{3} + \frac{y}{-4} + \frac{z}{2} = 1$$

$$\Rightarrow \quad 4x - 3y + 6z = 12$$

In vector form : $(x\hat{i} + y\hat{j} + z\hat{k}).(4\hat{i} - 3\hat{j} + 6\hat{k}) = 12$

i.e., $$\vec{r}.(4\hat{i} - 3\hat{j} + 6\hat{k}) = 12$$

This is the vector equation of the plane with intercept 3, – 4 and 2 on coordinate axis. **Ans.**

SECTION — B

7. Find the coordinates of the point where the line through the points A(3, 4, 1) and B(5, 1, 6) crosses the XZ-plane. Also find the angle which this line makes with the XZ-plane. [4]

Solution : The equation of the line passing through A and B is,

$$\frac{x-3}{5-3}=\frac{y-4}{1-4}=\frac{z-1}{6-1} \text{ or } \frac{x-3}{2}=\frac{y-4}{-3}=\frac{z-1}{5} \quad ...(i)$$

The coordinates of any point on this line are given by

$$\frac{x-3}{2}=\frac{y-4}{-3}=\frac{z-1}{5}=\lambda$$

$$\Rightarrow \quad x = 2\lambda + 3, y = -3\lambda + 4, z = 5\lambda + 1$$

So $(2\lambda + 3, -3\lambda + 4, 5\lambda + 1)$ are the coordinates of any point on the line passing through A and B. If it lies on XZ-plane then $y = 0$.

$$\therefore \quad -3\lambda + 4 = 0 \Rightarrow \lambda = \frac{4}{3}$$

So, the coordinates of required point are

$$\left(2\times\frac{4}{3}+3,-3\times\frac{4}{3}+4,5\times\frac{4}{3}+1\right) i.e., \left(\frac{17}{3},0,\frac{23}{3}\right)$$

Now, the line in equation (i) is parallel to the vector $\vec{b}=2\hat{i}-3\hat{j}+5\hat{k}$ and the XZ-plane is normal to the vector $\vec{n}=\hat{j}$. Therefore, the angle θ between them is given by

$$\sin\theta = \left|\frac{\vec{b}\cdot\vec{n}}{|\vec{b}|\cdot|\vec{n}|}\right|$$

$$= \left|\frac{(2\hat{i}-3\hat{j}+5\hat{k})(\hat{j})}{\sqrt{(2)^2+(-3)^2+(5)^2}\sqrt{(1)^2}}\right|$$

$$\Rightarrow \quad \sin\theta = \frac{3}{\sqrt{38}}$$

$$\Rightarrow \quad \theta = \sin^{-1}\left(\frac{3}{\sqrt{38}}\right)$$

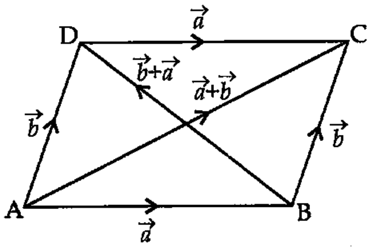
$$\Rightarrow \quad \theta = \sin^{-1}(0.4866)$$ **Ans.**

8. The two adjacent sides of a parallelogram are $2\hat{i}-4\hat{j}-5\hat{k}$ and $2\hat{i}+2\hat{j}+3\hat{k}$. Find the two unit vectors parallel to its diagonals. Using the diagonal vectors, find the area of the parallelogram. [4]

Solution : Let ABCD be a parallelogram such that

$$\vec{AB} = \vec{a} = 2\hat{i}-4\hat{j}-5\hat{k}$$

and $$\vec{BC} = \vec{b} = 2\hat{i}+2\hat{j}+3\hat{k}$$



Then,

$$\vec{AB} + \vec{BC} = \vec{AC}$$

$$\Rightarrow \quad \vec{AC} = \vec{a}+\vec{b} = 4\hat{i}-2\hat{j}-2\hat{k}$$

and $$\vec{AB} + \vec{BD} = \vec{AD}$$

$$\Rightarrow \quad \vec{BD} = \vec{AD} - \vec{AB}$$

$$\Rightarrow \quad \vec{BD} = \vec{b}-\vec{a} = 0\hat{i}+6\hat{j}+8\hat{k}$$

Now, $$\vec{AC} = 4\hat{i}-2\hat{j}-2\hat{k}$$

$$|\vec{AC}| = \sqrt{(4)^2+(-2)^2+(-2)^2} = \sqrt{24} = 2\sqrt{6}$$

and $$|\vec{BD}| = 0\hat{i}+6\hat{j}+8\hat{k}$$

$$|\vec{BD}| = \sqrt{(0)^2+(6)^2+(8)^2} = \sqrt{100} = 10$$

Unit vector along $\vec{AC}$

$$= \frac{\vec{AC}}{|\vec{AC}|} = \frac{1}{2\sqrt{6}}(4\hat{i}-2\hat{j}-2\hat{k})$$

$$= \frac{1}{\sqrt{6}}(2\hat{i}-\hat{j}-\hat{k})$$

Unit vector along $\vec{BD}$

$$= \frac{\vec{BD}}{|\vec{BD}|} = \frac{1}{10}(6\hat{j}+8\hat{k}) = \frac{1}{5}(3\hat{j}+4\hat{k})$$

Now, area of parallelogram

$$= \frac{1}{2}|\vec{AC}\times\vec{BD}|$$

$$\Rightarrow \quad \vec{AC}\times\vec{BD} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -2 & -2 \\ 0 & 6 & 8 \end{vmatrix}$$

$$= 4\hat{i} + 32\hat{j} + 24\hat{k}$$

Area of parallelogram

$$= \frac{1}{2}\left|\vec{AC}\times\vec{BD}\right|$$

$$= \sqrt{404} \text{ or } 2\sqrt{101} \text{ sq. units}$$ **Ans.**

9. In a game, a man wins ₹ 5 for getting a number greater than 4 and loses ₹ 1 otherwise, when a fair die is thrown. The man decided to throw

a die thrice but to quit as and when he gets a number greater than 4. Find the expected value of the amount he wins/loses. [4]

Solution : Let n denote the number of throws required to get a number greater than 4 and X denote the amount won/lost.

The man may get a number greater than 4 in the very first throw of the die or in second throw or in the third throw.

Let p = Probability of getting a number greater than 4

$$= \frac{2}{6}$$

$$q = 1-p = \frac{4}{6}$$

Thus, we have the following probability distribution for X.

Number of throws (n)	1	2	3	3
Amount won/lost (X)	5	4	3	– 3
Probability (P(X))	$\frac{2}{6}$	$\frac{4}{6}\times\frac{2}{6}$	$\frac{4}{6}\times\frac{4}{6}\times\frac{2}{6}$	$\frac{4}{6}\times\frac{4}{6}\times\frac{4}{6}$

$$E(X) = 5\times\frac{2}{6}+4\times\frac{4}{6}\times\frac{2}{6}+3\times\frac{4}{6}\times\frac{4}{6}\times\frac{2}{6}+(-3)\times\frac{4}{6}\times\frac{4}{6}\times\frac{4}{6}$$

$$= \frac{19}{9}$$

Expected amount he could wins is ₹ $\frac{19}{9}$ **Ans.**

OR

A bag contains 4 balls. Two balls are drawn at random (without replacement) and are found to be white. What is the probability that all balls in the bag are white ?

Solution : We know that the number of white balls can't be less than 2.

Now, there are different cases, for the number of white balls in the bag. The total cases are ${}^2C_2 + {}^3C_2 + {}^4C_2$.

$$= \frac{2!}{2!\times 0!}+\frac{3!}{2!\times 1!}+\frac{4!}{2!\times 2!}$$

$$= 1+3+6 = 10$$

∴ Probability of the case that there are 4 white balls

i.e., ${}^4C_2 = 6$

Hence, the probability that all balls in the bag are white is

$$\frac{6}{10} \text{ or } \frac{3}{5}.$$ **Ans.**

10. Differentiate $x^{\sin x} + (\sin x)^{\cos x}$ with respect to x. [4]

Solution : We have,

$$x^{\sin x} + (\sin x)^{\cos x}$$

Let $y = x^{\sin x} + (\sin x)^{\cos x}$

Taking log on both sides,

$$\log y = \sin x \log x + \cos x \log (\sin x)$$

$$\Rightarrow \quad y = e^{\sin x \log x} + e^{\cos x \log (\sin x)}$$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = e^{\sin x\log x}\frac{d}{dx}(\sin x\log x)+e^{\cos x\log(\sin x)}\frac{d}{dx}(\cos x\log(\sin x))$$

$$\Rightarrow \frac{dy}{dx} = x^{\sin x}\left\{\cos x\log x+\frac{\sin x}{x}\right\}+(\sin x)^{\cos x}\left\{-\sin x\log(\sin x)+\cos x.\frac{1}{\sin x}.\cos x\right\}$$

$[\because e^{\sin x \log x} = x^{\sin x} \text{ and } e^{\cos x \log (\sin x)} = (\sin x)^{\cos x}]$

$$= x^{\sin x}\left\{\cos x\log x+\frac{\sin x}{x}\right\}+(\sin x)^{\cos x}\left\{-\sin x\log(\sin x)+\frac{\cos^2 x}{\sin x}\right\}$$ **Ans.**

OR

If $y = 2 \cos (\log x) + 3 \sin (\log x)$, prove that $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0$.

Solution : Given, $y = 2 \cos (\log x) + 3 \sin (\log x)$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = -2\sin(\log x).\frac{1}{x}+3\cos(\log x).\frac{1}{x}$$

$$\Rightarrow x\frac{dy}{dx} = -2\sin(\log x)+3\cos(\log x)$$

Again differentiating both sides w.r.t. x, we get

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx} = -2\cos(\log x).\frac{1}{x}-3\sin(\log x).\frac{1}{x}$$

$$\Rightarrow \quad x^2\frac{d^2y}{dx^2}+x.\frac{dy}{dx} = -2\sin(\log x)+3\cos(\log x)$$

$$\Rightarrow \quad x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx} = -y$$

$$\Rightarrow x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0$$ **Hence Proved.**

11. If $x = a \sin 2t\ (1 + \cos 2t)$ and $y = b \cos 2t\ (1 - \cos 2t)$, find $\frac{dy}{dx}$ at $t = \frac{\pi}{4}$. [4]

Solution : We have, $x = a \sin 2t\ (1 + \cos 2t)$

$\Rightarrow \frac{dx}{dt} = 2a \cos 2t\,(1 + \cos 2t) - 2a \sin^2 2t$

and $y = b \cos 2t\,(1 - \cos 2t)$

$\Rightarrow \frac{dy}{dt} = 2b \cos 2t \sin 2t - 2b \sin 2t\,(1 - \cos 2t)$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= \frac{2b\cos 2t \sin 2t - 2b \sin 2t\,(1-\cos 2t)}{2a\cos 2t(1+\cos 2t) - 2a\sin^2 2t}$$

$$= \frac{b[\cos 2t \sin 2t - \sin 2t(1-\cos 2t)]}{a[\cos 2t(1+\cos 2t) - \sin^2 2t]}$$

Since, we know that at $t = \frac{\pi}{4}, \sin 2t = 1$ and $\cos 2t = 0$

So, $\left(\frac{dy}{dx}\right)_{t=\frac{\pi}{4}} = \frac{b[0.1 - 1(1-0)]}{a[0(1+0) - 1^2]} = \frac{b}{a}$ **Ans.**

12. The equation of tangent at (2, 3) on the curve $y^2 = ax^3 + b$ is $y = 4x - 5$. Find the value of a and b. [4]

Solution : Since the point (2, 3) lies on the curve $y^2 = ax^3 + b$

$\Rightarrow (3)^2 = a(2)^3 + b$

$\Rightarrow 9 = 8a + b$ ...(i)

and $y^2 = ax^3 + b$

On differentiating w.r.t. x,

$$2y\frac{dy}{dx} = 3ax^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{3ax^2}{2y}$$

$$\left(\frac{dy}{dx}\right)_{(2,3)} = \frac{3a(2)^2}{2(3)} = 2a$$

The required equation of the tangent at (2, 3) is

$(y - 3) = 2a\,(x - 2)$

$\Rightarrow y - 3 = 2ax - 4a$

$\Rightarrow y = 2ax + 3 - 4a$

Comparing the equation with $y = 4x - 5$, we get

$4 = 2a \Rightarrow a = 2$

From equation (i), we get

$9 = 8(2) + b \Rightarrow b = -7$ **Ans.**

13. Find : $\int \frac{x^2}{x^4 + x^2 - 2}dx$ [4]

Solution : Let,

$$I = \int \frac{x^2}{x^4+x^2-2}dx = \int \frac{x^2}{(x^2-1)(x^2+2)}dx$$

Let $x^2 = y$

Then, $\frac{y}{(y-1)(y+2)} = \frac{A}{(y-1)} + \frac{B}{(y+2)}$ ...(i)

$y = A\,(y+2) + B\,(y-1)$ ...(ii)

Putting $y = 1$ and $y = -2$ successively in (ii), we get

$$A = \frac{1}{3} \text{ and } B = \frac{2}{3}$$

Substituting the values of A and B in (i), we obtain

$$\frac{y}{(y-1)(y+2)} = \frac{1}{3(y-1)} + \frac{2}{3(y+2)}$$

Replacing y by x^2, we obtain

$$\frac{x^2}{(x^2-1)(x^2+2)} = \frac{1}{3(x^2-1)} + \frac{2}{3(x^2+2)}$$

$$\therefore I = \int \frac{x^2}{x^4+x^2-2}dx = \frac{1}{3}\int \frac{1}{(x^2-1)}dx + \frac{2}{3}\int \frac{1}{(x^2+2)}dx$$

$$= \frac{1}{3}\cdot\frac{1}{2}\log\left|\frac{x-1}{x+1}\right| + \frac{2}{3}\cdot\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C$$

$$= \frac{1}{6}\log\left|\frac{x-1}{x+1}\right| + \frac{\sqrt{2}}{3}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C$$ **Ans**

14. Evaluate : $\int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x}dx$ [4]

Solution : We have,

$$I = \int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x}dx \quad ...(i)$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sin^2\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right) + \cos\left(\frac{\pi}{2}-x\right)}dx$$

$$\left[\text{Using} : \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\cos^2 x}{\cos x + \sin x}dx \quad ...(ii)$$

Adding equation (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} + \frac{\cos^2 x}{\sin x + \cos x}dx$$

$$= \int_0^{\pi/2} \frac{1}{\sin x + \cos x}dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} \frac{1}{\frac{2\tan x/2}{1+\tan^2 x/2} + \frac{1-\tan^2 x/2}{1+\tan^2 x/2}}dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} \frac{1+\tan^2 x/2}{2\tan x/2 + 1 - \tan^2 x/2}dx$$

$$= \int_0^{\pi/2} \frac{\sec^2 x/2}{2\tan x/2 + 1 - \tan^2 x/2}dx$$

Let $\tan\frac{x}{2} = t$.

Then, $\sec^2\frac{x}{2}\cdot\frac{1}{2}dx = dt \Rightarrow \sec^2\frac{x}{2}dx = 2dt$

Also, $x = 0 \Rightarrow t = \tan 0 = 0$ and $x = \frac{\pi}{2} \Rightarrow t = \tan\frac{\pi}{4} = 1$

$\therefore \quad 2I = \int_0^1 \frac{2dt}{2t+1-t^2} = 2\int_0^1 \frac{1}{(\sqrt{2})^2-(t-1)^2}dt$

$\Rightarrow \quad 2I = 2\times\frac{1}{2\sqrt{2}}\left[\log\left|\frac{\sqrt{2}+t-1}{\sqrt{2}-t+1}\right|\right]_0^1$

$\Rightarrow \quad 2I = \frac{1}{\sqrt{2}}\left\{\log\left(\frac{\sqrt{2}}{\sqrt{2}}\right)-\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right\}$

$\Rightarrow \quad 2I = \frac{1}{\sqrt{2}}\left\{0-\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right\}$

$\Rightarrow \quad 2I = \frac{1}{\sqrt{2}}\left\{\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)^{-1}\right\}$

$\Rightarrow \quad 2I = \frac{1}{\sqrt{2}}\left\{\log\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)\right\}$

$\Rightarrow \quad I = \frac{1}{2\sqrt{2}}\log\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$ **Ans.**

OR

Evaluate : $\int_0^{3/2}|x\cos\pi x|\,dx$

Solution : We have, $0 < x < \frac{3}{2} \Rightarrow 0 < \pi x < \frac{3\pi}{2}$

Now, $0 < x < \frac{1}{2}$

$0 < \pi x < \frac{\pi}{2}$

$\Rightarrow \cos\pi x > 0$

$\Rightarrow x\cos\pi x > 0$

$\Rightarrow |x\cos\pi x| = x\cos\pi x$ for $0 < \pi x < \frac{\pi}{2}$

and $\frac{1}{2} < x < \frac{3}{2}$

$\Rightarrow \frac{\pi}{2} < \pi x < \frac{3\pi}{2}$

$\Rightarrow \cos\pi x < 0$

$\Rightarrow x\cos\pi x < 0$

$\Rightarrow |x\cos\pi x| = -x\cos\pi x$ for $\pi < \pi x < \frac{3\pi}{2}$

$\therefore \int_0^{3/2}|x\cos\pi x|dx$

$= \int_0^{1/2}|x\cos\pi x|dx + \int_{1/2}^{3/2}|x\cos\pi x|dx$

$= \int_0^{1/2}(x\cos\pi x)dx + \int_{1/2}^{3/2}(-x\cos\pi x)dx$

$= \int_0^{1/2}(x\cos\pi x)dx - \int_{1/2}^{3/2}(x\cos\pi x)dx$

$= \left[\left(\left|\frac{x\sin\pi x}{\pi}\right|_0^{1/2} - \int_0^{1/2}\frac{\sin\pi x}{\pi}dx\right) - \left(\left|\frac{x\sin\pi x}{\pi}\right|_{1/2}^{3/2} - \int_{1/2}^{3/2}\frac{\sin\pi x}{\pi}dx\right)\right]$

$= \left[\left\{\frac{1}{\pi}\left(\frac{1}{2}\frac{\sin\pi}{2}-0\right)+\left.\frac{\cos\pi x}{\pi^2}\right|_0^{1/2}\right\} - \left\{\frac{1}{\pi}\left(\frac{3}{2}\sin\frac{3\pi}{2}-\frac{1}{2}\sin\frac{\pi}{2}\right)+\left.\frac{\cos\pi x}{\pi^2}\right|_{1/2}^{3/2}\right\}\right]$

$= \frac{1}{2\pi}+\frac{1}{\pi^2}\left(\frac{\cos\pi}{2}-\cos 0\right)-\frac{1}{\pi}\left(\frac{3}{2}(-1)-\frac{1}{2}(1)\right) + \frac{1}{\pi^2}\left(\frac{\cos 3\pi}{2}-\frac{\cos\pi}{2}\right)\Big\}$

$= \frac{1}{2\pi}+\frac{1}{\pi^2}(0-1)-\frac{1}{\pi}\left(-\frac{3}{2}-\frac{1}{2}\right)+\frac{1}{\pi^2}(0-0)$

$= \frac{1}{2\pi}-\frac{1}{\pi^2}+\frac{2}{\pi}$

$= \frac{1}{2\pi}+\frac{2}{\pi}-\frac{1}{\pi^2}$

$= \frac{\pi+4\pi-2}{2\pi^2}$

$= \frac{5\pi-2}{2\pi^2}$ **Ans.**

15. Find : $\int(3x+1)\sqrt{4-3x-2x^2}\,dx$ **[4]**

Solution : Let $3x+1 = \lambda\frac{d}{dx}(4-3x-2x^2)+\mu$.

Then, $3x+1 = \lambda(-3-4x)+\mu$

$\Rightarrow 3x+1 = -4\lambda x+(-3\lambda+\mu)$

Comparing the coefficients of like powers of x, we get

$-4\lambda = 3$ and $-3\lambda+\mu = 1$

$\Rightarrow \lambda = \frac{-3}{4}$ and $\mu = \frac{-5}{4}$

Let $I = \int(3x+1)\sqrt{4-3x-2x^2}dx$

$I = \int\left\{-\frac{3}{4}(-3-4x)-\frac{5}{4}\right\}\sqrt{4-3x-2x^2}dx$

$= -\frac{3}{4}\int(-3-4x)\sqrt{4-3x-2x^2}dx - \frac{5}{4}\int\sqrt{4-3x-2x^2}\,dx$

$= -\frac{3}{4}\int\sqrt{t}\,dt - \frac{5}{4}\int\sqrt{-2\left(x^2+\frac{3}{2}x+\frac{9}{16}-\frac{9}{16}-2\right)}dx$

where, $t = 4 - 3x - 2x^2$

$\Rightarrow \quad dt = (-3-4x)\,dx$

$$= -\frac{3}{4}\left(\frac{t^{3/2}}{3/2}\right) + c_1 - \frac{5\sqrt{2}}{4}\int\sqrt{-\left(x+\frac{3}{4}\right)^2 + \left(\frac{\sqrt{41}}{4}\right)^2}\,dx$$

$$= -\frac{1}{2}(4-3x-2x^2)^{3/2} + c_1$$
$$-\frac{5}{2\sqrt{2}}\left\{\frac{1}{2}\left(x+\frac{3}{4}\right)\sqrt{\left(\frac{\sqrt{41}}{4}\right)^2 - \left(x+\frac{3}{4}\right)^2}\right\}$$
$$-\frac{5}{2\sqrt{2}}\times\frac{1}{2}\left(\frac{\sqrt{41}}{4}\right)^2\sin^{-1}\left(\frac{x+\frac{3}{4}}{\sqrt{41}/4}\right) + c_2$$

$$= -\frac{1}{2}(4-3x-2x^2)^{3/2} - \frac{5}{4\sqrt{2}}\left\{\left(x+\frac{3}{4}\right)\sqrt{\frac{4-3x-2x^2}{2}}\right\}$$
$$-\frac{5}{4\sqrt{2}}\times\frac{41}{16}\sin^{-1}\left(\frac{4x+3}{\sqrt{41}}\right) + c$$

where $(c = c_1 + c_2)$

$$= -\frac{1}{2}(4-3x-2x^2)^{3/2} - \frac{5}{8}\left(x+\frac{3}{4}\right)\sqrt{4-3x-2x^2}$$
$$+\frac{5\times 41}{4\sqrt{2}\times 16}\sin^{-1}\left(\frac{4x+3}{\sqrt{41}}\right) + c$$

$$= -\frac{1}{2}\left(4-3x-2x^2\right)^{3/2} - \frac{5}{8}\left(\frac{4x+3}{4}\right)\sqrt{4-3x-2x^2}$$
$$+\frac{5\times 41\sqrt{2}}{8\times 16}\sin^{-1}\left(\frac{4x+3}{\sqrt{41}}\right) + c$$

$$= -\frac{1}{2}\left(4-3x-2x^2\right)^{3/2} - \frac{5}{8}\left[\left(\frac{4x+3}{4}\right)\sqrt{4-3x-2x^2} + \frac{41\sqrt{2}}{16}\sin^{-1}\left(\frac{4x+3}{\sqrt{41}}\right) + c\right]$$ **Ans.**

16. Solve the differential equation :

$$y + x\frac{dy}{dx} = x - y\frac{dy}{dx}$$ **[4]**

Solution : We have, $y + x\frac{dy}{dx} = x - y\frac{dy}{dx}$

$$\Rightarrow \quad x\frac{dy}{dx} + y\frac{dy}{dx} = x - y$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x-y}{x+y} \qquad ...(i)$$

Which is a homogeneous differential equation.

Putting $y = Vx \Rightarrow \frac{dy}{dx} = V + x\frac{dV}{dx}$ in (i), we get

$$V + x\frac{dV}{dx} = \frac{x - Vx}{x + Vx}$$

$$\Rightarrow \quad V + x\frac{dV}{dx} = \frac{1-V}{1+V}$$

$$\Rightarrow \quad x\frac{dV}{dx} = \frac{1-V}{1+V} - V$$

$$\Rightarrow \quad x\frac{dV}{dx} = \frac{1-V-V-V^2}{1+V}$$

$$\Rightarrow \quad x\frac{dV}{dx} = \frac{1-2V-V^2}{1+V}$$

$$\Rightarrow \quad \frac{1+V}{1-2V-V^2}dV = \frac{dx}{x}, \quad x \neq 0$$

$$\Rightarrow \quad \frac{1+V}{V^2+2V-1}dV = \frac{-dx}{x} \qquad ...(ii)$$

Putting $t = V^2 + 2V - 1$

$\Rightarrow \quad dt = (2V + 2)\,dV$

$\Rightarrow \quad \frac{1}{2}dt = (V+1)\,dV$

Now, equation (ii) becomes

$$\frac{1}{2t}dt = \frac{-dx}{x}$$

On integrating above equation we get,

$$\frac{1}{2}\int\frac{1}{t}dt = -\int\frac{dx}{x}$$

$$\Rightarrow \quad \frac{1}{2}\log|t| = -\log|x| + \log C$$

$$\Rightarrow \frac{1}{2}\log|V^2+2V-1|^{1/2} = \log\left|\frac{C}{x}\right|$$

$$\Rightarrow \quad V^2 + 2V - 1 = \left(\frac{C}{x}\right)^2$$

$$\Rightarrow \quad \left(\frac{y}{x}\right)^2 + 2\left(\frac{2y}{x}\right) - 1 = \frac{C^2}{x^2} \qquad \left(\because V = \frac{y}{x}\right)$$

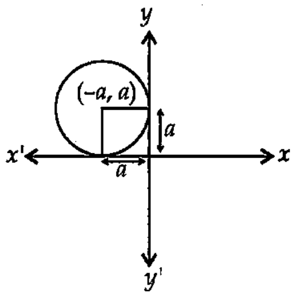
$$\Rightarrow \quad \frac{y^2}{x^2} + \frac{2y}{x} - 1 = \frac{C^2}{x^2}$$

$$\Rightarrow \quad y^2 + 2xy - x^2 = C^2$$ **Ans.**

17. Form the differential equation of the family of circles in the second quadrant and touching the coordinate axes. **[4]**

Solution : The equation of circles in the second quadrant which touch the coordinate axes is

$$(x+a)^2 + (y-a)^2 = a^2, \; a \in R \qquad ...(i)$$



where a is a parameter. This equation contains one arbitrary constant. So we shall differentiate it once only and we shall get a differential equation of first order.

Differentiating (i) w.r.t. x, we get

$$2(x+a)+2(y-a)\frac{dy}{dx} = 0$$

$$\Rightarrow \quad x+a+(y-a)\frac{dy}{dx} = 0$$

$$\Rightarrow \quad a = -\left(\frac{x+y\frac{dy}{dx}}{1-\frac{dy}{dx}}\right)$$

$$\Rightarrow \quad a = \frac{x+Py}{P-1}, \text{ where } P = \frac{dy}{dx}$$

Substituting the value of a in (i), we get

$$\left(x+\frac{x+Py}{P-1}\right)^2+\left(y-\frac{x+Py}{P-1}\right)^2 = \left(\frac{x+Py}{P-1}\right)^2$$

$$\Rightarrow (xP-x+x+yP)^2+(yP-y-x-yP)^2 = (x+yP)^2$$

$$\Rightarrow \quad (x+y)^2P^2+(x+y)^2 = (x+yP)^2$$

$$\Rightarrow \quad (x+y)^2(P^2+1) = (x+yP)^2$$

$$\Rightarrow \quad (x+y)^2\left[\left(\frac{dy}{dx}\right)^2+1\right] = \left(x+y\frac{dy}{dx}\right)^2$$

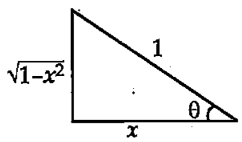
This is the required differential equation representing the given family of circles. **Ans.**

18. Solve the equation for x : $\sin^{-1}x+\sin^{-1}(1-x) = \cos^{-1}x$. **[4]**

Solution : We have, $\sin^{-1}x+\sin^{-1}(1-x) = \cos^{-1}x$

$$\Rightarrow \sin^{-1}(x\sqrt{1-(1-x)^2}+(1-x)\sqrt{1-x^2}) = \cos^{-1}x$$

$$\Rightarrow \sin^{-1}(x\sqrt{1-(1+x^2-2x)}+(1-x)\sqrt{1-x^2}) = \cos^{-1}x$$



Let $\cos^{-1}x = \theta$

$\Rightarrow \quad x = \cos\theta$

and $\sin\theta = \sqrt{1-x^2}$

$\Rightarrow \quad \theta = \sin^{-1}(\sqrt{1-x^2}) = \cos^{-1}x$

$$\Rightarrow \sin^{-1}(x\sqrt{2x-x^2}+(1-x)\sqrt{1-x^2}) = \sin^{-1}(\sqrt{1-x^2})$$

$$\Rightarrow \quad x\sqrt{2x-x^2}+(1-x)\sqrt{1-x^2} = \sqrt{1-x^2}$$

$$\Rightarrow \quad x\sqrt{2x-x^2} = \sqrt{1-x^2}.(1-1+x)$$

$$\Rightarrow \quad x\left[\sqrt{2x-x^2}-\sqrt{1-x^2}\right] = 0$$

$$\Rightarrow \quad x = 0 \text{ or } \sqrt{2x-x^2} = \sqrt{1-x^2}$$

$$\Rightarrow \quad x = 0 \text{ or } 2x-x^2 = 1-x^2$$

$$\Rightarrow \quad x = 0 \text{ or } x = \frac{1}{2}$$ **Ans.**

OR

If $\cos^{-1}\frac{x}{a}+\cos^{-1}\frac{y}{b} = \alpha$, prove that :

$$\frac{x^2}{a^2}-2\frac{xy}{ab}\cos\alpha+\frac{y^2}{b^2} = \sin^2\alpha$$

Solution : We have, $\cos^{-1}\frac{x}{a}+\cos^{-1}\frac{y}{b} = \alpha$

$$\Rightarrow \quad \cos^{-1}\left(\frac{x}{a}.\frac{y}{b}+\sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}}\right) = \alpha$$

$$\Rightarrow \quad \frac{xy}{ab}+\sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}} = \cos\alpha$$

$$\Rightarrow \quad \sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}} = \cos\alpha-\frac{xy}{ab}$$

On squaring both sides, we get

$$\left(1-\frac{x^2}{a^2}\right)\left(1-\frac{y^2}{b^2}\right) = \left(\cos\alpha-\frac{xy}{ab}\right)^2$$

$$\Rightarrow 1-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{x^2y^2}{a^2b^2} = \cos^2\alpha+\frac{x^2y^2}{a^2b^2}-\frac{2xy}{ab}\cos\alpha$$

$$\Rightarrow \quad 1-\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1-\sin^2\alpha-\frac{2xy}{ab}\cos\alpha$$

$$\Rightarrow \quad \frac{x^2}{a^2}-\frac{2xy}{ab}\cos\alpha+\frac{y^2}{b^2} = \sin^2\alpha$$ **Hence Proved.**

19. A trust invested some money in two types of bond. The first bond pays 10% interest and second bond pays 12% interest. The trust received ₹ 2,800 as interest. However, if trust had interchanged money in bonds, they would have got ₹ 100 less as interest. Using matrix method, find the amount invested by the trust. Interest received on this amount will be given to Helpage India as donation. Which value is reflected in this question ? **[4]**

Solution : Let the amount invested by the trust in first and second bond be x and y respectively.

$$\text{Interest from first bond} = \frac{10\times x\times 1}{100} = \frac{10x}{100}$$

Interest from second bond = $\frac{12 \times y \times 1}{100} = \frac{12y}{100}$

Interest received by trust = ₹ 2,800

According to the question,

$$\frac{10x}{100} + \frac{12y}{100} = 2{,}800$$

$$\Rightarrow \quad 10x + 12y = 2{,}80{,}000 \quad \text{...(i)}$$

and

$$\frac{12x}{100} + \frac{10y}{100} = 2{,}700$$

$$\Rightarrow \quad 12x + 10y = 2{,}70{,}000 \quad \text{...(ii)}$$

This system of equations can be written in matrix form as follows :

$$\begin{bmatrix} 10 & 12 \\ 12 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2{,}80{,}000 \\ 2{,}70{,}000 \end{bmatrix}$$

or $\quad AX = B,$

where $A = \begin{bmatrix} 10 & 12 \\ 12 & 10 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 2{,}80{,}000 \\ 2{,}70{,}000 \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 10 & 12 \\ 12 & 10 \end{vmatrix} = 100 - 144 = -44 \neq 0$

So, A^{-1} exists and the solution of the given system of equations is given by

$$X = A^{-1}B$$

Let c_{ij} be the cofactor of a_{ij} in $A = [a_{ij}]$. Then,

$c_{11} = 10, c_{12} = -12, c_{21} = -12, c_{22} = 10$

$$\therefore \text{adj } A = \begin{bmatrix} 10 & -12 \\ -12 & 10 \end{bmatrix}^T = \begin{bmatrix} 10 & -12 \\ -12 & 10 \end{bmatrix}$$

$$\text{So, } A^{-1} = \frac{1}{|A|}(\text{adj } A) = -\frac{1}{44}\begin{bmatrix} 10 & -12 \\ -12 & 10 \end{bmatrix}$$

Hence, the solution is given by

$$X = A^{-1}B = -\frac{1}{44}\begin{bmatrix} 10 & -12 \\ -12 & 10 \end{bmatrix}\begin{bmatrix} 2{,}80{,}000 \\ 2{,}70{,}000 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{44}\begin{bmatrix} 28{,}00{,}000 - 32{,}40{,}000 \\ -33{,}60{,}000 + 27{,}00{,}000 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{44}\begin{bmatrix} -4{,}40{,}000 \\ -6{,}60{,}000 \end{bmatrix} = \begin{bmatrix} 10{,}000 \\ 15{,}000 \end{bmatrix}$$

$\Rightarrow x = 10{,}000$ and $y = 15{,}000$

$\Rightarrow A = x + y = 10{,}000 + 15{,}000 =$ ₹ 25,000

Hence, the amount invested by the trust is ₹ 25,000.

Value : Giving help to those in need is a humanitarian act. **Ans.**

SECTION — C

20. There are two types of fertilisers 'A' and 'B'. 'A' consists of 12% nitrogen and 5% phosphoric acid whereas 'B' consists of 4% nitrogen and 5% phosphoric acid. After testing the soil conditions, farmer finds that he needs at least 12 kg of nitrogen and 12 kg of phosphoric acid for his crops. If 'A' costs ₹ 10 per kg and 'B' cost ₹ 8 per kg, then graphically determine how much of each type of fertiliser should be used so that nutrient requirements are met at a minimum cost. [6]

Solution : Let the quantity of fertiliser A and B be x and y respectively.

To minimize : Z = ₹ $(10x + 8y)$

Subject to the constraints :

$$\frac{12}{100}x + \frac{4}{100}y \geq 12$$

or $\quad 12x + 4y \geq 1200$

and

$$\frac{5x}{100} + \frac{5y}{100} \geq 12$$

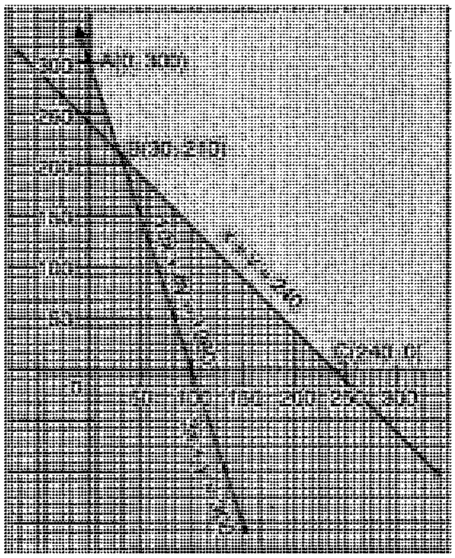
or $\quad 5x + 5y \geq 1200$

and $\quad x \geq 0, y \geq 0$

i.e., $\quad 3x + y \geq 300,$

$x + y \geq 240,$

$x \geq 0, y \geq 0$



Corner Points	$Z = 10x + 8y$
A (0,300)	$Z = 10 \times 0 + 8 \times 300 =$ ₹ 2400
B (30,210)	$Z = 10 \times 30 + 8 \times 210 =$ ₹ 1980
C (240,0)	$Z = 10 \times 240 + 8 \times 0 =$ ₹ 2400

The region of $10x + 8y < 1980$ has no point in common to the feasible region.

So, Z is minimum for $x = 30$ and $y = 210$ and the minimum value of Z is ₹ 1980.

Hence, the quantity of fertilizer A is 30 kg and of fertilizer B is 210 kg. **Ans.**

21. Five bad oranges are accidently mixed with 20 good ones. If four oranges are drawn one by one successively with replacement, then find the probability distribution of number of bad oranges drawn. Hence find the mean and variance of the distribution. [6]

Solution : Let X denotes the number of bad oranges in a draw of 4 oranges from a group of 20 good oranges and 5 bad oranges. Since there are 5 bad oranges in the group, therefore X can take values, 0, 1, 2, 3, 4.

Now, P(X = 0) = Probability of getting no bad orange.

P(X = 0) = Probability of getting 4 good oranges

$$= \left(\frac{20}{25}\right)^4 . {}^4C_0$$

P(X = 1) = Probability of getting one bad orange

$$= \frac{5}{25} \times \left(\frac{20}{25}\right)^3 . {}^4C_1$$

P(X = 2) = Probability of getting two bad oranges

$$= \left(\frac{5}{25}\right)^2 \times \left(\frac{20}{25}\right)^2 . {}^4C_2$$

P(X = 3) = Probability of getting three bad oranges

$$= \left(\frac{5}{25}\right)^3 \times \frac{20}{25} . {}^4C_3$$

P(X = 4) = Probability of getting four bad oranges

$$= \left(\frac{5}{25}\right)^4 . {}^4C_4$$

Computation of Mean and Variance

x_i	$p_i = p(X = x_i)$	$p_i x_i$	$p_i x_i^2$
0	$\frac{256}{625}$	0	0
1	$\frac{256}{625}$	$\frac{256}{625}$	$\frac{256}{625}$
2	$\frac{96}{625}$	$\frac{192}{625}$	$\frac{384}{625}$
3	$\frac{16}{625}$	$\frac{48}{625}$	$\frac{144}{625}$
4	$\frac{1}{625}$	$\frac{4}{625}$	$\frac{16}{625}$
		$\Sigma p_i x_i = \frac{500}{625}$	$\Sigma p_i x_i^2 = \frac{800}{625}$

We have, $\Sigma p_i x_i = \frac{500}{625} = \frac{4}{5}$ and $\Sigma p_i x_i^2 = \frac{800}{625} = \frac{32}{25}$

$$\therefore \quad \bar{X} = \text{Mean} = \Sigma p_i x_i = \frac{4}{5}$$

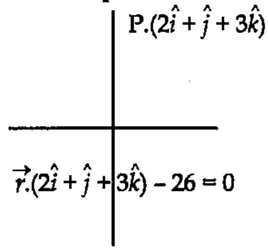
and $\quad \text{Var}(X) = \Sigma p_i x_i^2 - (\Sigma p_i x_i)^2 = \frac{32}{25} - \frac{16}{25} = \frac{16}{25}$

Hence, mean $= \frac{4}{5}$ and variance $= \frac{16}{25}$ **Ans.**

22. Find the position vector of the foot of perpendicular and the perpendicular distance from the point P with position vector $2\hat{i} + 3\hat{j} + 4\hat{k}$ to the plane $\vec{r}.(2\hat{i} + \hat{j} + 3\hat{k}) - 26 = 0$. Also find image of P in the plane. [6]

Solution : Let L be the foot of the perpendicular drawn from $P(2\hat{i} + 3\hat{j} + 4\hat{k})$ on the plane $\vec{r}.(2\hat{i} + \hat{j} + 3\hat{k}) - 26 = 0$

We know that the position vector of the foot of perpendicular from the point P with position vector $\vec{a}$ from the plane $\vec{r}.\vec{n} = d$ is given by



$$\vec{PL} = \frac{[d - (\vec{a}.\vec{n})]\vec{n}}{|\vec{n}|^2}$$

$$= \frac{[26 - (2\hat{i} + 3\hat{j} + 4\hat{k})(2\hat{i} + \hat{j} + 3\hat{k})](2\hat{i} + \hat{j} + 3\hat{k})]}{(\sqrt{4+1+9})^2}$$

$$= \frac{[26 - (4+3+12)](2\hat{i} + \hat{j} + 3\hat{k})}{14} = \frac{7}{14}(2\hat{i} + \hat{j} + 3\hat{k})$$

$$= \hat{i} + \frac{1}{2}\hat{j} + \frac{3}{2}\hat{k}$$

and the perpendicular distance is given by $\frac{|\vec{a}.\vec{n} - d|}{|\vec{n}|}$

So, required distance

$$= \frac{|(2\hat{i} + 3\hat{j} + 4\hat{k})(2\hat{i} + \hat{j} + 3\hat{k}) - 26|}{\sqrt{14}}$$

$$= \frac{7}{\sqrt{14}}$$

Let Q be the image of the point $P(2\hat{i} + 3\hat{j} + 4\hat{k})$ to the plane $\vec{r} \cdot (2\hat{i} + \hat{j} + 3\hat{k}) - 26 = 0$. Then PQ is normal to the plane.

Therefore, equation of line PQ is

$$\vec{r} = (2\hat{i} + 3\hat{j} + 4\hat{k}) + \lambda(2\hat{i} + \hat{j} + 3\hat{k})$$

Since Q lies on line PQ. So, let the position vector of Q be $(2\hat{i}+3\hat{j}+4\hat{k})+\lambda(2\hat{i}+\hat{j}+3\hat{k})$

$$=(2+2\lambda)\hat{i}+(3+\lambda)\hat{j}+(4+3\lambda)\hat{k}$$

Since, R is the mid-point of PQ. Therefore, position vector of R is

$$\frac{[(2+2\lambda)\hat{i}+(3+\lambda)\hat{j}+(4+3\lambda)\hat{k}]+(2\hat{i}+3\hat{j}+4\hat{k})}{2}$$

$$=(2+\lambda)\hat{i}+(3+\lambda/2)\hat{j}+(4+3\lambda/2)\hat{k}$$

Since R lies on the plane $\vec{r}.[2\hat{i}+\hat{j}+3\hat{k}]-26=0$

$$\therefore \left\{(2+\lambda)\hat{i}+(3+\lambda/2)\hat{j}+(4+3\lambda/2)\hat{k}\right\}.(2\hat{i}+\hat{j}+3\hat{k})-26=0$$

$$\Rightarrow \quad 4+2\lambda+3+\frac{\lambda}{2}+12+\frac{9\lambda}{2}-26=0$$

$$\Rightarrow \quad \lambda = \frac{7}{7}=1$$

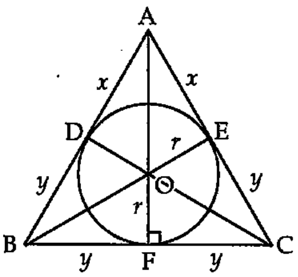
Thus, the position vector of Q is

$$4\hat{i}+4\hat{j}+7\hat{k}$$ **Ans.**

23. Show that the binary operation * on A = R – {– 1} defined as $a * b = a + b + ab$ for all $a, b \in$ A is commutative and associative on A. Also find the identity element of * in A and prove that every element of A is invertible. [6]**

24. Prove that the least perimeter of an isosceles triangle in which a circle of radius r can be inscribed is $6\sqrt{3}r$. [6]

Solution : Let ABC be an isosceles triangle with AB = AC and a circle with centre O and radius r, touching sides AB, BC, CA at D, F, E respectively.



In ΔABC

Let AD = AE = x, BD = BF = y and CF = CE = y

(∵ Tangents drawn from an external point are equal)

Now, ar (ΔABC) = ar (ΔAOB) + ar (ΔAOC) + ar (ΔBOC)

$$\Rightarrow \frac{1}{2}\times 2y\left(r+\sqrt{r^2+x^2}\right) = \frac{1}{2}\{2yr+(x+y)r+(x+y)r\}$$

$$\Rightarrow \quad y\left(r+\sqrt{r^2+x^2}\right) = \frac{1}{2}\{2yr+2(x+y)r\}$$

$$\Rightarrow \quad y\left(r+\sqrt{r^2+x^2}\right) = r(y+x+y)$$

$$\Rightarrow \quad yr+y\left(\sqrt{r^2+x^2}\right) = 2yr+rx$$

$$\Rightarrow \quad \left(\sqrt{r^2+x^2}\right)y = rx+yr$$

Squaring both sides, we get

$$y^2(r^2+x^2) = r^2x^2+y^2r^2+2r^2xy$$

$$\Rightarrow \quad y^2r^2+y^2x^2 = r^2x^2+y^2r^2+2r^2xy$$

$$\Rightarrow \quad y^2x = r^2x+2r^2y$$

$$\Rightarrow \quad x = \frac{2r^2y}{y^2-r^2}$$

Now, P (Perimeter of ΔABC) = $2x + 4y$

$$\Rightarrow \quad P = \frac{4r^2y}{y^2-r^2}+4y$$

Differentiate above equation w.r.t. y, we get

$$\frac{dP}{dy} = \frac{4r^2(y^2-r^2)-4r^2y(2y)}{(y^2-r^2)^2}+4$$

$$= \frac{4r^2\left[y^2-r^2-2y^2\right]}{(y^2-r^2)^2}+4$$

$$\Rightarrow \quad \frac{dP}{dy} = \frac{-4r^2(r^2+y^2)}{(y^2-r^2)^2}+4$$

For maxima and minima of P, put

$$\Rightarrow \quad \frac{dP}{dy} = 0$$

$$\Rightarrow \quad = \frac{-4r^2(r^2+y^2)}{(y^2-r^2)^2}+4=0$$

$$\Rightarrow \quad r^2(r^2+y^2) = (y^2-r^2)^2$$

$$\Rightarrow \quad r^4+r^2y^2 = y^4+r^4-2y^2r^2$$

$$\Rightarrow \quad 3y^2r^2 = y^4$$

$$\Rightarrow \quad y^2 = 3r^2$$

$$\Rightarrow \quad y = \sqrt{3}r$$

Now, again differentiate w.r. to y, we get $\frac{d^2P}{dy^2}$

$$= \frac{-4r^2(2y)(y^2-r^2)^2+4r^2(r^2+y^2)2(y^2-r^2)\times 2y}{(y^2-r^2)^4}$$

**Answer is not given due to the change in present syllabus

$$= \frac{4r^2(y^2-r^2)+[-2y(y^2-r^2)+4y(r^2+y^2)]}{(y^2-r^2)^4}$$

$$= \frac{4r^2y(y^2-r^2)+y[-2y^2+2r^2+4r^2+4y^2]}{(y^2-r^2)^4}$$

$$\Rightarrow \frac{d^2P}{dy^2} = \frac{4r^2y[2y^2+6r^2]}{(y^2-r^2)^3}$$

$$\left.\frac{d^2P}{dy^2}\right|_{y=\sqrt{3}r} = \frac{6\sqrt{3}}{r} => 0$$

Hence, perimeter P of ΔABC is least for $y = \sqrt{3}\, r$

and least perimeter is $P = 4y + \dfrac{4r^2 y}{y^2-r^2}$

$$= 4\sqrt{3} + \frac{4r^2\sqrt{3}r}{2r^2}$$

$$= 6\sqrt{3}r \text{ Hence Proved.}$$

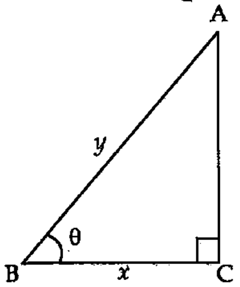
OR

If the sum of lengths of hypotenuse and a side of a right angled triangle is given, show that area of triangle is maximum, when the angle between them is $\frac{\pi}{3}$.

Solution: Let ABC be a right angled triangle with base $BC = x$, $AB = y$ such that $x + y = k$ (constant).

Let θ be the angle between base and hypotenuse.

Let A be the area of the tringle. Then,



$$A = \frac{1}{2} \times BC \times AC$$

$$= \frac{1}{2}x\sqrt{y^2-x^2}$$

$$\Rightarrow \quad A^2 = \frac{x^2}{4}(y^2-x^2) \quad (\because y = k-x)$$

$$\Rightarrow \quad A^2 = \frac{x^2}{4}[(k-x)^2-x^2]$$

$$\Rightarrow \quad A^2 = \frac{k^2x^2-2kx^3}{4} \quad \text{...(i)}$$

Differentiating w.r.t. x, we get

$$2A\frac{dA}{dx} = \frac{2k^2x-6kx^2}{4}$$

$$\Rightarrow \quad \frac{dA}{dx} = \frac{k^2x-3kx^2}{4A}$$

For maximum or minimum, we have

$$\frac{dA}{dx} = 0 \Rightarrow \frac{k^2x-3kx^2}{4A} = 0 \Rightarrow x = \frac{k}{3}$$

Again differentiating (ii) w.r.t. x, we get

$$2\left(\frac{dA}{dx}\right)^2 + 2A\frac{d^2A}{dx^2} = \frac{2k^2-12kx}{4} \quad \text{....(iii)}$$

Putting $\frac{dA}{dx} = 0$ and $x = \frac{k}{3}$ in (iii), we get

$$\frac{d^2A}{dx^2} = \frac{-k^2}{4A} < 0$$

Thus, A is maximum when $= \frac{k}{3}$

Now, and $x = \frac{k}{3}$ and $y = k - x = k - \frac{k}{3} = \frac{2k}{3}$

$$\therefore \quad \cos\theta = \frac{x}{y} = \frac{k/3}{2k/3} = \frac{1}{2}$$

$$\theta = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3} \text{ Hence Proved.}$$

25. Prove that the curves $y^2 = 4x$ and $x^2 = 4y$ divide the area of square bounded by $x = 0$, $x = 4$, $y = 4$ and $y = 0$ into three equal parts. [6]

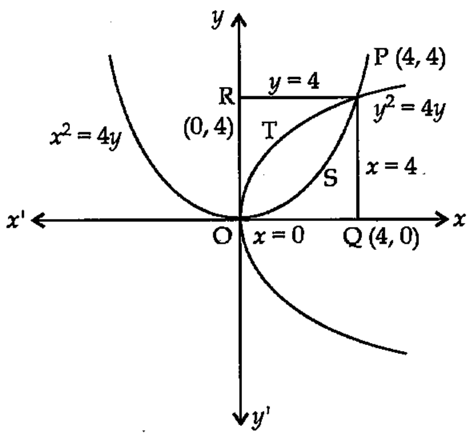
Solution : Let A_1, A_2 and A_3 denote areas OSPQO, OSPTO and OTPRO respectively.

To prove : $A_1 = A_2 = A_3$,

$$\text{Now,} \quad A_1 = \int_0^4 \frac{x^2}{4}dx$$

$$= \frac{1}{4}\int_0^4 x^2 dx$$

$$= \frac{1}{4}\left[\frac{x^3}{3}\right]_0^4 = \frac{1}{4} \times \frac{64}{3} = \frac{16}{3} \text{ sq. units}$$



A_2 = (Area bounded by $y^2 = 4x$) – (Area bounded by $x^2 = 4y$)

$$A_2 = \int_0^4 \left(\sqrt{4x} - \frac{x^2}{4}\right)dx$$

$$= \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right)dx$$

$$= \left[\frac{4}{3}x^{3/2} - \frac{x^3}{12}\right]_0^4$$

$$= \left(\frac{4}{3}\times 8 - \frac{64}{12}\right) = \frac{16}{3} \text{ sq. units}$$

and $$A_3 = \int_0^4 \frac{y^2}{4}dy$$

$$= \frac{1}{4}\left[\frac{y^3}{3}\right]_0^4 = \frac{1}{4}\times\frac{64}{3} = \frac{16}{3} \text{ sq. units.}$$

Hence, $A_1 = A_2 = A_3$. **Hence Proved.**

26. Using properties of determinants, show that ΔABC is isosceles if : [6]

$$\begin{vmatrix} 1 & 1 & 1 \\ 1+\cos A & 1+\cos B & 1+\cos C \\ \cos^2 A+\cos A & \cos^2 B+\cos B & \cos^2 C+\cos C \end{vmatrix} = 0$$

Solution We have,

$$\begin{vmatrix} 1 & 1 & 1 \\ 1+\cos A & 1+\cos B & 1+\cos C \\ \cos^2 A+\cos A & \cos^2 B+\cos B & \cos^2 C+\cos C \end{vmatrix} = 0$$

Applying $C_2 \to C_2 - C_1$ and $C_3 - C_3 \to C_1$

$$\begin{vmatrix} 1 & 0 & 0 \\ 1+\cos A & \cos B - \cos A & \cos C - \cos A \\ \cos^2 A+\cos A & (\cos B-\cos A)(1+\cos A+\cos B) & (\cos C-\cos A)(1+\cos A+\cos C) \end{vmatrix} = 0$$

Taking (cos B – cos A) and (cos C – cos A) as common from C_2 and C_3 respectively.

⇒ (cos B – cos A) (cos C – cos A)

$$\begin{vmatrix} 1 & 0 & 0 \\ 1+\cos A & 1 & 1 \\ \cos^2 A+\cos A & 1+\cos A+\cos B & 1+\cos A+\cos C \end{vmatrix} = 0$$

⇒ Expanding along R_1

(cos B – cos A) (cos C – cos A)

[1 + cos A + cos C – 1 – cos A – cos B] = 0

⇒(cos B – cos A) (cos C – cos A) (cos C – cos B) = 0

Either cos B = cos A or cos C = cos A or cos C = cos B

i.e., either BC = AC or BC = AB or AC = AB

Hence, Δ ABC is isosceles. **Hence Proved.**

OR

A shopkeeper has 3 varieties of pens 'A', 'B' and 'C'. Meenu purchased 1 pen of each variety for a total of ₹ 21. Jeevan purchased 4 pens of 'A' variety, 3 pens of 'B' variety and 2 pens of 'C' variety for ₹ 60. While Shikha purchased 6 pens of 'A' variety, 2 pens of 'B' variety and 3 pens of 'C' variety for ₹ 70. Using matrix method, find cost of each variety of pen.

Solution : Let the cost of each variety of pen be ₹ x, ₹ y and ₹ z respectively. Then,

$$x + y + z = 21$$
$$4x + 3y + 2z = 60$$
and $$6x + 2y + 3z = 70$$

This system of equations can be written in matrix form as follows :

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & 2 \\ 6 & 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 21 \\ 60 \\ 70 \end{bmatrix}$$

or $$AX = B$$

where $A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & 2 \\ 6 & 2 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 21 \\ 60 \\ 70 \end{bmatrix}$

Now, $$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 3 & 2 \\ 6 & 2 & 3 \end{vmatrix}$$

$$= 1(9-4) - 1(12-12) + 1(8-18)$$
$$= -5 \neq 0$$

So, A^{-1} exists and the solution of the given system of equation is given by

$$X = A^{-1}B$$

Let c_{ij} be the cofactor of a_{ij} in $A = [a_{ij}]$. Then,

$c_{11} = 5, c_{12} = 0, c_{13} = -10, c_{21} = -1, c_{22} = -3, c_{23} = 4, c_{31} = -1, c_{32} = 2, c_{33} = -1$

$$\therefore \text{adj } A = \begin{bmatrix} 5 & 0 & -10 \\ -1 & -3 & 4 \\ -1 & 2 & -1 \end{bmatrix}^T = \begin{bmatrix} 5 & -1 & -1 \\ 0 & -3 & 2 \\ -10 & 4 & -1 \end{bmatrix}$$

So, $A^{-1} = \frac{1}{|A|}(\text{adj } A) = -\frac{1}{5}\begin{bmatrix} 5 & -1 & -1 \\ 0 & -3 & 2 \\ -10 & 4 & -1 \end{bmatrix}$

Hence, the solution is given by

$$X = A^{-1}B = -\frac{1}{5}\begin{bmatrix} 5 & -1 & -1 \\ 0 & -3 & 2 \\ -10 & 4 & -1 \end{bmatrix}\begin{bmatrix} 21 \\ 60 \\ 70 \end{bmatrix}$$

$$= -\frac{1}{5}\begin{bmatrix} 105-60-70 \\ 0-180+140 \\ -210+240-70 \end{bmatrix}$$

$$= -\frac{1}{5}\begin{bmatrix} -25 \\ -40 \\ -40 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 8 \end{bmatrix}$$

Hence, the cost of each variety of pen are ₹ 5, ₹ 8 and ₹ 8 respectively. **Ans.**

●●

All questions are same in Outside Delhi Set II and Set III

Mathematics 2016 (Delhi) SET I

Time allowed : 3 hours **Maximum marks : 100**

SECTION — A

1. Find the maximum value of $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+\sin\theta & 1 \\ 1 & 1 & 1+\cos\theta \end{vmatrix}$. **[1]**

Soution : Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+\sin\theta & 1 \\ 1 & 1 & 1+\cos\theta \end{vmatrix}$

On applying $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$, we get

$$\Delta = \begin{vmatrix} 0 & -\sin\theta & 0 \\ 0 & \sin\theta & -\cos\theta \\ 1 & 1 & 1+\cos\theta \end{vmatrix}$$

On expanding along R_1, we get

$$\Delta = 0 + \sin\theta\,(0 + \cos\theta) + 0$$
$$= \sin\cos\theta$$
$$= \frac{1}{2}\sin 2\theta$$

Now as $-1 \le \sin 2\theta \le 1$ for all $\theta \in R$

$\Rightarrow -\frac{1}{2} \le \frac{1}{2}\sin 2\theta \le \frac{1}{2}$ for all $\theta \in R$

Clearly, maximum value of Δ is $\frac{1}{2}$. **Ans.**

2. If A is a square matrix such that $A^2 = I$, then find the simplified value of $(A - I)^3 + (A + I)^3 - 7A$. **[1]**

Solution : Given,

$$(A-I)^3 + (A+I)^3 - 7A$$
$$= A^3 - I^3 - 3A^2I + 3AI^2 + A^3 + I^3 + 3A^2I + 3AI^2 - 7A$$
$$= 2A^3 + 6AI^2 - 7A$$
$$= 2A.A^2 + 6AI^2 - 7A$$
$$= 2AI + 6AI - 7A$$
$$= 8A - 7A$$
$$= A$$ **Ans.**

3. Matrix $A = \begin{bmatrix} 0 & 2b & -2 \\ 3 & 1 & 3 \\ 3a & 3 & -1 \end{bmatrix}$ **is given to be symmetric, find values of a and b.** **[1]**

Solution : We have, $A = \begin{bmatrix} 0 & 2b & -2 \\ 3 & 1 & 3 \\ 3a & 3 & -1 \end{bmatrix}$

It is given that the matrix is symmetric.

$\therefore \quad A = A'$

$$\Rightarrow \begin{bmatrix} 0 & 2b & -2 \\ 3 & 1 & 3 \\ 3a & 3 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 3a \\ 2b & 1 & 3 \\ -2 & 3 & -1 \end{bmatrix}$$

Now, by equality of matrices, we get

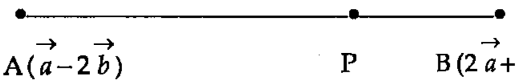
$$2b = 3$$
$$\Rightarrow \quad b = \frac{3}{2}$$
and
$$3a = -2$$
$$\Rightarrow \quad a = \frac{-2}{3}$$

Therefore, $a = \frac{-2}{3}$ and $b = \frac{3}{2}$ **Ans.**

4. Find the position vector of a point which divides the join of points with position vectors $\vec{a} - 2\vec{b}$ and $2\vec{a} + \vec{b}$ externally in the ratio 2 : 1. [1]

Solution : Let A and B be the given points with position vectors $\vec{a} - 2\vec{b}$ and $2\vec{a} + \vec{b}$ respectively.

Let P be the point dividing AB in the ratio 2 : 1 externally.

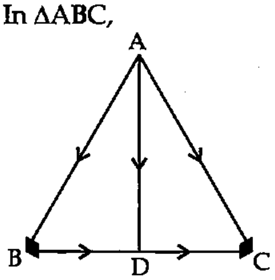


$\therefore$ Position vector of $= \frac{2\times(2\vec{a}+\vec{b})-1\times(\vec{a}-2\vec{b})}{2-1}$

$= 3\vec{a}+4\vec{b}$ **Ans.**

5. The two vectors $\hat{j}+\hat{k}$ and $3\hat{i}-\hat{j}+4\hat{k}$ represent the two sides AB and AC, respectively of a ΔABC. Find the length of the median through A. [1]

Solution : In ΔABC,



Using the triangle law of vector addition, we have

$$\vec{BC} = \vec{AC} - \vec{AB}$$

$$= (3\hat{i}-\hat{j}+4\hat{k})-(\hat{j}+\hat{k})$$

$$= 3\hat{i}-2\hat{j}+3\hat{k}$$

$$\therefore \vec{BD} = \frac{1}{2}\vec{BC} = \frac{3}{2}\hat{i}-\hat{j}+\frac{3}{2}\hat{k}$$

(since AD is the median)

In ΔABD, using the triangle law of vector addition, we have

$$\vec{AD} = \vec{AB}+\vec{BD}$$

$$= (\hat{j}+\hat{k})+\left(\frac{3}{2}\hat{i}-\hat{j}+\frac{3}{2}\hat{k}\right)$$

$$= \frac{3}{2}\hat{i}+0\hat{j}+\frac{5}{2}\hat{k}$$

$$\therefore AD = \sqrt{\left(\frac{3}{2}\right)^2+0^2+\left(\frac{5}{2}\right)^2} = \frac{1}{2}\sqrt{34}$$

Hence, the length of the median through A is $\frac{1}{2}\sqrt{34}$ units. **Ans.**

6. Find the vector equation of a plane which is at a distance of 5 units from the origin and its normal vector is $2\hat{i}-3\hat{j}+6\hat{k}$. [1]

Solution : Here, $d = 5$ units and $\vec{n} = 2\hat{i}-3\hat{j}+6\hat{k}$

$$\therefore \quad \hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{2\hat{i}-3\hat{j}+6\hat{k}}{\sqrt{4+9+36}} = \frac{2\hat{i}-3\hat{j}+6\hat{k}}{\sqrt{49}}$$

$$\Rightarrow \quad \hat{n} = \frac{2}{7}\hat{i}-\frac{3}{7}\hat{j}+\frac{6}{7}\hat{k}$$

Hence, the required equation of the plane is

$$\vec{r}\cdot\left(\frac{2}{7}\hat{i}-\frac{3}{7}\hat{j}+\frac{6}{7}\hat{k}\right) = 5 \qquad [\because \vec{r}\cdot\hat{n} = d]$$

or $\vec{r}\cdot(2\hat{i}-3\hat{j}+6\hat{k}) = 35$ **Ans.**

SECTION — B

7. Prove that :

$$\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{8} = \frac{\pi}{4} \qquad [4]$$

Solution LHS =

$$\left(\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}\right)+\left(\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{8}\right)$$

$$= \tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{7}}{1-\frac{1}{5}\times\frac{1}{7}}\right)+\tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{8}}{1-\frac{1}{3}\times\frac{1}{8}}\right)$$

$$\left[\because \tan^{-1}A+\tan^{-1}B = \tan^{-1}\left(\frac{A+B}{1-AB}\right)\right]$$

$$= \tan^{-1}\frac{6}{17}+\tan^{-1}\frac{11}{23}$$

$$= \tan^{-1}\left(\frac{\frac{6}{17}+\frac{11}{23}}{1-\frac{6}{17}\times\frac{11}{23}}\right)$$

$$= \tan^{-1}\left(\frac{325}{325}\right)$$

$$= \tan^{-1}(1)$$

$$= \frac{\pi}{4}$$

Hence Proved.

OR

Solve for x :

$$\mathbf{2\tan^{-1}(\cos x) = \tan^{-1}(2\,\text{cosec}\, x)}$$

Solution : Given, $2\tan^{-1}(\cos x) = \tan^{-1}(2\text{cosec}\, x)$

$$\Rightarrow \quad \tan^{-1}\left(\frac{2\cos x}{1-\cos^2 x}\right) = \tan^{-1}(2\,\text{cosec}\, x)$$

$$\left[\because 2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]$$

$\Rightarrow \quad \frac{2\cos x}{\sin^2 x} = 2 \text{ cosec } x = \frac{2}{\sin x}$

$\Rightarrow \quad \cos x = \sin x$

$\Rightarrow \quad \tan x = 1$

$\Rightarrow \quad x = \frac{\pi}{4}$ **Ans.**

8. The monthly incomes of Aryan and Babban are in the ratio 3 : 4 and their monthly expenditures are in the ratio 5 : 7. If each saves ₹ 15,000 per month, find their monthly incomes using matrix method. This problem reflects which value ? [4]

Solution : Let the monthly incomes of Aryan and Babban be $3x$ and $4x$ respectively.

Suppose their monthly expenditures are $5y$ and $7y$ respectively.

Since each saves ₹ 15,000 per month.

Monthly saving of Aryan : $3x - 5y = 15{,}000$

Monthly saving of Babban : $4x - 7y = 15{,}000$

The above system of equations can be written in the matrix form as follows :

$$\begin{bmatrix} 3 & -5 \\ 4 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 15000 \\ 15000 \end{bmatrix}$$

AX = B, where $A = \begin{bmatrix} 3 & -5 \\ 4 & -7 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 15000 \\ 15000 \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 3 & -5 \\ 4 & -7 \end{vmatrix} = -21-(-20) = -1 \neq 0$

$$\text{adj } A = \begin{bmatrix} -7 & -4 \\ 5 & 3 \end{bmatrix}^T = \begin{bmatrix} -7 & 5 \\ -4 & 3 \end{bmatrix}$$

So, $\quad A^{-1} = \frac{1}{|A|}\text{adj } A = -1\begin{bmatrix} -7 & 5 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 7 & -5 \\ 4 & -3 \end{bmatrix}$

$\therefore \quad x = A^{-1}B$

$\Rightarrow \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 & -5 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} 15000 \\ 15000 \end{bmatrix}$

$\Rightarrow \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 105000-75000 \\ 60000-45000 \end{bmatrix}$

$\Rightarrow \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 30000 \\ 15000 \end{bmatrix}$

$\Rightarrow x = 30{,}000$ and $y = 15{,}000$

Therefore,

Monthly income of Aryan = 3 × 30,000 = ₹ 90,000

Monthly income of Babban = 4 × 30,000 = ₹ 1,20,000

Value : Saving in good time helps us to survive in bad times. **Ans.**

9. If $x = a\sin 2t(1+\cos 2t)$ and $y = b\cos 2t(1-\cos 2t)$, find the values of $\frac{dy}{dx}$ at $t = \frac{\pi}{4}$ and $t = \frac{\pi}{3}$. [4]

Solution : We have, $x = a \sin 2t\ (1 + \cos 2t)$

$\Rightarrow \quad \frac{dx}{dt} = 2a\cos 2t\ (1+\cos 2t) - 2a\sin^2 2t$

and, $\quad y = b\cos 2t\ (1-\cos 2t)$

$\Rightarrow \quad \frac{dy}{dt} = 2b\cos 2t\sin 2t - 2b\sin 2t\ (1-\cot 2t)$

$\therefore \quad \frac{dy}{dx} = \frac{dy}{dt}\times\frac{dt}{dx}$

$$= \frac{2b\cos 2t\sin 2t - 2b\sin 2t(1-\cos 2t)}{2a\cos 2t(1+\cos 2t) - 2a\sin^2 2t}$$

$$= \frac{b[\cos 2t\sin 2t - \sin 2t(1-\cos 2t)]}{a[\cos 2t(1+\cos 2t) - \sin^2 2t]}$$

Since we know that at $t = \frac{\pi}{4}$, $\sin 2t = 1$ and $\cos 2t = 0$,

So, $\left(\frac{dy}{dx}\right)_{t=\frac{\pi}{4}} = \frac{b[0.1-1(1-0)]}{a[0(1+0)-1^2]} = \frac{b}{a}$

Also, we know that at $t = \frac{\pi}{3}$, $\sin 2t = \frac{\sqrt{3}}{2}$ and $\cos 2t = -\frac{1}{2}$

So $\left(\frac{dy}{dx}\right)_{t=\frac{\pi}{3}} = \frac{b\left[\left(-\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)-\left(\frac{\sqrt{3}}{2}\right)\left(1+\frac{1}{2}\right)\right]}{a\left[\left(-\frac{1}{2}\right)\left(1-\frac{1}{2}\right)-\left(\frac{\sqrt{3}}{2}\right)^2\right]}$

$$= \frac{b\left[-\frac{\sqrt{3}}{4}-\frac{3\sqrt{3}}{4}\right]}{a\left[-\frac{1}{4}-\frac{3}{4}\right]}$$

$$= \frac{b[-\sqrt{3}]}{a[-1]} = \sqrt{3}\left(\frac{b}{a}\right)$$ **Ans.**

OR

If $y = x^x$ prove that $\frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x} = 0$.

Solution : Given, $y = x^x$

Taking log on both sides, we get

$\log y = \log (x^x)$

$\Rightarrow \quad \log y = x \log x$

On differentiating w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = \log x + x \times \frac{1}{x} = 1 + \log x$$

$$\Rightarrow \quad \frac{dy}{dx} = x^x(1+\log x)$$

$$\Rightarrow \quad \frac{dy}{dx} = y(1+\log x)$$

Again differentiating w.r.t. x, we get

$$\Rightarrow \quad \frac{d^2y}{dx^2} = (1+\log x)\frac{dy}{dx} + y\frac{d}{dx}(1+\log x)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = (1+\log x).x^x(1+\log x) + x^x \times \frac{1}{x}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = x^x(1+\log x)^2 + x^{x-1}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{1}{y}\left(\frac{dy}{dx}\right)^2 + \frac{y}{x}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x} = 0$$ **Hence Proved.**

10. Find the values of p and q for which

$$f(x) = \begin{cases} \dfrac{1-\sin^3 x}{3\cos^2 x} & , \text{ if } x < \dfrac{\pi}{2} \\ p & , \text{ if } x = \dfrac{\pi}{2} \\ \dfrac{q(1-\sin x)}{(\pi-2x)^2} & , \text{ if } x > \dfrac{\pi}{2} \end{cases}$$ **[4]**

is continuous at $x = \pi/2$.

Solution: Given, $f(x) = \begin{cases} \dfrac{1-\sin^3 x}{3\cos^2 x} & , \text{ if } x < \dfrac{\pi}{2} \\ p & , \text{ if } x = \dfrac{\pi}{2} \\ \dfrac{q(1-\sin x)}{(\pi-2x)^2} & , \text{ if } x > \dfrac{\pi}{2} \end{cases}$

$f(x)$ is continuous at $x = \frac{\pi}{2}$, then LHL = RHL = $f(a)$

i.e. $\lim_{x\to\frac{\pi}{2}^-} f(x) = \lim_{x\to\frac{\pi}{2}^+} f(x) = f\left(\frac{\pi}{2}\right)$

$$\lim_{x\to\frac{\pi}{2}^-} f(x) = \lim_{x\to\frac{\pi}{2}} \left(\frac{1-\sin^3 x}{3\cos^2 x}\right)$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{(1-\sin x)(1+\sin^2 x+\sin x)}{3[1-\sin^2 x]}$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{(1-\sin x)(1+\sin^2 x+\sin x)}{3(1+\sin x)(1-\sin x)}$$

$$\lim_{x\to\frac{\pi}{2}^-} f(x) = \lim_{x\to\frac{\pi}{2}} \frac{1+\sin^2 x+\sin x}{3(1+\sin x)} = \frac{1+1+1}{3(2)} = \frac{1}{2}$$

Let $x = \frac{\pi}{2} + \theta$ as $x \to \frac{\pi}{2}$, $\theta \to 0$

$$\lim_{x\to\frac{\pi}{2}^+} f(x) = \lim_{\theta\to 0} q\frac{\left[1-\sin\left(\frac{\pi}{2}-\theta\right)\right]}{(2\theta)^2} = \frac{q}{4}\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta^2}$$

$$= \frac{q}{4}\lim_{\theta\to 0}\frac{2\sin^2\frac{\theta}{2}}{\theta^2} = \frac{q}{2}\lim_{\theta\to 0}\frac{\sin^2\frac{\theta}{2}}{4\times\left(\frac{\theta}{2}\right)^2} = \frac{q}{8}$$

Now, $\lim_{x\to\frac{\pi}{2}^-} f(x) = \lim_{x\to\frac{\pi}{2}^+} f(x) = f\left(\frac{\pi}{2}\right)$

$$\Rightarrow \quad \frac{1}{2} = p = \frac{q}{8}$$

$$\Rightarrow \quad p = \frac{1}{2} \text{ and } q = 4$$ **Ans.**

11. Show that the equation of normal at any point on the curve $x = 3\cos t - \cos^3 t$ and $y = 3\sin t - \sin^3 t$ is $4(y\cos^3 t - x\sin^3 t) = 3\sin 4t$. – $\sin^3 t$ is **[4]**

Solution : Given,

$$x = 3\cos t - \cos^3 t$$

$$\Rightarrow \quad \frac{dy}{dt} = -3\sin t + 3\cos^2 t\sin t$$

and $\quad y = 3\sin t - \sin^3 t$

$$\Rightarrow \quad \frac{dy}{dt} = 3\cos t - 3\sin^2 t\cos t$$

Slope of the tangent,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3\cos t - 3\sin^2 t\cos t}{-3\sin t + 3\cos^2 t\sin t}$$

$$= \frac{3\cos t[\cos^2 t]}{-3\sin t[\sin^2 t]}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-\cos^3 t}{\sin^3 t}$$

$\therefore$ Slope of the normal $= \frac{-dy}{dx} = \frac{\sin^3 t}{\cos^3 t}$

The equation of the normal is given by

$$\frac{y-(3\sin t - \sin^3 t)}{x-(3\cos t - \cos^3 t)} = \frac{\sin^3 t}{\cos^3 t}$$

$\Rightarrow \quad y\cos^3 t - 3\sin t\cos^3 t + \sin^3 t\cos^3 t$

$= x\sin^3 t - 3\cos t\sin^3 t + \sin^3 t\cos^3 t$

$\Rightarrow \quad y\cos^3 t - x\sin^3 t = 3(\sin t\cos^3 t - \cos t\sin^3 t)$

$\Rightarrow \quad y\cos^3 t - x\sin^3 t = 3\sin t\cos t(\cos^2 t - \sin^2 t)$

$\Rightarrow \quad y\cos^3 t - x\sin^3 t = \frac{3\times 2}{2}\ \sin t\cos t\cos 2t$

$\Rightarrow \quad y\cos^3 t - x\sin^3 t = \frac{3}{2}\sin 2t\cos 2t$

$\Rightarrow \quad y\cos^3 t - x\sin^3 t = \frac{3\times 2}{2\times 2}\sin 2t\cos 2t$

$\Rightarrow \quad y\cos^3 t - x\sin^3 t = \frac{3}{4}\sin 4t$

$\Rightarrow 4(y\cos^3 t - x\sin^3 t) = 3\sin 4t$ **Hence Proved.**

12. Find $\int \frac{(3\sin\theta - 2)\cos\theta}{5-\cos^2\theta - 4\sin\theta}d\theta$ **[4]**

Solution : Let $I = \int \frac{(3\sin\theta-2)\cos\theta}{5-\cos^2\theta-4\sin\theta}.d\theta$

$\Rightarrow \quad I = \int \frac{(3\sin\theta-2)\cos\theta}{4+1-\cos^2\theta-4\sin\theta}d\theta$

$\Rightarrow \quad I = \int \frac{(3\sin\theta-2)\cos\theta.d\theta}{4+\sin^2\theta-4\sin\theta}$

$[\because 1-\cos^2\theta = \sin^2\theta]$

$\Rightarrow \quad I = \int \frac{(3\sin\theta-2)\cos\theta.d\theta}{(\sin\theta-2)^2}$

Put $\sin\theta = t$

$\Rightarrow \quad \cos\theta.\ d\theta = dt$

$\therefore \quad I = \int \frac{(3t-2)}{(t-2)^2}dt$

Consider,

$$\frac{3t-2}{(t-2)^2} = \frac{A}{(t-2)} + \frac{B}{(t-2)^2}$$

$$3t-2 = A(t-2) + B$$

On comparing, we get

$$A = 3 \text{ and } B = 4$$

$$I = \int\left(\frac{3}{t-2} + \frac{4}{(t-2)^2}\right)dt$$

$\Rightarrow \quad I = 3\log|t-2| - \frac{4}{(t-2)} + c$

$\Rightarrow \quad I = 3\log|\sin\theta - 2| - \frac{4}{(\sin\theta-2)} + c$

Ans.

OR

Evaluate $\int_0^\pi e^{2x}.\sin\left(\frac{\pi}{4}+x\right)dx$

Solution : Let $I = \int_0^\pi e^{2x}.\sin\left(\frac{\pi}{4}+x\right)dx$

$$I = \left[\sin\left(\frac{\pi}{4}+x\right).\frac{e^{2x}}{2}\right]_0^\pi - \int_0^\pi \cos\left(\frac{\pi}{4}+x\right).\frac{e^{2x}}{2}dx$$

$\Rightarrow I = \sin\frac{5\pi}{4}.\frac{e^{2\pi}}{2} - \sin\frac{\pi}{4}\times\frac{1}{2} - \frac{1}{2}$

$$\left[\cos\left(\frac{\pi}{4}+x\right).\frac{e^{2x}}{2}\right]_0^\pi + \frac{1}{2}\int_0^\pi -\sin\left(\frac{\pi}{4}+x\right).\frac{e^{2x}}{2}dx$$

$\Rightarrow I = \frac{1}{2}\sin\frac{5\pi}{4}.e^{2\pi} - \frac{1}{2}\sin\frac{\pi}{4} - \frac{1}{4}$

$$\left[\cos\frac{5\pi}{4}.e^{2\pi} - \cos\frac{\pi}{4}\right] - \frac{1}{2}\times\frac{1}{2}\int_0^\pi \sin\left(\frac{\pi}{4}+x\right).e^{2x}.dx$$

$\Rightarrow \quad I = \frac{1}{2}\left(-\frac{1}{\sqrt{2}}\right)e^{2\pi} - \frac{1}{2}\times\frac{1}{\sqrt{2}} - \frac{1}{4}\left(\frac{-1}{\sqrt{2}}e^{2\pi} - \frac{1}{\sqrt{2}}\right) - \frac{1}{4}I$

$\Rightarrow I + \frac{I}{4} = -\frac{1}{2\sqrt{2}}e^{2\pi} - \frac{1}{2\sqrt{2}} + \frac{1}{4\sqrt{2}}e^{2\pi} + \frac{1}{4\sqrt{2}}$

$\Rightarrow \frac{5I}{4} = \frac{-2e^{2\pi} - 2 + e^{2\pi} + 1}{4\sqrt{2}} = \frac{-e^{2\pi}-1}{4\sqrt{2}}$

$\Rightarrow 5I = -\left(\frac{e^{2\pi}+1}{\sqrt{2}}\right)$

$\Rightarrow I = -\frac{1}{5}\left(\frac{e^{2\pi}+1}{\sqrt{2}}\right)$ **Ans.**

13. Find $\int \frac{\sqrt{x}}{\sqrt{a^3-x^3}}dx.$ **[4]**

Solution : Let $I = \int \frac{\sqrt{x}}{\sqrt{a^3-x^3}}\cdot dx$

Put $x^{3/2} = t$

$\Rightarrow \quad \frac{3}{2}\sqrt{x}.dx = dt$

$\Rightarrow \quad \sqrt{x}.dx = \frac{2}{3}dt$

Putting the values in I, we get

$$I = \frac{2}{3}\int \frac{1}{\sqrt{(a^{3/2})^2 - t^2}}\cdot dt$$

$\Rightarrow \quad I = \frac{2}{3}\sin^{-1}\left(\frac{t}{a^{3/2}}\right) + c$

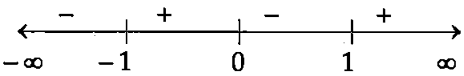
or $\quad I = \frac{2}{3}\sin^{-1}\left(\frac{x}{a}\right)^{3/2} + c$ **Ans.**

14. Evaluate : $\int_{-1}^{2}|x^3 - x|\,dx.$ **[4]**

Solution : Let $I = \int_{-1}^{2}|x^3-x|\,dx.$

$$f(x) = x^3 - x$$
$$f(x) = x^3 - x = x(x-1)(x+1)$$

The signs of $f(x)$ for the different values are shown in the figure given below :



$$f(x) > 0 \text{ for all } x \in (-1, 0) \cup (1, 2)$$
$$f(x) < 0 \text{ for all } x \in (0, 1)$$

Therefore,

$$|x^3 - x| = \begin{cases} x^3 - x, & x \in (-1,0) \cup (1,2) \\ -(x^3 - x), & x \in (0,1) \end{cases}$$

$$\therefore \text{I} = \int_{-1}^{2} |x^3 - x|\, dx$$
$$= \int_{-1}^{0} |x^3 - x|\, dx + \int_{0}^{1} |x^3 - x|\, dx + \int_{1}^{2} |x^3 - x|\, dx$$
$$= \int_{-1}^{0} (x^3 - x) dx - \int_{0}^{1} (x^3 - x) dx + \int_{1}^{2} (x^3 - x) dx$$
$$= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} - \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{0}^{1} + \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{1}^{2}$$
$$= -\left(\frac{1}{4} - \frac{1}{2}\right) - \left(\frac{1}{4} - \frac{1}{2}\right) + \left(\frac{16}{4} - \frac{4}{2}\right) - \left(\frac{1}{4} - \frac{1}{2}\right)$$
$$= \frac{3}{4} + (4 - 2)$$
$$= \frac{11}{4}$$

Ans.

15. Find the particular solution of the differential equation $(1-y^2)(1+\log x)\,dx + 2xy\,dy = 0$, given that $y = 0$ when $x = 1$. **[4]**

Solution : The given differential equation is,

$$(1 - y^2)(1 + \log x).dx + 2xy\, dy = 0$$

$$\Rightarrow \frac{(1+\log x)}{x} \cdot dx = \frac{-2y}{(1-y^2)} \cdot dy$$

On integrating both side, we have

$$\Rightarrow \int \frac{1+\log x}{x} \cdot dx = \int \frac{-2y}{(1-y^2)} \cdot dy$$

In first integral,

put $1 + \log x = t$

$$\Rightarrow \frac{1}{x} \cdot dx = dt$$

Also, in second integral,

put $1 - y^2 = u$

$$\Rightarrow -2y.dy = du$$

$$\therefore \int t.dt = \int \frac{1}{u}.du$$

$$\Rightarrow \frac{t^2}{2} - \log|u| = c$$

or $\frac{1}{2}(1 + \log x)^2 - \log|1 - y^2| = c$

It is given that $y = 0$ when $x = 1$

So, $\frac{1}{2}(1 + \log 1)^2 - \log|1 - 0^2| = c$

$$\Rightarrow c = \frac{1}{2}$$

$$\therefore \frac{(1+\log x)^2}{2} - \log|1 - y^2| = \frac{1}{2}$$

or $(1 + \log x)^2 - 2 \log|1 - y^2| = 1$

It is the required particular solution. **Ans.**

16. Find the general solution of the following differential equation :

$$(1+y^2) + (x - e^{\tan^{-1} y})\frac{dy}{dx} = 0 \qquad \textbf{[4]}$$

Solution : The given differential equation is,

$$(1+y^2) + (x - e^{\tan^{-1} y})\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dx}{dy} + \left(\frac{1}{1+y^2}\right)x = \frac{e^{\tan^{-1} y}}{1+y^2}$$

This is a linear differential equation with

$$\text{P} = \frac{1}{1+y^2}$$

and $$\text{Q} = \frac{e^{\tan^{-1} y}}{1+y^2}$$

$$\therefore \text{I.F.} = e^{\int \frac{1}{1+y^2} dx} = e^{\tan^{-1} y}$$

So, the required solution is :

$$x e^{\tan^{-1} y} = \int e^{\tan^{-1} y} . \frac{e^{\tan^{-1} y}}{1+y^2} \cdot dy$$

Put $\tan^{-1} y = t$

$$\Rightarrow \frac{1}{1+y^2} \cdot dy = dt$$

$$\therefore x e^{\tan^{-1} y} = \int e^{2t}.dt$$

$$\Rightarrow x e^{\tan^{-1} y} = \frac{1}{2} e^{2t} + \text{C}$$

$$\Rightarrow x e^{\tan^{-1} y} = \frac{1}{2} e^{2\tan^{-1} y} + \text{C}$$

$$\Rightarrow x = \frac{1}{2} e^{\tan^{-1} y} + \text{C} e^{-\tan^{-1} y}$$ **Ans.**

17. Show that the vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar if $\vec{a} + \vec{b}, \vec{b} + \vec{c}$ and $\vec{c} + \vec{a}$ are coplanar. **[4]**

Solution : As $\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}$ are coplanar.

So, $[\vec{a}+\vec{b} \quad \vec{b}+\vec{c} \quad \vec{c}+\vec{a}]=0$ ...(i)

Now, if $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar vectors

then, $[\vec{a} \quad \vec{b} \quad \vec{c}]=0$

Consider $[\vec{a}+\vec{b} \quad \vec{b}+\vec{c} \quad \vec{c}+\vec{a}]$

$=[(\vec{a}+\vec{b})\times(\vec{b}+\vec{c})].(\vec{c}+\vec{a})$

$=[\vec{a}\times\vec{b}+\vec{a}\times\vec{c}+\vec{b}\times\vec{b}+\vec{b}\times\vec{c}].(\vec{c}+\vec{a})$

$=(\vec{a}\times\vec{b}).\vec{c}+(\vec{a}\times\vec{b}).\vec{a}+(\vec{a}\times\vec{c}).\vec{c}+(\vec{a}\times\vec{c}).\vec{a}$

$+(\vec{b}\times\vec{c}).\vec{c}+(\vec{b}\times\vec{c}).\vec{a}$

$=(\vec{a}\times\vec{b}).\vec{c}+0+0+0+0+(\vec{b}\times\vec{c}).\vec{a}$

$=[\vec{a} \quad \vec{b} \quad \vec{c}]+[\vec{b} \quad \vec{c} \quad \vec{a}]$

$=[\vec{a} \quad \vec{b} \quad \vec{c}]+[\vec{a} \quad \vec{b} \quad \vec{c}]$

$=2[\vec{a} \quad \vec{b} \quad \vec{c}]$

From equation (i)

$=2[\vec{a} \quad \vec{b} \quad \vec{c}] = [\vec{a}+\vec{b} \quad \vec{b}+\vec{c} \quad \vec{c}+\vec{a}]=0$

$\Rightarrow \quad [\vec{a} \quad \vec{b} \quad \vec{c}] = 0$

$\therefore$ $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar if $\vec{a}+\vec{b}, \vec{b}+\vec{c}$ and $\vec{c}+\vec{a}$ are coplanar. **Hence Proved**

18. Find the vector and cartesian equations of the line through the point (1, 2, –4) and perpendicular to the two lines.

$$\vec{r}=(8\hat{i}-19\hat{j}+10\hat{k})+\lambda(3\hat{i}-16\hat{j}+7\hat{k})$$

and $\vec{r}=(15\hat{i}+29\hat{j}+5\hat{k})+\mu(3\hat{i}+8\hat{j}-5\hat{k})$

Solution : The equations of the given lines are

$\vec{r}=(8\hat{i}-19\hat{j}+10\hat{k})+\lambda(3\hat{i}-16\hat{j}+7\hat{k})$ ......(i)

$\vec{r}=(15\hat{i}+29\hat{j}+5\hat{k})+\mu(3\hat{i}+8\hat{j}-5\hat{k})$ .....(ii)

Normal parallel to (i) is

$\vec{n_1}=3\hat{i}-16\hat{j}+7\hat{k}$

Normal parallel to (ii) is

$\vec{n_2}=3\hat{i}+8\hat{j}-5\hat{k}$

The required line is perpendicular to the given lines. So the normal $\vec{n}$ parallel to the required line perpendicular to $\vec{n_1}$ and $\vec{n_2}$.

$$\therefore \vec{n}=\vec{n_1}\times\vec{n_2}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & -16 & 7\\ 3 & 8 & -5\end{vmatrix}=24\hat{i}+36\hat{j}+72\hat{k}$$

Thus, the vector equation of the required line is

$\vec{r}=(\hat{i}+2\hat{j}-4\hat{k})+\gamma(24\hat{i}+36\hat{j}+72\hat{k})$

$\Rightarrow \vec{r}=(\hat{i}+2\hat{j}-4\hat{k})+k(2\hat{i}+3\hat{j}+6\hat{k})$

(where $k=12\gamma$)

Also, the cartesian equation of the required line is

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z+4}{6}$$ **Ans.**

19. Three persons A, B and C apply for a job of Manager in a Private Company. Chances of their selection (A, B and C) are in the ratio 1 : 2 : 4. The probabilities that A, B and C can introduce changes to improve profits of the company are 0.8, 0.5 and 0.3 respectively. If the change does not take place, find the probability that it is due to the appointment of C. [4]

Solution : Let E_1, E_2 and E_3 be the events denoting the selection of A, B and C as managers respectively.

$P(E_1)$ = Probability of selection of A = $\frac{1}{7}$

$P(E_2)$ = Probability of selection of B = $\frac{2}{7}$

$P(E_3)$ = Probability of selection of C = $\frac{4}{7}$

Let A be the event denoting the change not taking place.

$P(A/E_1)$ = Probability that A does not introduce change $=0.2$

$P(A/E_2)$ = Probability that B does not introduce change $=0.5$

$P(A/E_3)$ = Probability that C does not introduce change $=0.7$

$\therefore$ Required probability $= P(E_3/A)$

By Bayes' theorem, we have

$P(E_3/A) =$

$$\frac{P(E_3)P(A/E_3)}{P(E_1)P(A/E_1)+P(E_2)P(A/E_2)+P(E_3)P(A/E_3)}$$

$$=\frac{\frac{4}{7}\times0.7}{\frac{1}{7}\times0.2+\frac{2}{7}\times0.5+\frac{4}{7}\times0.7}$$

$$= \frac{2.8}{0.2+1+2.8}$$

$$= \frac{2.8}{4} = 0.7$$ **Ans.**

OR

A and B throw a pair of dice alternately. A wins the game if he gets a total of 7 and B wins the game if he gets a total of 10. If A starts the game, then find the probability that B wins.

Solution : Total of 7 on the dice can be obtained in the following ways :

(1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3)

Probability of getting a total of 7 $= \frac{6}{36} = \frac{1}{6}$

Probability of not getting a total of 7 $= 1 - \frac{1}{6} = \frac{5}{6}$

Total of 10 on the dice can be obtained in the following ways :

(4, 6), (6, 4), (5, 5)

Probability of getting a total of 10

$$= \frac{3}{36} = \frac{1}{12}$$

Probability of not getting a total of 10

$$= 1 - \frac{1}{12} = \frac{11}{12}$$

Let E and F be the two events, defined as follows :

E = Getting a total of 7 in a single throw of a dice

F = Getting a total of 10 in a single throw of a dice

$$P(E) = \frac{1}{6}, P(\bar{E}) = \frac{5}{6}$$

$$P(F) = \frac{1}{12}, P(\bar{F}) = \frac{11}{12}$$

A wins if he gets a total of 7 in 1^{st}, 3^{rd} or 5^{th}.... throws.

Probability of A getting a total of 7 in the1st throw $= \frac{1}{6}$

A will get the 3^{rd} throw if he fails in the 1st throw and B fails in the 2^{nd} throw.

Probability of A getting a total of 7 in the 3^{rd} throw

$$= P(\bar{E})\, P(\bar{F})\, P(E) = \frac{5}{6} \times \frac{11}{12} \times \frac{1}{6}$$

Similarly, probability of getting a total of 7 in the 5^{th} throw $= P(\bar{E})\, P(\bar{F})\, P(\bar{E})\, P(\bar{E})\, P(E)$

$$= \frac{5}{6} \times \frac{11}{12} \times \frac{5}{6} \times \frac{11}{12} \times \frac{1}{6} \text{ and so on}$$

Probability of winning of A

$$= \frac{1}{6} + \left(\frac{5}{6} \times \frac{11}{12} \times \frac{1}{6}\right) + \left(\frac{5}{6} \times \frac{11}{12} \times \frac{5}{6} \times \frac{11}{12} \times \frac{1}{6}\right) + \ldots$$

$$= \frac{\frac{1}{6}}{1 - \frac{5}{6} \times \frac{11}{12}} = \frac{12}{17}$$

∴ Probability of winning of B = 1 – Probability of winning of A

$$= 1 - \frac{12}{17} = \frac{5}{17}$$ **Ans.**

SECTION — C

20. Let $f : N \to N$ be a function defined as $f(x) = 9x^2 + 6x - 5$. Show that $f : N \to S$, where S is the range of f, is invertible. Find the inverse of f and hence find $f^{-1}(43)$ and $f^{-1}(163)$. [6]

Solution : Given,

$$f(x) = 9x^2 + 6x - 5$$

Let $x_1, x_2 \in N$ and $f(x_1) = f(x_2)$

$\Rightarrow \quad 9x_1^2 + 6x_1 - 5 = 9x_2^2 + 6x_2 - 5$

$\Rightarrow \quad 9(x_1^2 - x_2^2) + 6(x_1 - x_2) = 0$

$\Rightarrow \quad (x_1 - x_2)\,[9(x_1 + x_2) + 6] = 0$

$\Rightarrow \quad x_1 - x_2 = 0$

as $\quad 9x_1 + 9x_2 + 6 \neq 0 \quad [x_1, x_2 \in N]$

$\Rightarrow \quad x_1 = x_2$

f is one-one function.

Let $\quad y = 9x^2 + 6x - 5$

$\Rightarrow \quad y = (3x+1)^2 - 1 - 5 = (3x+1)^2 - 6$

$\Rightarrow \quad (3x+1)^2 = y + 6$

$\Rightarrow \quad 3x + 1 = \sqrt{y+6}$

$\Rightarrow \quad x = \frac{\sqrt{y+6}-1}{3}$ as $x \in N$

$\Rightarrow \quad \sqrt{y+6} - 1 > 0$

$\Rightarrow \quad y + 6 > 1$

$\Rightarrow \quad y > -5$ and $y \in N$

So, the function is invertible if the range of the function $f(x)$ is {1, 2, 3, ...}.

$f : N \to S$ is onto as codomain = Range

Hence f is invertible.

Therefore, the inverse of the function $f(x)$ is $f^{-1}(y)$, *i.e.*, x.

Now, $\quad f^{-1}(y) = \frac{\sqrt{y+6}-1}{3}$

$$\Rightarrow \quad f^{-1}(43) = \frac{\sqrt{43+6}-1}{3} = 2$$

and $$f^{-1}(163) = \frac{\sqrt{163+6}-1}{3} = 4$$ **Ans.**

21. Prove that $\begin{vmatrix} yz-x^2 & zx-y^2 & xy-z^2 \\ zx-y^2 & xy-z^2 & yz-x^2 \\ xy-z^2 & yz-x^2 & zx-y^2 \end{vmatrix}$ **is visible by $(x+y+z)$ and hence find the quotient.** **[6]**

Solution : Let $\Delta = \begin{vmatrix} yz-x^2 & zx-y^2 & xy-z^2 \\ zx-y^2 & xy-z^2 & yz-x^2 \\ xy-z^2 & yz-x^2 & zx-y^2 \end{vmatrix}$

Applying $C_1 \to C_1 - C_2$, $C_2 \to C_2 - C_3$

$$\Delta = \begin{vmatrix} yz-x^2-zx+y^2 & zx-xy-y^2+z^2 & xy-z^2 \\ zx-xy-y^2+z^2 & xy-yz-z^2+x^2 & yz-x^2 \\ xy-yz-z^2+x^2 & yz-zx-x^2+y^2 & zx-y^2 \end{vmatrix}$$

$$\Delta = \begin{vmatrix} (y-x)(x+y+z) & (z-y)(x+y+z) & xy-z^2 \\ (z-y)(x+y+z) & (x-z)(x+y+z) & yz-x^2 \\ (x-z)(x+y+z) & (y-x)(x+y+z) & zx-y^2 \end{vmatrix}$$

Taking $(x + y + z)$ common from C_1 and C_2 both

$$\Delta = (x+y+z)^2 \begin{vmatrix} (y-x) & (z-y) & xy-z^2 \\ (z-y) & (x-z) & yz-x^2 \\ (x-z) & (y-x) & zx-y^2 \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$

$$\Delta = (x+y+z)^2 \begin{vmatrix} 0 & 0 & xy+yz+zx-x^2-y^2-z^2 \\ (z-y) & (x-y) & yz-x^2 \\ (x-z) & (y-x) & zx-y^2 \end{vmatrix}$$

Expanding along R_1, we get

$$\Delta = (x+y+z)^2 \{ (xy+yz+zx-x^2-y^2-z^2) [(z-y)(y-x)-(x-z)^2]\}$$

$$\Rightarrow \Delta = (x+y+z)^2 \{(xy+yz+zx-x^2-y^2-z^2)(xy+yz+zx-x^2-y^2-z^2)\}$$

$$\Delta = (x+y+z)^2 (xy+zy+zx-x^2-y^2-z^2)^2$$

Hence Δ is divisible by $(x+y+z)$ and the quotient is $(x+y+z)(xy+yz+zx-x^2-y^2-z^2)^2$

Hence Proved.

OR

Using elementary transformations, find the inverse of the matrix A = $\begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ **and use it to solve the following system of linear equations :**

$$8x + 4y + 3z = 19$$
$$2x + y + z = 5$$
$$x + 2y + 2z = 7$$

Solution: Here, $A = \begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

Using A = IA, we have

$$\begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_1 \leftrightarrow R_3$, We get

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \\ 8 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A$$

Applying $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 8R_1$, We get

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & -3 & -3 \\ 0 & -12 & -13 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & 0 & -8 \end{bmatrix} A$$

Applying $R_2 \to \frac{R_2}{-3}$, we get

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & -12 & -13 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -\frac{1}{3} & \frac{2}{3} \\ 1 & 0 & -8 \end{bmatrix} A$$

Applying $R_1 \to R_1 - 2R_2$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -12 & -13 \end{bmatrix} = \begin{bmatrix} 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{2}{3} \\ 1 & 0 & -8 \end{bmatrix} A$$

Applying $R_3 \to R_3 + 12R_2$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{2}{3} \\ 1 & -4 & 0 \end{bmatrix} A$$

Applying $R_3 \to -R_3$, we get

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}0 & \frac{2}{3} & -\frac{1}{3}\\ 0 & -\frac{1}{3} & \frac{2}{3}\\ -1 & 4 & 0\end{bmatrix}A$$

Applying $R_2 \to R_2 - R_3$, we get

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}0 & \frac{2}{3} & -\frac{1}{3}\\ 1 & -\frac{13}{3} & \frac{2}{3}\\ -1 & 4 & 0\end{bmatrix}A$$

Thus, we have

$$A^{-1} = \begin{bmatrix}0 & \frac{2}{3} & -\frac{1}{3}\\ 1 & -\frac{13}{3} & \frac{2}{3}\\ -1 & 4 & 0\end{bmatrix}$$

The given system of equations is

$$8x + 4y + 3z = 19$$
$$2x + y + z = 5$$
$$x + 2y + 2z = 7$$

This system of equations can be written as $AX = B$,

where $A = \begin{bmatrix}8 & 4 & 3\\ 2 & 1 & 1\\ 1 & 2 & 2\end{bmatrix}, X = \begin{bmatrix}x\\ y\\ z\end{bmatrix}$ and $B = \begin{bmatrix}19\\ 5\\ 7\end{bmatrix}$

$\therefore \quad X = A^{-1}B$

$$\Rightarrow \quad \begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0 & \frac{2}{3} & -\frac{1}{3}\\ 1 & -\frac{13}{3} & \frac{2}{3}\\ -1 & 4 & 0\end{bmatrix}\begin{bmatrix}19\\ 5\\ 7\end{bmatrix}$$

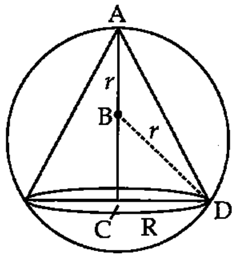
$$\Rightarrow \quad \begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0+\frac{10}{3}-\frac{7}{3}\\ 19-\frac{65}{3}+\frac{14}{3}\\ -19+20+0\end{bmatrix} = \begin{bmatrix}1\\ 2\\ 1\end{bmatrix}$$

$\therefore x = 1, y = 2$ and $z = 1$. **Ans.**

22. Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$. Also find maximum volume in terms of volume of the sphere. [6]

Solution : A sphere of fixed radius (r) is given.

Let R and h be the radius and the height of the cone respectively.



The volume (V) of the cone is given by,

$$V = \frac{1}{3}\pi R^2 h$$

Now, from the right triangle BCD, we have

$$BC = \sqrt{r^2 - R^2}$$

$$\therefore \quad h = r + \sqrt{r^2 - R^2}$$

$$V = \frac{1}{3}\pi R^2 r + \frac{1}{3}\pi R^2\sqrt{r^2 - R^2}$$

$$\frac{dV}{dR} = \frac{2}{3}\pi R r + \frac{2}{3}\pi R\sqrt{r^2 - R^2} + \frac{\pi R^2}{3}\cdot\frac{(-2R)}{2\sqrt{r^2 - R^2}}$$

$$= \frac{2}{3}\pi R r + \frac{2}{3}\pi R\sqrt{r^2 - R^2} - \frac{\pi R^3}{3\sqrt{r^2 - R^2}}$$

$$= \frac{2}{3}\pi R r + \frac{2\pi R(r^2 - R^2) - \pi R^3}{3\sqrt{r^2 - R^2}}$$

$$= \frac{2}{3}\pi R r + \frac{2\pi R r^2 - 3\pi R^3}{3\sqrt{r^2 - R^2}}$$

Now, $\frac{dV}{dR} = 0$

$$\Rightarrow \quad \frac{2\pi r R}{3} = \frac{3\pi R^3 - 2\pi R r^2}{3\sqrt{r^2 - R^2}}$$

$$\Rightarrow \quad 2r\sqrt{r^2 - R^2} = 3R^2 - 2r^2$$

$$\Rightarrow \quad 4r^2(r^2 - R^2) = (3R^2 - 2r^2)^2$$

$$\Rightarrow \quad 4r^4 - 4r^2R^2 = 9R^4 + 4r^4 - 12R^2r^2$$

$$\Rightarrow \quad 9R^4 - 8r^2R^2 = 0$$

$$\Rightarrow \quad 9R^2 = 8r^2$$

$$\Rightarrow \quad R^2 = \frac{8r^2}{9}$$

Now, $\frac{d^2V}{dR^2}$

$$= \frac{2\pi r}{3} + \frac{3\sqrt{r^2 - R^2}(2\pi r^2 - 9\pi R^2) - +(2\pi R r^2 - 3\pi R^3)(-6R)\frac{1}{2\sqrt{r^2 - R^2}}}{9(r^2 - R^2)}$$

$$3\sqrt{r^2-R^2}(2\pi r^2-9\pi R^2)$$

$$=\frac{2\pi r}{3}+\frac{(2\pi R r^2-3\pi R^3)(6R)\frac{1}{2\sqrt{r^2-R^2}}}{9(r^2-R^2)}$$

No, when $R^2 = \frac{8r^2}{9}$, Clearly $\frac{d^2V}{dR^2} < 0.$

∴ The volume is maximum when

$$R^2 = \frac{8r^2}{9}.$$

Height of the cone

$$= r + \sqrt{r^2 - \frac{8r^2}{9}} = r + \sqrt{\frac{r^2}{9}} = r + \frac{r}{3} = \frac{4r}{3}.$$

Hence, it can be seen that the altitude of a right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$.

Let volume of the sphere be $V_s = \frac{4}{3}\pi r^3$.

$$r = \sqrt[3]{\frac{3V_s}{4\pi}}$$

∴ Volume of cone, $V = \frac{1}{3}\pi R^2 h$

$$\Rightarrow \quad V = \frac{1}{3}\pi \frac{8r^2}{9} \times \frac{4r}{3} \qquad \left(\because R^2 = \frac{8r^2}{9}\right)$$

$$\Rightarrow \quad V = \frac{32\pi r^3}{81} = \frac{8}{27}\left(\frac{4}{3}\pi r^3\right)$$

∴ Maximum volume of cone in terms of sphere =

$\frac{8 \text{ Volume of sphere}}{27}$ **Hence Proved.**

OR

Find the intervals in which $f(x) = \sin 3x - \cos 3x$, $0 < x < \pi$, is strictly increasing or strictly decreasing.

Solution : Consider the function

$$f(x) = \sin 3x - \cos 3x$$

$$\Rightarrow f'(x) = 3\cos 3x + 3\sin 3x$$

$$= 3(\sin 3x + \cos 3x)$$

$$= 3\sqrt{2}\left\{\sin 3x \cos\left(\frac{\pi}{4}\right) + \cos 3x \sin\left(\frac{\pi}{4}\right)\right\}$$

$$= 3\sqrt{2}\left\{\sin\left(3x + \frac{\pi}{4}\right)\right\}$$

For the increasing interval $f'(x) > 0$.

$$3\sqrt{2}\left\{\sin\left(3x+\frac{\pi}{4}\right)\right\} > 0$$

$$\sin\left(3x+\frac{\pi}{4}\right) > 0$$

$$\Rightarrow \quad 0 < 3x + \frac{\pi}{4} < \pi$$

$$\Rightarrow \quad \frac{-\pi}{4} < 3x < \frac{3\pi}{4}$$

$$\Rightarrow \quad \frac{-\pi}{12} < x < \frac{\pi}{4}$$

as $0 < x < \pi$ is given

$\Rightarrow 0 < x < \pi/4$

Also, $\sin\left(3x+\frac{\pi}{4}\right) > 0$

when, $2\pi < 3x + \frac{\pi}{4} < 3\pi$

Therefore, intervals in which function is strictly increasing is $0 < x < \frac{\pi}{4}$ and $\frac{7\pi}{12} < x < \frac{11\pi}{12}$.

Similarly, for the decreasing interval $f'(x) < 0$

$$3\sqrt{2}\left\{\sin\left(3x+\frac{\pi}{4}\right)\right\} < 0$$

$$\sin\left(3x+\frac{\pi}{4}\right) < 0$$

$$\Rightarrow \quad \pi < 3x + \frac{\pi}{4} < 2\pi$$

$$\Rightarrow \quad \frac{3\pi}{4} < 3x < \frac{7\pi}{4}$$

$$\Rightarrow \quad \frac{\pi}{4} < x < \frac{7\pi}{12}$$

Also, $\quad \sin\left(3x+\frac{\pi}{4}\right) < 0$

When $\quad 3\pi < 3x + \frac{\pi}{4} < 4\pi,$

$$\Rightarrow \quad \frac{11\pi}{4} < 3x < \frac{15\pi}{4}$$

$$\Rightarrow \quad \frac{11\pi}{12} < x < \frac{15\pi}{12}$$

But $0 < x < \pi$ so $\frac{11\pi}{12} < x < \pi$

The function is strictly decreasing in $\frac{\pi}{4} < x < \frac{7\pi}{12}$

and $\frac{11\pi}{12} < x < \pi.$ **Ans.**

23. Using integration find the area of the region $\{(x, y) : x^2 + y^2 \le 2ax, y^2 \ge ax, x, y \ge 0\}$ [6]

Solution : We have, $\{(x, y) : x^2 + y^2 \le 2ax, y^2 \ge ax, x, y \ge 0\}$

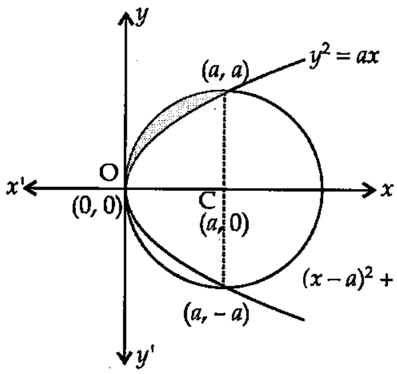
Consider $\quad x^2 + y^2 = 2ax \quad$ ...(i)

$$y^2 = ax \quad ...(ii)$$
$$x = 0, y = 0$$

Solving equation (i) and (ii), we get

$$x^2 + ax = 2ax$$
$$\Rightarrow \quad x^2 - ax = 0$$
$$\Rightarrow \quad x(x-a) = 0$$
$$\therefore \quad x = 0, a$$

So, points of intersections of (i) and (ii) are (0, 0) and $(a, \pm a)$. Also, equation (i) can be written as, $(x-a)^2 + (y-0)^2 = a^2$ whose centre is at $(a, 0)$ and radius is of 'a' units.



$\therefore$ Require area

$$= \int_0^a y_1 dx - \int_0^a y_2 dx$$

$$= \int_0^a \sqrt{a^2-(x-a)^2}.dx - \sqrt{a}\int_0^a \sqrt{x}.dx$$

$$= \left[\frac{x-a}{2}\sqrt{a^2-(x-a)^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right)\right]_0^a - \frac{2}{3}\sqrt{a}[x^{3/2}]_0^a$$

$$= [0+0] - \left[0 + \frac{a^2}{2}\left(-\frac{\pi}{2}\right)\right] - \frac{2}{3}\sqrt{a}[a^{3/2} - 0]$$

$$= \left(\frac{\pi}{4} - \frac{2}{3}\right)a^2 \text{ sq. units}$$ **Ans.**

24. Find the coordinate of the point P where the line through A(3, – 4, – 5) and B (2, – 3, 1) crosses the plane passing through three points L (2, 2, 1), M (3, 0, 1) and N (4, – 1, 0). Also, find the ratio in which P divides the line segment AB. [6]

Solution : The equation of the plane passing through three given points can be given by

$$\begin{vmatrix} x-2 & y-2 & z-1 \\ x-3 & y-0 & z-1 \\ x-4 & y+1 & z-0 \end{vmatrix} = 0$$

Performing elementary row operations $R_2 \to R_1 - R_2$ and $R_3 \to R_1 - R_3$, we get

$$\begin{vmatrix} x-2 & y-2 & z-1 \\ 3-2 & 0-2 & 0 \\ 4-2 & -1-2 & -1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-2 & y-2 & z-1 \\ 1 & -2 & 0 \\ 2 & -3 & -1 \end{vmatrix} = 0$$

Solving the above determinant, we get

$$(x-2)(2-0) - (y-2)(-1-0) + (z-1)(-3+4) = 0$$
$$\Rightarrow \quad (2x-4) + (y-2) + (z-1) = 0$$
$$\Rightarrow \quad 2x + y + z - 7 = 0$$

Therefore, the equation of the plane is

$$2x + y + z - 7 = 0$$

Now, the equation of the line passing through two given points is

$$\frac{x-3}{2-3} = \frac{y+4}{-3+4} = \frac{z+5}{1+5} = \lambda$$
$$\Rightarrow \quad \frac{x-3}{-1} = \frac{y+4}{1} = \frac{z+5}{6} = \lambda$$
$$\Rightarrow \quad x = (-\lambda+3), y = (\lambda-4), z = (6\lambda-5)$$

At the point of intersection, these points satisfy the equation of the plane $2x + y + z - 7 = 0$

Putting the values of x, y and z in the equation of the plane, we get the value of λ

$$2(-\lambda+3) + (\lambda-4) + (6\lambda-5) - 7 = 0$$
$$\Rightarrow \quad -2\lambda + 6 + \lambda - 4 + 6\lambda - 5 - 7 = 0$$
$$\Rightarrow \quad 5\lambda = 10$$
$$\Rightarrow \quad \lambda = 2$$

Thus, the point of intersection is P (1, – 2, 7).

Now, let P divide the line AB in the ratio $m : n$.

By the section formula, we have

$$1 = \frac{2m+3n}{m+n}$$
$$\Rightarrow \quad m + 2n = 0$$
$$\Rightarrow \quad m = -2n$$
$$\Rightarrow \quad \frac{m}{n} = \frac{-2}{1}$$

Hence, P divides externally the line segment AB in the ratio 2 : 1. **Ans.**

25. An urn contains 3 white and 6 red balls. Four balls are drawn one by one with replacement from the urn. Find the probability distribution of the number of red balls drawn. Also find mean and variance of the distribution. [6]

Solution : Let X denote the total number of red balls when four balls are drawn one by one with replacement.

P (getting a red ball in one draw) = $\frac{6}{9} = \frac{2}{3}$

P (getting a white ball in one draw) = $\frac{1}{3}$

X	0	1	2	3	4
P(X)	$\left(\frac{1}{3}\right)^4$	$\frac{2}{3}\left(\frac{1}{3}\right)^3 \cdot {}^4C_1$	$\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2 \cdot {}^4C_2$	$\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right) \cdot {}^4C_3$	$\left(\frac{2}{3}\right)^4$
	$\frac{1}{81}$	$\frac{8}{81}$	$\frac{24}{81}$	$\frac{32}{81}$	$\frac{16}{81}$

Using the formula for mean, we have

$$\overline{X} = \Sigma P_i X_i$$

Mean

$$(\overline{X}) = \left(0 \times \frac{1}{81}\right) + 1\left(\frac{8}{81}\right) + 2\left(\frac{24}{81}\right) + 3\left(\frac{32}{81}\right) + 4\left(\frac{16}{81}\right)$$

$$= \frac{1}{81}(8 + 48 + 96 + 64)$$

$$= \frac{216}{81}$$

$$= \frac{8}{3}$$

Using the formula for variance, we have

$$\text{Var}(X) = \Sigma P_i X_i^2 - (\Sigma P_i X_i)^2$$

$$\Rightarrow \text{Var}(X) = \left\{\left(0 \times \frac{1}{81}\right) + 1\left(\frac{8}{81}\right) + 4\left(\frac{24}{81}\right) + 9\left(\frac{32}{81}\right) + 16\left(\frac{16}{81}\right)\right\} - \left(\frac{8}{3}\right)^2$$

$$= \frac{648}{81} - \frac{64}{9}$$

$$= \frac{8}{9}$$

Hence, the mean of the disribution is $\frac{8}{3}$ and the variance of the disribution is $\frac{8}{9}$ **Ans.**

26. A manufacturer produces two products A and B. Both the products are processed on two different machines. The available capacity of first machine is 12 hours and that of second machine is 9 hours per day. Each unit of product A requires 3 hours on both machines and each unit of product B requires 2 hours on first machine and 1 hour on second machine. Each unit of product A is sold at ₹ 7 profit and B at a profit of ₹ 4. Find the production level per day for maximum profit graphically. [6]

Solution : Let the numbers of units of products A and B to be produced be x and y, respectively.

Product	Machine	
	I (h)	II (h)
A	3	3
B	2	1

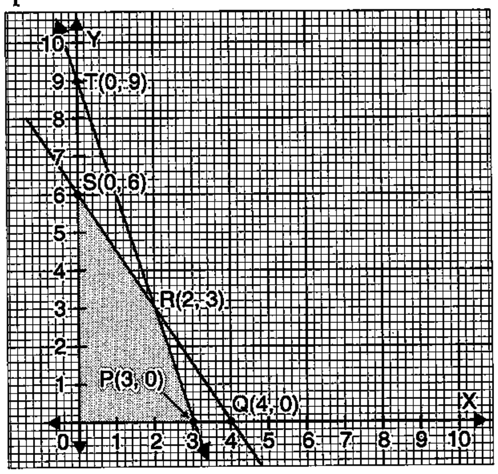
Total profit : $Z = 7x + 4y$

We have to maximize $Z = 7x + 4y$.

Subject to constraints :

$$3x + 2y \le 12$$
$$3x + y \le 9$$
$$\Rightarrow \quad x \ge 0 \text{ and } y \ge 0$$

The given information can be graphically expressed as follows :



Values of $Z = 7x + 4y$ at the corner points are S follows :

Corner Points	$Z = 7x + 4y$
S (0,6)	24
R (2,3)	**26** ← Maximum
P (3,0)	21

Therefore, the manufacturer has to produce 2 units of product A and 3 units of product B for the maximum profit of ₹ 26. **Ans.**

●●

All questions are same in Outside Delhi Set II and Set III

Mathematics 2017 (Outside Delhi)

SET I

Time allowed : 3 hours **Maximum marks : 100**

SECTION — A

1. If for any 2 × 2 square matrix A,

$$A(\text{adj } A) = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix},$$

then write the value of $|A|$. [1]

Solution : We have,

$$A(\text{adj } A) = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$$

As,

$$A(\text{adj } A) = |A|\, I$$

$$\begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix} = 8\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

On comparing, we get

$$|A| = 8.$$ **Ans.**

2. Determine the value of 'k' for which the following function is continuous at $x = 3$.

$$f(x) = \begin{cases} \dfrac{(x+3)^2 - 36}{x-3}, & x \neq 3 \\ k, & x = 3 \end{cases}$$ [1]

Solution : Given,

$$f(x) = \begin{cases} \dfrac{(x+3)^2 - 36}{x-3}, & x \neq 3 \\ k, & x = 3 \end{cases}$$

Since $f(x)$ is continuous at $x = 3$

$\therefore \quad \lim_{x \to 3} f(x) = f(3)$

$\Rightarrow \quad \lim_{x \to 3} f(x) = k$

$\Rightarrow \quad \lim_{x \to 3} \dfrac{(x+3)^2 - 36}{x-3} = k$

$\Rightarrow \quad \lim_{x+3 \to 6} \dfrac{(x+3)^2 - 6^2}{(x+3)-6} = k$

$\Rightarrow \quad \lim_{x \to 3} x+3+6 = k$

$\Rightarrow \quad 12 = k$

Thus, $f(x)$ is continuous at $x = 3$; if $k = 12$. **Ans.**

3. Find : $\displaystyle\int \frac{\sin^2 x - \cos^2 x}{\sin x \cos x}\, dx.$ [1]

Solution : We have,

$$\int \frac{\sin^2 x - \cos^2 x}{\sin x \cos x}\, dx = -2\int \frac{\cos^2 x - \sin^2 x}{2 \sin x \cos x}\, dx$$

$$= -2\int \frac{\cos 2x}{\sin 2x}\, dx$$

$$= -2\int \cot 2x\, dx$$

$$= -\log|\sin 2x| + C$$

Ans.

4. Find the distance between the planes $2x - y + 2z = 5$ and $5x - 2.5y + 5z = 20$. [1]

Solution : Since, the direction ratios of the normal to the given planes are proportional.

i.e., $\frac{2}{5} = \frac{-1}{-2.5} = \frac{2}{5}$

Thus, the given planes are parallel.

Now, let $P(x_1, y_1, z_1)$ be any point on $2x - y + 2z - 5 = 0$

Then, $2x_1 - y_1 + 2z_1 - 5 = 0$

or $5x_1 - 2.5y_1 + 5z_1 - 12.5 = 0$ ...(i)

The length of the perpendicular from $P(x_1, y_1, z_1)$ to $5x - 2.5y + 5z - 20 = 0$,

$$= \left| \frac{5x_1 - 2.5y_1 + 5z_1 - 20}{\sqrt{(5)^2 + (-2.5)^2 + (5)^2}} \right|$$

$$= \left| \frac{12.5 - 20}{\sqrt{25 + 6.25 + 25}} \right| \quad \text{[From eq. (i)]}$$

$$= \left| \frac{-7.5}{7.5} \right| = 1 \text{ unit}$$

Therefore, the distance between the given planes is 1 unit. **Ans.**

SECTION — B

5. If A is a skew-symmetric matrix of order 3, then prove that det A = 0. [2]

Solution : Given, A is a skew-symmetric matrix of order 3.

So, $A^T = -A$

Now, $|A^T| = |-A|$

$|A^T| = (-1)^3|A|$

$[\because |kA| = k^n |A|$ where n is order of A]

$|A| = -|A| \quad [\ |A^T| = |A|]$

$\Rightarrow |A| + |A| = 0$

$\therefore 2|A| = 0$ or $|A| = 0$.

i.e., det A $= 0$ **Hence Proved.**

6. Find the value of c in Rolle's theorem for the function $f(x) = x^3 - 3x$ in $[-\sqrt{3}, 0]$ [2]

Solution : We know that the polynomial function $f(x) = x^3 - 3x$ is everywhere continuous and differentiable.

So, $f(x)$ is continuous on $[-\sqrt{3}, 0]$.

Also, $f(x)$ is differentiable on $(-\sqrt{3}, 0)$

Now, $f(-\sqrt{3}) = (-\sqrt{3})^3 - 3(-\sqrt{3})$

$= -3\sqrt{3} + 3\sqrt{3} = 0$

and $f(0) = (0)^3 - 3(0) = 0$

$\therefore f(-\sqrt{3}) = f(0)$

Thus, all the three conditions of Rolle's theorem are satisfied.

Now, there must exist $c \in (-\sqrt{3}, 0)$ such that $f'(c) = 0$

Now, $f'(x) = 3x^2 - 3$

Then, $f'(c) = 0$

$\Rightarrow 3c^2 - 3 = 0$

$\Rightarrow 3(c^2 - 1) = 0$

$\Rightarrow c = \pm 1$

$c \neq 1$ as $1 \notin (-\sqrt{3}, 0)$

$\therefore c = -1 \notin (-\sqrt{3}, 0)$

Thus, required value of c is -1. **Ans.**

7. The volume of a cube is increasing at the rate of 9 cm³/s. How fast is its surface area increasing when the length of an edge is 10 cm ? [2]

Solution : Let, the side of the cube be a cm

then, volume of cube (V) $= a^3$

Differentiating V w.r.t. t, we get

$$\frac{dV}{dt} = 3a^2 \frac{da}{dt}$$

$$\Rightarrow 3a^2 \frac{da}{dt} = 9 \text{ cm}^3/\text{s}. \quad \left[\text{Given}, \frac{dV}{dt} = 9 \text{ cm}^3/\text{s}\right]$$

$$\Rightarrow \frac{da}{dt} = \frac{3}{a^2} \text{ cm/s}$$

and surface area of cube (S) $= 6a^2$

$$\frac{dS}{dt} = 12a \frac{da}{dt}$$

$$\frac{dS}{dt} = 12a \times \frac{3}{a^2} = \frac{36}{a} \text{ cm}^2/\text{s}$$

When $a = 10$

$$\therefore \left[\frac{dS}{dt}\right]_{a=10} = \frac{36}{10} = 3.6 \text{ cm}^2/\text{s}.$$ **Ans.**

8. Show that the function $f(x) = x^3 - 3x^2 + 6x - 100$ is increasing on R. [2]

Solution : We have,

$f(x) = x^3 - 3x^2 + 6x - 100$

then, $f'(x) = 3x^2 - 6x + 6$

$= 3(x^2 - 2x + 1) + 3$

$= 3(x - 1)^2 + 3 > 0$ for all $x \in$ R.

Hence, the function $f(x)$ is increasing on R.

Hence Proved.

9. The x-coordinate of a point on the line joining the points P(2, 2, 1) and Q (5, 1, – 2) is 4. Find its z-coordinate. [2]

Solution : Given, the points P(2, 2, 1) and Q (5, 1, – 2) of a line.

Then, equation of line PQ,

$$\frac{x-2}{5-2} = \frac{y-2}{1-2} = \frac{z-1}{-2-1}$$

$$\Rightarrow \quad \frac{x-2}{3} = \frac{y-2}{-1} = \frac{z-1}{-3} = \lambda$$

∴ Coordinates of any point of line PQ is

$$(3\lambda+2, -\lambda+2, -3\lambda+1)$$

Now, we have the x-coordinate as 4.

$$\Rightarrow \quad 3\lambda + 2 = 4$$

$$\Rightarrow \quad 3\lambda = 2$$

$$\therefore \quad \lambda = \frac{2}{3}$$

∴ z coordinate is $-3\lambda + 1$ *i.e.*, – 1. **Ans.**

10. A die, whose faces are marked 1, 2, 3 in red and 4, 5, 6 in green, is tossed. Let A be the event "number obtained is even" and B be the event "number obtained is red". Find if A and B are independent events. [2]

Solution : Since, A be the event of number obtained is even

then, $A = \{2, 4, 6\}$

and B be the event of number obtained is red

then, $B = \{1, 2, 3\}$

∴ $A \cap B = \{2\}$

So, $P(A) = \frac{3}{6} = \frac{1}{2}$; $P(B) = \frac{3}{6} = \frac{1}{2}$; $P(A \cap B) = \frac{1}{6}$

Now, $P(A \cap B) \neq P(A) \cdot P(B)$

$$\frac{1}{6} \neq \frac{1}{4}$$

Hence, the events A and B are not independent events. **Ans.**

11. Two tailors, A and B earn ₹ 300 and ₹ 400 per day respectively. A can stitch 6 shirts and 4 pairs of trousers while B can stitch 10 shirts and 4 pairs of trousers per day. To find how many days should each of them work if it is desired to produce at least 60 shirts and 32 pairs of trousers at a minimum labour cost, formulate this as an LPP. [2]

Solution : Let x and y be the number of days for which the tailors A and B work respectively.

Total cost per day = ₹ $(300x + 400y)$

Let Z denote the total cost in rupees, then,

$$Z = 300x + 400y$$

Since in one day 6 shirts are stitched by tailor A and 10 shirts are stitched by tailor B and it is desired to produce atleast 60 shirts.

$$\therefore \quad 6x + 10y \geq 60$$

It is given that 4 pairs of trousers are stitched by each tailor A and B per day to produce atleast 32 pairs of trousers.

$$\therefore \quad 4x + 4y \geq 32$$

Finally, the no. of shirts and pair of trousers cannot be negative.

$$\therefore \quad x \geq 0, \; y \geq 0$$

Thus, mathematical formulation of the given LPP is as follows :

Minimize $Z = 300x + 400y$

Subject to constraints :

$$6x + 10y \geq 60$$
$$4x + 4y \geq 32$$
$$x \geq 0, \; y \geq 0$$

Ans.

12. Find : $\int \frac{dx}{5 - 8x - x^2}$. [2]

Solution : We have,

$$\int \frac{dx}{5-8x-x^2} = \int \frac{dx}{5-8x-x^2+(4)^2-(4)^2}$$

$$= \int \frac{dx}{21-[(4)^2+8x+(x)^2]}$$

$$= \int \frac{dx}{(\sqrt{21})^2-(x+4)^2}$$

$$= \frac{1}{2\sqrt{21}} \log \left| \frac{\sqrt{21}+x+4}{\sqrt{21}-x-4} \right| + C$$ **Ans.**

SECTION — C

13. If $\tan^{-1}\frac{x-3}{x-4} + \tan^{-1}\frac{x+3}{x+4} = \frac{\pi}{4}$, then find the value of x. [4]

Solution : We have,

$$\tan^{-1}\left(\frac{x-3}{x-4}\right) + \tan^{-1}\left(\frac{x+3}{x+4}\right) = \frac{\pi}{4}$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{x-3}{x-4}\right) + \tan^{-1}\left(\frac{x+3}{x+4}\right) = \tan^{-1} 1$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{x-3}{x-4}\right) = \tan^{-1}1 - \tan^{-1}\left(\frac{x+3}{x+4}\right)$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{x-3}{x-4}\right) = \tan^{-1}\left(\frac{1-\frac{x+3}{x+4}}{1+1\times\frac{x+3}{x+4}}\right)$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{x-3}{x-4}\right) = \tan^{-1}\left(\frac{x+4-x-3}{x+4+x+3}\right)$$

$$\Rightarrow \quad \frac{x-3}{x-4} = \frac{1}{2x+7}$$

$$\Rightarrow \quad (2x+7)(x-3) = x-4$$

$$\Rightarrow \quad 2x^2-6x+7x-21 = x-4$$

$$\Rightarrow \quad 2x^2+x-21 = x-4$$

$$\Rightarrow \quad 2x^2 = 17$$

$$\Rightarrow \quad x = \pm\sqrt{\frac{17}{2}}$$

Ans.

14. Using properties of determinants, prove that

$$\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a-1)^3.$$

[4]

Solution : Let $\Delta = \begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix}$ = L. H. S.

Applying, $R_1 \to R_1 - R_2$, $R_2 \to R_2 - R_3$

$$\Delta = \begin{vmatrix} a^2-2a & a-1 & 0 \\ 2a-1 & a-2 & 1 \\ 3 & 3 & 1 \end{vmatrix}$$

$$= (a-1)^2 \begin{vmatrix} a+1 & 1 & 0 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{vmatrix}$$

[Taking $(a-1)$ common from R_1 and R_2]

Now, expanding along C_3,

$$\Delta = (a-1)^2\{(a+1)\cdot 1-(2)1\}$$

$$= (a-1)^2(a-1) = (a-1)^3$$

= R. H. S. **Hence Proved.**

OR

Find matrix A such that

$$\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix} A = \begin{bmatrix} -1 & -8 \\ 1 & -2 \\ 9 & 22 \end{bmatrix}$$

Solution : We have,

$$\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix}_{3\times 2} A = \begin{bmatrix} -1 & -8 \\ 1 & -2 \\ 9 & 22 \end{bmatrix}_{3\times 2}$$

$\therefore$ A is of order 2×2

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Then, $\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -1 & -8 \\ 1 & -2 \\ 9 & 22 \end{bmatrix}$

$$\begin{bmatrix} 2a-c & 2b-d \\ a & b \\ -3a+4c & -3b+4d \end{bmatrix} = \begin{bmatrix} -1 & -8 \\ 1 & -2 \\ 9 & 22 \end{bmatrix}$$

By equality of matrices, on comparing, we get

$$2a-c = -1$$
$$2b-d = -8$$
$$a = 1$$
$$b = -2$$
$$-3a+4c = 9$$
$$-3b+4d = 22$$

On solving the equations, we get

$a = 1, \quad b = -2, \quad c = 3, \quad d = 4$

Hence, $A = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$ **Ans.**

15. If $x^y + y^x = a^b$, then find $\frac{dy}{dx}$. **[4]**

Solution : We have, $x^y + y^x = a^b$

$$\Rightarrow \quad e^{\log x^y} + e^{\log y^x} = a^b$$

$$\Rightarrow \quad e^{y\log x} + e^{x\log y} = a^b$$

On differentiating both sides w.r.t. x, we get

$$e^{y\log x}\left(y\cdot\frac{1}{x}+\log x\cdot\frac{dy}{dx}\right)+e^{x\log y}\left(x\cdot\frac{1}{y}\frac{dy}{dx}+\log y\right)=0$$

$$\Rightarrow \quad x^y\left(\frac{y}{x}+\log x\frac{dy}{dx}\right)+y^x\left(\frac{x}{y}\frac{dy}{dx}+\log y\right)=0$$

$$\Rightarrow \quad yx^{y-1}+x^y\log x\frac{dy}{dx}+xy^{x-1}\frac{dy}{dx}+y^x\log y=0$$

$$\Rightarrow \quad \frac{dy}{dx}(x^y\log x+xy^{x-1}) = -(y^x\log y+yx^{y-1})$$

$$\therefore \quad \frac{dy}{dx} = -\left(\frac{y^x\log y+y\cdot x^{y-1}}{x^y\log x+x\cdot y^{x-1}}\right)$$

Ans.

OR

If $e^y(x+1)=1$, then show that $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$.

Solution : We have,

$$e^y(x+1) = 1$$

$$\Rightarrow \quad e^y = \frac{1}{x+1}$$

$$\Rightarrow \quad \log e^y = \log_e\left(\frac{1}{x+1}\right)$$

[Taking log of both]

$$\Rightarrow \quad y = -\log(x+1)$$

On differentiating y w.r.t. x, we get

$$\frac{dy}{dx} = -\frac{1}{x+1} \quad \text{...(i)}$$

Again, differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{1}{(x+1)^2}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \left(-\frac{1}{(x+1)}\right)^2$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2 \quad \text{[using (i)]}$$

Hence Proved.

16. Find : $\int \frac{\cos\theta}{(4+\sin^2\theta)(5-4\cos^2\theta)}d\theta$ **[4]**

Solution :

$$\text{Let I} = \int \frac{\cos\theta}{(4+\sin^2\theta)(5-4\cos^2\theta)}d\theta$$

$$= \int \frac{\cos\theta}{(4+\sin^2\theta)(5-4(-\sin^2\theta+1)]}d\theta$$

$$= \int \frac{\cos\theta}{(4+\sin^2\theta)(1+4\sin^2\theta)}d\theta$$

Substituting $\sin\theta = y \Rightarrow \cos\theta\, d\theta = dy$

$$\text{I} = \int \frac{1}{(4+y^2)(1+4y^2)}dy$$

$$\text{Let } \frac{1}{(4+y^2)(1+4y^2)} = \frac{\text{A}}{(4+y^2)} + \frac{\text{B}}{(1+4y^2)}$$

$$1 = \text{A}(1+4y^2) + \text{B}(4+y^2)$$

Putting $y = 0$, we get

$$1 = \text{A} + 4\text{B} \quad \text{...(i)}$$

Putting $y = 1$, we get

$$1 = 5\text{A} + 5\text{B} \quad \text{...(ii)}$$

Solving (i) and (ii), we get

$$\text{A} = \frac{-1}{15} \text{ and } \text{B} = \frac{4}{15}$$

$$\therefore \quad \text{I} = \int\left(\frac{-1}{15(4+y^2)} + \frac{4}{15(1+4y^2)}\right)dy$$

$$\Rightarrow \quad \text{I} = \frac{-1}{15}\int\frac{dy}{((2)^2+y^2)} + \frac{4}{15}\int\frac{dy}{((1)^2+(2y)^2)}$$

$$\Rightarrow \quad \text{I} = \frac{-1}{15}\times\frac{1}{2}\tan^{-1}\frac{y}{2} + \frac{4}{15}\times\frac{1}{2}\tan^{-1}2y + \text{C}$$

$$\Rightarrow \quad \text{I} = \frac{-1}{30}\tan^{-1}\left(\frac{\sin\theta}{2}\right) + \frac{2}{15}\tan^{-1}(2\sin\theta) + \text{C}$$

Ans.

17. Evaluate : $\int_0^{\pi} \frac{x\tan x}{\sec x + \tan x}dx.$ **[4]**

Solution : Let $\text{I} = \int_0^{\pi}\frac{x\tan x}{\sec x+\tan x}dx. \quad \text{...(i)}$

Now,

$$\text{I} = \int_0^{\pi}\frac{(\pi-x)\tan(\pi-x)}{\sec(\pi-x)+\tan(\pi-x)}$$

$$\Rightarrow \quad \text{I} = \int_0^{\pi}\frac{(\pi-x)\tan x}{\sec x+\tan x}dx \quad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2\text{I} = \int_0^{\pi}\frac{x\tan x}{\sec x+\tan x}dx + \int_0^{\pi}\frac{(\pi-x)\tan x}{\sec x+\tan x}dx$$

$$\Rightarrow \quad 2\text{I} = \pi\int_0^{\pi}\frac{\tan x}{\sec x+\tan x}dx$$

$$\Rightarrow \quad \text{I} = \frac{\pi}{2}\int_0^{\pi}\frac{\tan x}{\sec x+\tan x}\times\frac{\sec x-\tan x}{\sec x-\tan x}dx$$

$$\Rightarrow \quad \text{I} = \frac{\pi}{2}\int_0^{\pi}\frac{(\sec x\tan x-\tan^2 x)}{\sec^2 x-\tan^2 x}dx$$

$$\Rightarrow \quad \text{I} = \frac{\pi}{2}\int_0^{\pi}(\sec x\tan x-\sec^2 x+1)\,dx$$

$$\Rightarrow \quad \text{I} = \frac{\pi}{2}\left[\sec x-\tan x+x\right]_0^{\pi}$$

$$\Rightarrow \quad \text{I} = \frac{\pi}{2}[(\sec\pi-\tan\pi+\pi) - (\sec 0-\tan 0+0]$$

$$\Rightarrow \quad \text{I} = \frac{\pi}{2}[(-1+\pi)-(1)]$$

$$\therefore \quad \text{I} = \frac{\pi(\pi-2)}{2}$$

Ans.

OR

Evaluate : $\int_1^4\{|x-1|+|x-2|+|x-4|\}\,dx$

Solution : Let $\text{I} = \int_1^4\{|x-1|+|x-2|+|x-4|\}\,dx$

$$\Rightarrow \quad \text{I} = \int_1^4|x-1|\,dx + \int_1^4|x-2|\,dx + \int_1^4|x-4|\,dx$$

$$\Rightarrow \quad \text{I} = \int_1^4|x-1|\,dx + \int_1^2|x-2|\,dx + \int_2^4|x-2|\,dx + \int_1^4|x-4|\,dx$$

$$\Rightarrow \quad I = \int_1^4 (x-1)\,dx - \int_1^2 (x-2)\,dx + \int_2^4 (x-2)dx - \int_1^4 (x-4)\,dx$$

$$\Rightarrow \quad I = \frac{1}{2}\left[(x-1)^2\right]_1^4 - \frac{1}{2}\left[(x-2)^2\right]_1^2 + \frac{1}{2}\left[(x-2)^2\right]_2^4 - \frac{1}{2}\left[(x-4)^2\right]_1^4$$

$$= \frac{1}{2}(9-0) - \frac{1}{2}(0-1) + \frac{1}{2}(4-0) - \frac{1}{2}(0-9)$$

$$\Rightarrow \quad I = \frac{9}{2} + \frac{1}{2} + \frac{4}{2} + \frac{9}{2}$$

$$\therefore \quad I = \frac{23}{2}$$ **Ans.**

18. Solve the differential equation

$(\tan^{-1} x - y)\,dx = (1 + x^2)\,dy.$ **[4]**

Solution : We have,

$$(\tan^{-1}x - y)\,dx = (1+x^2)\,dy$$

i.e.,
$$\frac{dy}{dx} = \frac{\tan^{-1}x - y}{1+x^2}$$

$$\Rightarrow \quad \frac{dy}{dx} + \left(\frac{1}{1+x^2}\right)y = \frac{\tan^{-1}x}{1+x^2}$$

which is a linear differential equation of the form,

$$\frac{dy}{dx} + Py = Q$$

where, $P = \frac{1}{1+x^2}$ and $Q = \frac{\tan^{-1}x}{1+x^2}$

Now, $\text{I. F.} = e^{\int P dx} = e^{\int \frac{1}{1+x^2}dx} = e^{\tan^{-1}x}$

Then, required solution is :

$$ye^{\tan^{-1}x} = \int e^{\tan^{-1}x}\frac{\tan^{-1}x}{1+x^2}dx + C$$

Putting, $\tan^{-1}x = t$

$$\Rightarrow \quad \frac{1}{1+x^2}dx = dt$$

$$\Rightarrow \quad ye^t = \int e^t\, t dt + C$$

$$\Rightarrow \quad ye^t = t\int e^t dt - \int\left(\frac{d}{dt}(t)\int e^t\, dt\right)dt + C$$

$$\Rightarrow \quad ye^t = te^t - e^t + C$$

$$\Rightarrow \quad ye^t = (t-1)e^t + C$$

$$\Rightarrow \quad ye^{\tan^{-1}x} = (\tan^{-1}x - 1)\,e^{\tan^{-1}x} + C$$

$[\because \ t = \tan^{-1}x]$

$$\therefore \quad y = \tan^{-1}x - 1 + Ce^{-\tan^{-1}x}$$

which is the required solution. **Ans.**

19. Show that the points A, B, C with position vectors $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} - 3\hat{j} - 5\hat{k}$ and $3\hat{i} - 4\hat{j} - 4\hat{k}$ respectively, are the vertices of a right-angled triangle. Hence, find the area of the triangle. **[4]**

Solution : Given, the position vectors of the points A, B and C are $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} - 3\hat{j} - 5\hat{k}$ and $3\hat{i} - 4\hat{j} - 4\hat{k}$, respectively.

Then, $\vec{OA} = 2\hat{i} - \hat{j} + \hat{k}$,

$\vec{OB} = \hat{i} - 3\hat{j} - 5\hat{k}$

and $\vec{OC} = 3\hat{i} - 4\hat{j} - 4\hat{k}$

Now, $\vec{AB} = \vec{OB} - \vec{OA}$

$$= (\hat{i} - 3\hat{j} - 5\hat{k}) - (2\hat{i} - \hat{j} + \hat{k})$$

$$= -\hat{i} - 2\hat{j} - 6\hat{k}$$

$$\vec{BC} = \vec{OC} - \vec{OB}$$

$$= (3\hat{i} - 4\hat{j} - 4\hat{k}) - (\hat{i} - 3\hat{j} - 5\hat{k})$$

$$= 2\hat{i} - \hat{j} + \hat{k}$$

and $\vec{CA} = \vec{OA} - \vec{OC}$

$$= (2\hat{i} - \hat{j} + \hat{k}) - (3\hat{i} - 4\hat{j} - 4\hat{k})$$

$$= -\hat{i} + 3\hat{j} + 5\hat{k}$$

$$|\vec{AB}|^2 = (-1)^2 + (-2)^2 + (-6)^2 = 1 + 4 + 36 = 41$$

$$|\vec{BC}|^2 = (2)^2 + (-1)^2 + (1)^2 = 4 + 1 + 1 = 6$$

and $|\vec{CA}|^2 = (-1)^2 + (3)^2 + (5)^2 = 1 + 9 + 25 = 35$

$$\therefore \quad |\vec{AB}|^2 = |\vec{BC}|^2 + |\vec{CA}|^2$$

Hence, A, B, C are the vertices of a right angled triangle.

Now, area of $\Delta ABC = \frac{1}{2}\left|\vec{BC} \times \vec{CA}\right|$

$$\therefore \quad \vec{BC} \times \vec{CA} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ -1 & 3 & 5 \end{vmatrix}$$

$$= -8\hat{i} - 11\hat{j} + 5\hat{k}$$

$$|\vec{BC}\times\vec{CA}| = \sqrt{(-8)^2+(-11)^2+(5)^2}$$

$$= \sqrt{64+121+25} = \sqrt{210}$$

Hence, Area of $\Delta ABC = \frac{\sqrt{210}}{2}$ square units. **Ans.**

20. Find the value of λ, if four points with position vectors $3\hat{i}+6\hat{j}+9\hat{k}$, $\hat{i}+2\hat{j}+3\hat{k}$, $2\hat{i}+3\hat{j}+\hat{k}$ and $4\hat{i}+6\hat{j}+\lambda\hat{k}$ are coplanar. [4]

Solution : Let, the four points be A, B, C and D with position vectors $3\hat{i}+6\hat{j}+9\hat{k}$, $\hat{i}+2\hat{j}+3\hat{k}$, $2\hat{i}+3\hat{j}+\hat{k}$ and $4\hat{i}+6\hat{j}+\lambda\hat{k}$, respectively.

Then, $\vec{OA} = 3\hat{i}+6\hat{j}+9\hat{k}$

$\vec{OB} = \hat{i}+2\hat{j}+3\hat{k}$

$\vec{OC} = 2\hat{i}+3\hat{j}+\hat{k}$

and $\vec{OD} = 4\hat{i}+6\hat{j}+\lambda\hat{k}$

Then, $\vec{AB} = \vec{OB}-\vec{OA}$

$= (\hat{i}+2\hat{j}+3\hat{k})-(3\hat{i}+6\hat{j}+9\hat{k})$

$= -2\hat{i}-4\hat{j}-6\hat{k}$

$\vec{AC} = \vec{OC}-\vec{OA}$

$= (2\hat{i}+3\hat{j}+\hat{k})-(3\hat{i}+6\hat{j}+9\hat{k})$

$= -\hat{i}-3\hat{j}-8\hat{k}$

and $\vec{AD} = \vec{OD}-\vec{OA}$

$= (4\hat{i}+6\hat{j}+\lambda\hat{k})-(3\hat{i}+6\hat{j}+9\hat{k})$

$= \hat{i}+(\lambda-9)\hat{k}$

Since, the points are coplanar, then

$$[\vec{AB}, \vec{AC}, \vec{AD}] = 0$$

i.e.
$$\begin{bmatrix} -2 & -4 & -6 \\ -1 & -3 & -8 \\ 1 & 0 & \lambda-9 \end{bmatrix} = 0$$

$\Rightarrow -2(-3\lambda+27)+4(-\lambda+9+8)-6(3) = 0$

$\Rightarrow 6\lambda-54-4\lambda+68-18 = 0$

$\Rightarrow 2\lambda-4 = 0$

$\Rightarrow 2\lambda = 4$

$\therefore \lambda = 2$ **Ans.**

21. There are 4 cards numbered 1, 3, 5 and 7, one number on one card. Two cards are drawn at random without replacement. Let X denote the sum of the numbers on the two drawn cards. Find the mean and variance of X. [4]

Solution : Given, X denote the sum of the numbers on the two drawn cards.
Then, X can take values 4, 6, 8, 10, 12
and sample space (S) = {(1,3), (1,5), (1,7), (3,1), (3,5), (3, 7), (5, 1), (5, 3), (5, 7), (7, 1), (7, 3), (7, 5)}

So, the probability distribution of X is as given below :

X	4	6	8	10	12
P(X)	$\frac{2}{12}$	$\frac{2}{12}$	$\frac{4}{12}$	$\frac{2}{12}$	$\frac{2}{12}$

Computation of Mean and Variance :

x_i	$P(X=x_i) = p_i$	p_ix_i	$p_ix_i^2$
4	$\frac{2}{12}$	$\frac{8}{12}$	$\frac{32}{12}$
6	$\frac{2}{12}$	$\frac{12}{12}$	$\frac{72}{12}$
8	$\frac{4}{12}$	$\frac{32}{12}$	$\frac{256}{12}$
10	$\frac{2}{12}$	$\frac{20}{12}$	$\frac{200}{12}$
12	$\frac{2}{12}$	$\frac{24}{12}$	$\frac{288}{12}$
		$\Sigma p_ix_i = \frac{96}{12}$	$\Sigma p_ix_i^2 = \frac{848}{12}$

We have, $\Sigma p_ix_i = \frac{96}{12} = 8$

$\therefore$ Mean, $\bar{X} = \Sigma p_ix_i = 8$

Now, $\text{Var}(X) = \Sigma p_ix_i^2 - (\Sigma p_ix_i)^2$

$= \frac{848}{12} - (8)^2$

$= \frac{848}{12} - 64$

$= \frac{80}{12} = \frac{20}{3}$

Hence, Mean = 8 and Variance = $\frac{20}{3}$ **Ans.**

22. Of the students in a school, it is known that 30% have 100% attendance and 70% students are irregular. Previous year results report that

70% of all students who have 100% attendance attain A grade and 10% irregular students attain A grade in their annual examination. At the end of the year, one student is chosen at random from the school and he was found to have an A grade. What is the probability that the student has 100% attendance ? Is regularity required only in school ? Justify your answer. [4]

Solution : Consider the following events :

A : the student has grade A.

E_1 : the student has 100% attendance.

E_2 : the student is irregular.

Then, probability of the students having 100% attendance:

$$P(E_1) = 30\% = 0.3$$

Similarly, $P(E_2) = 70\% = 0.7$

Now, by previous year report, the probability of the students having grade A who have 100% attendance :

$$P(A/E_1) = 70\% = 0.7$$

and the probability of the students having grade A who are irregular :

$$P(A/E_2) = 10\% = 0.1$$

Then, the probability of the student having 100% attendance who already has attain A grade

$$= P(E_1/A)$$

By Bayes' theorem,

$$P(E_1/A) = \frac{P(A/E_1)\,P(E_1)}{P(A/E_1)\,P(E_1) + P(A/E_2)\,P(E_2)}$$

$$= \frac{0.7 \times 0.3}{0.7 \times 0.3 + 0.1 \times 0.7}$$

$$= \frac{21}{21+7}$$

$$= \frac{21}{28} = \frac{3}{4}$$

$= 45\%$

No, regularity is required in school as well as in life.

It helps to be disciplined in every aspect of life.

Or, when you work regularly, inspiration strikes regularly. **Ans.**

23. Maximise $Z = x + 2y$

subject to the constraints :

$$x + 2y \geq 100 \qquad 2x - y \leq 0$$

$$2x + y \leq 200 \qquad x, y \geq 0$$

Solve the above LPP graphically. [4]

Solution : Given,

Maximise $Z = x + 2y$

Subject to the constraints :

$$x + 2y \geq 100 \qquad 2x - y \leq 0$$

$$2x + y \leq 200 \qquad x, y \geq 0$$

Converting the inequations into equations we obtain the lines

$$x + 2y = 100,$$

$$2x - y = 0,$$

$$2x + y = 200.$$

Then, $x + 2y = 100$

x	0	100
y	50	0

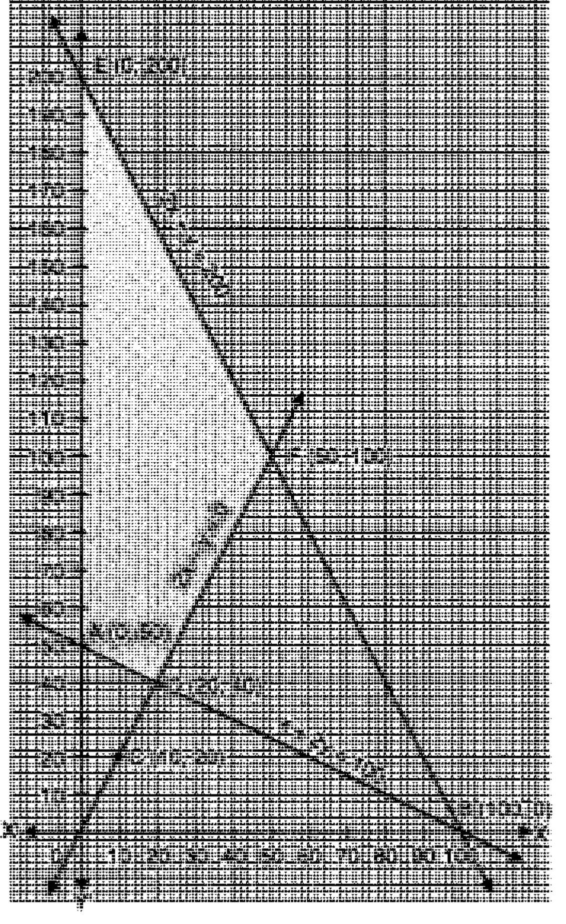
$2x - y = 0$

x	10	20
y	20	40

and $2x + y = 200$

x	0	100
y	200	0

Plotting these points on the graph, we get the shaded feasible region *i.e.*, ADFEA.



Corner points	Value of $Z = x + 2y$
A (0, 50)	$(0) + 2(50) = 100$
D (20, 40)	$20 + 2(40) = 100$
F (50, 100)	$50 + 2(100) = 250$
E (0, 200)	$0 + 2(200) = 400$

Clearly, the maximum value of Z is 400 at (0, 200).

Ans.

SECTION — D

24. Determine the product:

$$\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$$ **and use it to solve the system of equations $x - y + z = 4, x - 2y - 2z = 9, 2x + y + 3z = 1$.** **[6]**

Solution :

Let $A = \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}$

and $B = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$

Then, $AB = \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$

$$= \begin{bmatrix} -4+4+8 & 4-8+4 & -4-8+12 \\ -7+1+6 & 7-2+3 & -7-2+9 \\ 5-3-2 & -5+6-1 & 5+6-3 \end{bmatrix}$$

$\Rightarrow \quad AB = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = 8I$

$\Rightarrow \quad AB = 8I$

$\Rightarrow ABB^{-1} = 8IB^{-1}$

$\Rightarrow \quad \frac{1}{8}A = B^{-1}$ ...(i)

Now, consider the given system of equations :

$$x - y + z = 4$$
$$x - 2y - 2z = 9$$
$$2x + y + 3z = 1$$

which can be expressed as $BX = C$,

where, $B = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $C = \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$

$X = B^{-1}C$

$X = \frac{1}{8}AC$ [using (i)]

$\therefore \quad X = \frac{1}{8}\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$

$$= \frac{1}{8}\begin{bmatrix} -16+36+4 \\ -28+9+3 \\ 20-27-1 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} 24 \\ -16 \\ -8 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$$

$\therefore \quad x = 3, \quad y = -2, \quad z = -1$ **Ans.**

25. Consider $f : R - \left\{-\frac{4}{3}\right\} \to R - \left\{\frac{4}{3}\right\}$ given by $f(x) = \frac{4x+3}{3x+4}$. Show that f is bijective. Find the inverse of f and hence find $f^{-1}(0)$ and x such that $f^{-1}(x) = 2$. **[6]**

Solution : For one-one :

Let $x, y \in R - \left\{-\frac{4}{3}\right\}$

and $f(x) = f(y)$

$$\frac{4x+3}{3x+4} = \frac{4y+3}{3y+4}$$

$\Rightarrow \quad (4x+3)(3y+4) = (3x+4)(4y+3)$

$\Rightarrow \quad 12xy + 16x + 9y + 12 = 12xy + 9x + 16y + 12$

$\Rightarrow \quad 16x + 9y - 9x - 16y = 0$

$\Rightarrow \quad 7x - 7y = 0$

$\Rightarrow \quad 7(x - y) = 0$

$\Rightarrow \quad x - y = 0$

$\Rightarrow \quad x = y$

Hence, f is one-one function.

For onto :

Let $y \in R - \left\{\frac{4}{3}\right\}$, then

$$f(x) = y$$

$$\Rightarrow \quad \frac{4x+3}{3x+4} = y$$

$$\Rightarrow \quad 4x+3 = 3xy+4y$$

$$\Rightarrow \quad (4-3y)x = 4y-3$$

$$\Rightarrow \quad x = \frac{4y-3}{4-3y} \quad \text{...(i)}$$

As $y \in R - \left\{\frac{4}{3}\right\}, x = \frac{4y-3}{4-3y} \in R$

Also $\frac{4y-3}{4-3y} \neq -\frac{4}{3}$

Since if, $\frac{4y-3}{4-3y} = -\frac{4}{3}$

$$\Rightarrow \quad 12y-9 = -16+12y$$

$$\Rightarrow \quad 9 = 16 \text{ which is not possible.}$$

Thus, $x = \frac{4y-3}{4-3y} \in R - \left\{-\frac{4}{3}\right\}$ such that

$$f(x) = f\left(\frac{4y-3}{4-3y}\right)$$

$$\Rightarrow \quad f(x) = \frac{4\left(\frac{4y-3}{4-3y}\right)+3}{3\left(\frac{4y-3}{4-3y}\right)+4}$$

$$= \frac{16y-12+12-9y}{12y-9+16-12y}$$

$$= \frac{7y}{7}$$

$$\Rightarrow \quad f(x) = y$$

Hence, every element $y \in R - \left\{-\frac{4}{3}\right\}$ has its pre image $x \in R - \left\{-\frac{4}{3}\right\}$

Hence, f is onto.

$\Rightarrow f$ is one-one and onto, so f is invertible.

Now, $f(x) = y$

$$\Rightarrow \quad f^{-1}(y) = x$$

$$\Rightarrow \quad f^{-1}(y) = \frac{4y-3}{4-3y} \quad \text{[from eq. (i)]}$$

$$f^{-1}(0) = \frac{4\times 0-3}{4-3\times 0}$$

$$= \frac{-3}{4}$$

Also given, $f^{-1}(x) = 2$

$$\Rightarrow \quad \frac{4x-3}{4-3x} = 2$$

$$\Rightarrow \quad 4x-3 = 8-6x$$

$$\Rightarrow \quad 10x = 11$$

$$\Rightarrow \quad x = \frac{11}{10}$$ **Ans.**

OR

Let A = Q × Q and let * be a binary operation on A defined by (a, b) * (c, d) = (ac, b + ad) for (a, b), (c, d) ∈A. Determine, whether * is commutative and associative. Then, with respect to * on A.**

(i) Find the identity element in A
(ii) Find the invertible elements of A.

26. Show that the surface area of a closed cuboid with square base and given volume is minimum, when it is a cube. **[6]**

Solution : Let the length and breadth of the cuboid of square base be x and height be y.

Then, volume of the cuboid (V) = x^2y

$$\Rightarrow \quad y = \frac{V}{x^2} \quad \text{...(i)}$$

Now, surface area of cuboid,

$$S = 2(x^2+yx+yx)$$

$$\Rightarrow \quad S = 2(x^2+2xy)$$

$$\Rightarrow \quad S = 2\left(x^2+2x\left(\frac{V}{x^2}\right)\right) \text{[using (i)]}$$

$$\Rightarrow \quad S = 2\left(x^2+\frac{2V}{x}\right)$$

Now, differentiating S w.r.t. x, we get

$$\frac{dS}{dx} = 2\left(2x-\frac{2V}{x^2}\right)$$

For maxima and minima, we have

$$\frac{dS}{dx} = 0$$

$$\Rightarrow \quad 2\left(2x-\frac{2V}{x^2}\right) = 0$$

$$\Rightarrow \quad 4\left(x-\frac{V}{x^2}\right) = 0$$

$$x = \frac{V}{x^2} \Rightarrow x = \sqrt[3]{V}$$

Now, $\frac{dS}{dx} = 2\left(2x-\frac{2V}{x^2}\right) = 4\left(x-\frac{V}{x^2}\right)$

Again differentiating w.r.t. x, we get

$$\frac{d^2S}{dx^2} = 4\left(1+\frac{2V}{x^3}\right)$$

**Anwer is not given due to the change in present syllabus

$$\Rightarrow \quad \left[\frac{d^2S}{dx^2}\right]_{x=\sqrt[3]{V}} = 4\left(1+\frac{2V}{V}\right) = 12 > 0$$

Thus, S is minimum when $x = \sqrt[3]{V}$.

Putting $x = \sqrt[3]{V}$ in (i), we get

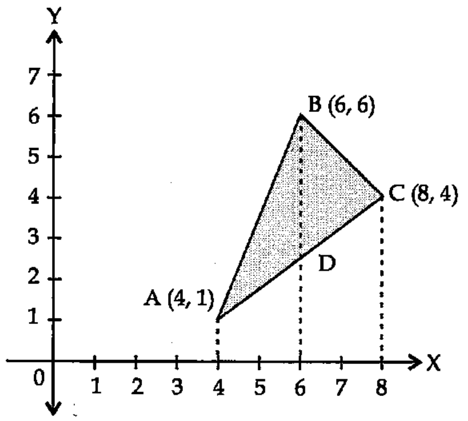
$$y = \frac{V}{x^2} = \frac{x^3}{x^2} = x$$

$$\Rightarrow \quad y = x$$

Hence, it is a cube since the length, breadth and height of a cube are equal. **Hence Proved.**

27. Using the method of integration, find the area of the triangle ABC, coordinates of whose vertices are A (4, 1), B (6, 6) and C (8, 4). [6]

Solution : We have, A(4, 1), B (6, 6) and C(8, 4) as the vertices of a triangle ABC.



Then, equation of AB is

$$\Rightarrow \quad \frac{x-4}{6-4} = \frac{y-1}{6-1}$$

$$\Rightarrow \quad \frac{x-4}{2} = \frac{y-1}{5}$$

$$\Rightarrow \quad 5x - 20 = 2y - 2$$

$$\Rightarrow \quad 5x - 2y - 18 = 0$$

$$\Rightarrow \quad y = \frac{5x-18}{2} \qquad \text{...(i)}$$

Equation of BC is,

$$\frac{x-6}{8-6} = \frac{y-1}{6-1}$$

$$\Rightarrow \quad \frac{x-6}{2} = \frac{y-6}{-2}$$

$$\Rightarrow \quad -2x + 12 = 2y - 12$$

$$\Rightarrow \quad -2x - 2y + 24 = 0$$

$$\Rightarrow \quad x + y - 12 = 0$$

$$\Rightarrow \quad y = 12 - x \qquad \text{...(ii)}$$

and equation of CA is

$$\frac{x-8}{4-8} = \frac{y-4}{1-4}$$

$$\Rightarrow \quad \frac{x-8}{-4} = \frac{y-4}{-3}$$

$$\Rightarrow \quad -3x + 24 = -4y + 16$$

$$-3x + 4y + 8 = 0 \qquad \text{...(iii)}$$

$$\Rightarrow \quad y = \frac{3x-8}{4}$$

Clearly, Area of ΔABC = Area of trapezium ABDE + Area of trapezium BDFC – Area of trapezium ACFE

Hence, area of ΔABC

$$= \int_4^6 \left(\frac{5x-18}{2}\right)dx + \int_6^8 (12-x)dx - \int_4^8 \left(\frac{3x-8}{4}\right)dx$$

$$= \left[\frac{5x^2}{4} - 9x\right]_4^6 + \left[12x - \frac{x^2}{2}\right]_6^8 - \left[\frac{3x^2}{8} - 2x\right]_4^8$$

$$= \left(\frac{5(6)^2}{4} - 9(6)\right) - \left(\frac{5(4)^2}{4} - 9(4)\right) + \left(12(8) - \frac{(8)^2}{2}\right) - \left(12(6) - \frac{(6)^2}{2}\right) - \left(\frac{3(8)^2}{8} - 2(8)\right) + \left(\frac{3(4)^2}{8} - 2(4)\right)$$

$$= (45 - 54) - (20 - 36) + (96 - 32) - (72 - 18) - (24 - 16) + (6 - 8)$$

$$= -9 + 16 + 64 - 54 - 8 - 2$$

$= 7$ sq. units. **Ans.**

OR

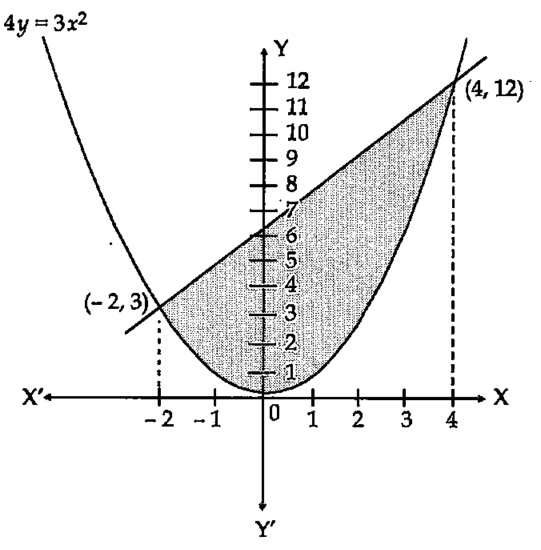
Find the area enclosed between the parabola $4y = 3x^2$ and the straight line $3x - 2y + 12 = 0$.

Solution : Given, the equation $4y = 3x^2$ and $3x - 2y + 12 = 0$.

i.e.,
$$y = \frac{3x^2}{4} \qquad \text{...(i)}$$

and
$$y = \frac{3x}{2} + 6 \qquad \text{...(ii)}$$

Solving these equations, we get



$$\frac{3x^2}{4} = \frac{3x}{2}+6$$

$\Rightarrow \quad 3x^2-6x-24 = 0$

$\Rightarrow \quad x^2-2x-8 = 0$

$\Rightarrow \quad x^2-4x+2x-8 = 0$

$\Rightarrow \quad x(x-4)+2(x-4) = 0$

$\Rightarrow \quad (x+2)(x-4) = 0$

$\Rightarrow \quad x = -2, 4$

At $\quad x = -2, \ y = 3$

and at $x = 4, \ y = 12$

The points of intersection are (4, 12) and (– 2, 3).

$$\therefore \text{Required area} = \int_{-2}^{4}\left(\frac{3x}{2}+6\right)dx - \int_{-2}^{4}\frac{3x^2}{4}\,dx$$

$$= \left[\frac{3x^2}{4}+6x\right]_{-2}^{4} - \frac{[x^3]_{-2}^{4}}{4}$$

$$= (12+24-3+12) - \frac{(64+8)}{4}$$

$$= 48-3-18$$

$= 27$ sq. units. **Ans.**

28. Find the particular solution of the differential equation $(x-y)\dfrac{dy}{dx} = (x+2y)$, given that $y = 0$ when $x = 1$. [6]

Solution : We have,

$$(x-y)\frac{dy}{dx} = (x+2y)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x+2y}{x-y} \quad \text{...(i)}$$

Putting $\quad y = vx$

and $\quad \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{x+2vx}{x-vx}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{1+2v}{1-v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{1+2v}{1-v} - v$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{1+2v-v+v^2}{1-v}$$

$$\frac{1-v}{1+v+v^2}\,dv = \frac{dx}{x}$$

On integrating both sides, we get

$$\int\frac{1-v}{1+v+v^2}\,dv = \int\frac{dx}{x}$$

$$\Rightarrow \quad \frac{1}{2}\int\frac{3-1-2v}{1+v+v^2}\,dv = \int\frac{dx}{x}$$

$$\Rightarrow \quad \frac{3}{2}\int\frac{1}{1+v+v^2}\,dv - \frac{1}{2}\int\frac{1+2v}{1+v+v^2}\,dv = \int\frac{dx}{x}$$

$$\Rightarrow \frac{3}{2}\int\frac{dv}{\left(v+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} - \frac{1}{2}\int\frac{1+2v}{1+v+v^2}\,dv = \int\frac{dx}{x}$$

$$\Rightarrow \frac{3}{2}\cdot\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2v+1}{\sqrt{3}}\right) - \frac{1}{2}\log|1+v+v^2| = \log|x| + C$$

$$\Rightarrow \sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right) - \frac{1}{2}\log|x^2+xy+y^2| = C \quad \text{...(ii)}$$

Now, given $y = 0$, when $x = 1$

So,

$$\sqrt{3}\tan^{-1}\left(\frac{2(0)+1}{\sqrt{3}}\right) - \frac{1}{2}\log|1+0+0| = C$$

$$\Rightarrow \quad C = \sqrt{3}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\sqrt{3}\pi}{6}$$

Putting the value of C in (ii), we get

$$\sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right) - \frac{1}{2}\log|x^2+xy+y^2| = \frac{\sqrt{3}\pi}{6}$$

which is the required solution. **Ans.**

29. Find the coordinates of the point where the line through the points (3, – 4, – 5) and (2, – 3, 1), crosses the plane determined by the points (1, 2, 3), (4, 2, –3) and (0, 4, 3). [6]

Solution : Equation of the plane determined by the points (1, 2, 3), (4, 2, – 3) and (0, 4, 3) is

$$\begin{bmatrix} x-1 & y-2 & z-3 \\ 4-1 & 2-2 & -3-3 \\ 0-1 & 4-2 & 3-3 \end{bmatrix} = 0$$

$$\begin{bmatrix} x-1 & y-2 & z-3 \\ 3 & 0 & -6 \\ -1 & 2 & 0 \end{bmatrix} = 0$$

$$(x-1)(0+12) - (y-2)(0-6) + (z-3)(6-0) = 0$$

$$\Rightarrow \quad 12x-12+6y-12+6z-18 = 0$$

$$\Rightarrow \quad 12x+6y+6z-42 = 0$$

$\Rightarrow \quad 2x + y + z - 7 = 0$

$\Rightarrow \quad 2x + y + z = 7$ ...(i)

Now, equation of line through (3, – 4, – 5) and (2, – 3, 1) is

$$\frac{x-3}{2-3} = \frac{y+4}{-3+4} = \frac{z+5}{1+5} = \lambda$$

i.e., $$\frac{x-3}{-1} = \frac{y+4}{1} = \frac{z+5}{6} = \lambda$$

∴ Coordinates of any point on line is :

P $(-\lambda + 3, \lambda - 4, 6\lambda - 5)$

Now, the point P crosses the plane.

∴ It satisfies the equation (i) of plane.

$$2(-\lambda + 3) + (\lambda - 4) + (6\lambda - 5) = 7$$

$\Rightarrow \quad -2\lambda + 6 + \lambda - 4 + 6\lambda - 5 = 7$

$\Rightarrow \quad 5\lambda - 3 = 7$

$\Rightarrow \quad 5\lambda = 10$

$\therefore \quad \lambda = 2$

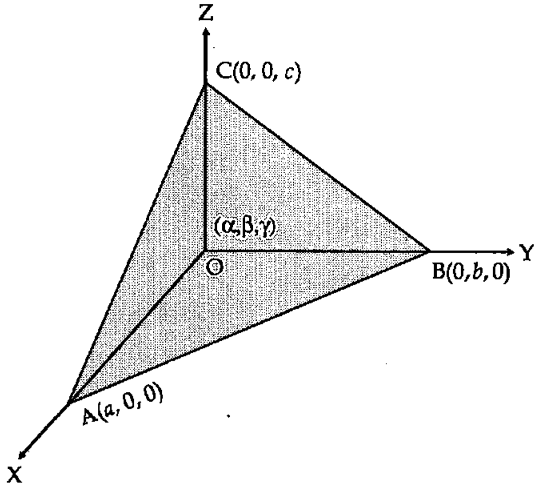
Hence, the point of intersection is (1, – 2, 7). **Ans.**

OR

A variable plane which remains at a constant distance 3*p* from the origin cuts the coordinate axes at A, B, C. Show that the locus of the centroid of triangle ABC is

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{p^2}$$

Solution : Let the coordinates of A, B, C are $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$



∴ The equation of plane is :

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \quad \text{...(i)}$$

Since, the distance of plane is equal to 3*p* from the origin.

Then, $$3p = \frac{\left|\frac{0}{a} + \frac{0}{b} + \frac{0}{c} - 1\right|}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}}$$

$$\Rightarrow \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{9p^2} \quad \text{...(ii)}$$

Let the centroid of ΔABC be (x, y, z)

$$= \left(\frac{a+0+0}{3}, \frac{0+b+0}{3}, \frac{0+0+c}{3}\right)$$

$$= \left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right)$$

$\Rightarrow \quad a = 3x, \quad b = 3y, \quad c = 3z$

Putting the value of *a*, *b* and *c* in (ii), we get

$$\frac{1}{(3x)^2} + \frac{1}{(3y)^2} + \frac{1}{(3z)^2} = \frac{1}{9p^2}$$

$$\frac{1}{9x^2} + \frac{1}{9y^2} + \frac{1}{9z^2} = \frac{1}{9p^2}$$

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{p^2}$$

Hence, the required locus of the centroid is :

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{p^2}.$$

Hence Proved.

Mathematics 2017 (Outside Delhi) SET II

Time allowed : 3 hours **Maximum marks : 100**

Note : Except for the following questions, all the remaining questions have been asked in previous set.

SECTION — B

12. The length x, of a rectangle is decreasing at the rate of 5 cm/minute and the width y, is increasing at the rate of 4 cm/minute. When $x = 8$ cm and $y = 6$ cm, find the rate of change of the area of the rectangle. [2]

Solution : We have,

$$\frac{dx}{dt} = -5 \text{ cm/min} \quad ...(i)$$

and $$\frac{dy}{dt} = 4 \text{ cm/min} \quad ...(ii)$$

Now, area of the rectangle, $A = xy$

$$\Rightarrow \frac{dA}{dt} = x\frac{dy}{dt} + y\frac{dx}{dt}$$

$$\frac{dA}{dt} = x(4) + y(-5)$$

[using (i) and (ii)]

$$\therefore \frac{dA}{dt} = 4x - 5y$$

When $x = 8$ cm and $y = 6$ cm,

$$\therefore \left[\frac{dA}{dt}\right]_{\text{at } x=8,\, y=6} = 4(8) - 5(6)$$

$$= 32 - 30 = 2 \text{ cm}^2/\text{min}$$

Hence, the rate of change of the area of the rectangle is2 cm^2/min. **Ans.**

SECTION — C

20. Find : $\int \frac{\sin\theta\, d\theta}{(4+\cos^2\theta)(2-\sin^2\theta)}$. [4]

Solution : Let $I = \int \frac{\sin\theta}{(4+\cos^2\theta)(2-\sin^2\theta)} d\theta$

$$= \int \frac{\sin\theta}{(4+\cos^2\theta)[(2-(1-\cos^2\theta)]} d\theta$$

$$= \int \frac{\sin\theta}{(4+\cos^2\theta)(1+\cos^2\theta)} d\theta$$

Putting $\cos\theta = t$

$\Rightarrow -\sin\theta\, d\theta = dt$

$$\therefore \quad I = \int \frac{-dt}{(4+t^2)(1+t^2)}$$

Let $\frac{-1}{(4+t^2)(1+t^2)} = \frac{A}{4+t^2} + \frac{B}{1+t^2}$

$$-1 = A(1+t^2) + B(4+t^2)$$

Putting $t = 0$, we get

$$-1 = A + 4B \quad ...(i)$$

Putting $t = 1$, we get

$$-1 = 2A + 5B \quad ...(ii)$$

Solving (i) and (ii), we get

$$A = \frac{1}{3} \text{ and } B = \frac{-1}{3}$$

$$\therefore \quad I = \frac{1}{3}\int\frac{1}{4+t^2}dt - \frac{1}{3}\int\frac{1}{1+t^2}dt$$

$$\Rightarrow \quad I = \frac{1}{3}\times\frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right) - \frac{1}{3}\times\tan^{-1}t + C$$

$$\Rightarrow \quad I = \frac{1}{6}\tan^{-1}\left(\frac{t}{2}\right) - \frac{1}{3}\times\tan^{-1}t + C$$

$$\Rightarrow \quad I = \frac{1}{6}\tan^{-1}\left(\frac{\cos\theta}{2}\right) - \frac{1}{3}\times\tan^{-1}(\cos\theta) + C$$

Ans.

21. Solve the following linear programming problem graphically :

Maximise $Z = 34x + 45y$

under the following constraints

$$x + y \le 300$$
$$2x + 3y \le 70$$
$$x \ge 0, y \ge 0 \quad [4]$$

Solution : We have,

Maximise $Z = 34x + 45y$

Subject to the constraints :

$$x + y \le 300$$
$$2x + 3y \le 70$$
$$x \ge 0, y \ge 0$$

Converting the given inequalities into equations, we obtain the following equations :

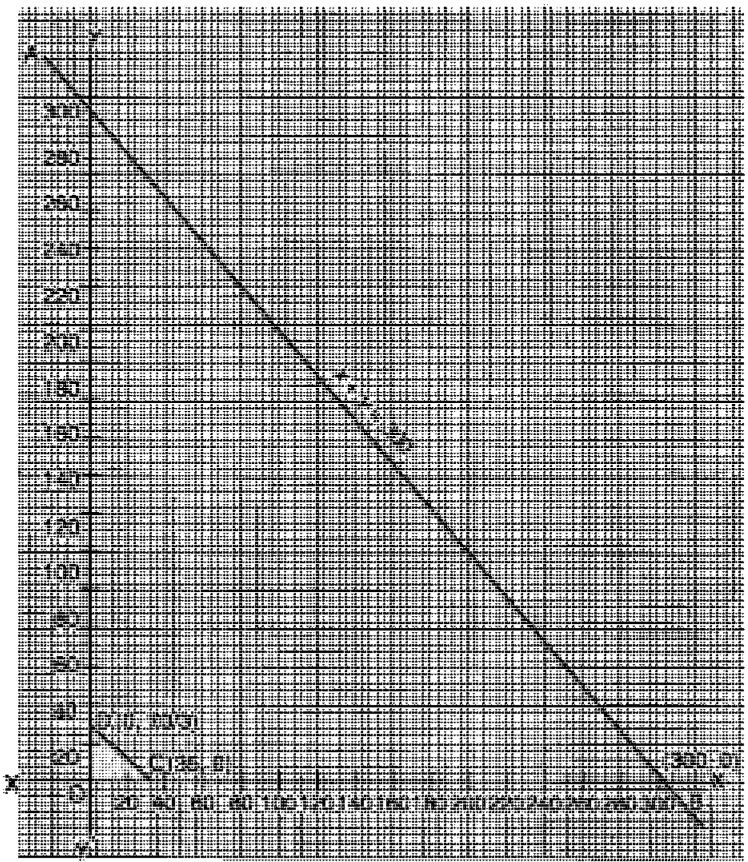
$$x + y = 300$$
$$2x + 3y = 70$$

Then, $x + y = 300$

x	0	300
y	300	0

and $2x + 3y = 70$

x	0	35
y	70/3	0



Plotting these points on the graph, we get the shaded feasible region *i.e.*, OCDO.

Corner point	Value of $Z = 34x + 45y$
O (0, 0)	$34(0) + 45(0) = 0$
C (35, 0)	$34(35) + 45(0) = 1190$
D (0, 70/3)	$34(0) + 45\,(70/3) = 1050$

Clearly, the maximum value of Z is 1190 at (35, 0). **Ans.**

22. Find the value of x such that the points A (3, 2, 1), B (4, x, 5), C(4, 2, – 2) and D (6, 5, – 1) are coplanar. [4]

Solution : Given, the points A(3, 2, 1), B (4, x, 5), C (4, 2, – 2) and D (6, 5, – 1).

$$\therefore \quad \vec{AB} = 4\hat{i} + x\hat{j} + 5\hat{k} - (3\hat{i} + 2\hat{j} + \hat{k}) = \hat{i} + (x-2)\hat{j} + 4\hat{k}$$

and
$$\vec{AC} = 4\hat{i} + x\hat{j} - 2\hat{k} - (3\hat{i} + 2\hat{j} + \hat{k}) = \hat{i} - 3\hat{k}$$

$$\vec{AD} = 6\hat{i} + 5\hat{j} - \hat{k} - (3\hat{i} + 2\hat{j} + \hat{k}) = 3\hat{i} + 3\hat{j} - 2\hat{k}$$

Since, the points are coplanar, then

$$\left[\vec{AB}, \vec{AC}, \vec{AD}\right] = 0$$

$$\begin{vmatrix} 1 & x-2 & 4 \\ 1 & 0 & -3 \\ 3 & 3 & -2 \end{vmatrix} = 0$$

$$\Rightarrow \quad 1\,(0 + 9) - (x-2)\,(-2 + 9) + 4\,(3 - 0) = 0$$

$$\Rightarrow \quad 9 - 7x + 14 + 12 = 0$$

$$\Rightarrow \quad 7x = 35$$

$$\therefore \quad x = 5 \text{ Ans.}$$

23. Find the general solution of the differential equation :

$$y\,dx - (x + 2y^2)\,dy = 0 \qquad [4]$$

Solution : We have,

$$ydx - (x + 2y^2)dy = 0$$

$$\Rightarrow \quad ydx = (x + 2y^2)dy$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{x + 2y^2}{y}$$

$$\Rightarrow \quad \frac{dx}{dy} + \left(\frac{-1}{y}\right)x = 2y$$

which is a linear differential equation of the form,

$$\frac{dx}{dy} + Px = Q$$

where
$$P = \frac{-1}{y}$$

and $\qquad Q = 2y$

Now, $\quad$ I. F. $= e^{\int P dy} = e^{\int \frac{-1}{y} dy} = e^{\log(1/y)} = \frac{1}{y}$

So, the required solution is

$$x \cdot \frac{1}{y} = \int 2y \cdot \frac{1}{y} dy + C$$

$\therefore \quad \frac{x}{y} = 2y + C$ is the required solution.

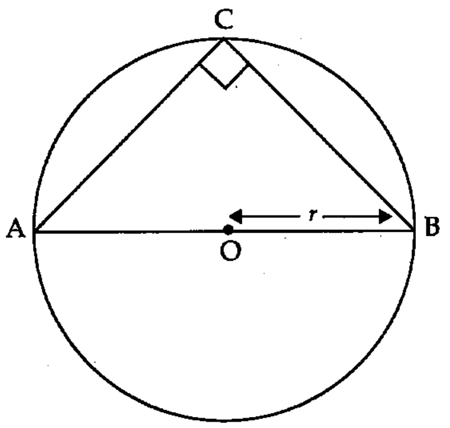
Ans.

SECTION — D

28. AB is the diameter of a circle and C is any point on the circle. Show that the area of triangle ABC is maximum, when it is an isosceles triangle.[6]

Solution : Let r be the radius of the circle then,

$$AB = 2r \quad (\because \text{AB is diameter})$$



Let, $\qquad BC = x$ units

We know that angle subtended by diameter in a circle is right angle

$\therefore \qquad \angle C = 90°$

Then, $\qquad AC = \sqrt{(AB)^2 - (BC)^2}$

$$AC = \sqrt{(2r)^2 - (x)^2} = \sqrt{4r^2 - x^2} \quad ...(i)$$

Now, area of ΔABC

$$A = \frac{1}{2}(AC)(BC)$$

$$A = \frac{1}{2}\sqrt{4r^2 - x^2}\,(x)$$

Differentiating A w.r.t. x, we get

$$\Rightarrow \quad \frac{dA}{dx} = \frac{1}{2}\left\{\sqrt{4r^2 - x^2} + x\frac{1}{2\sqrt{4r^2 - x^2}} \cdot \frac{d}{dx}(4r^2 - x^2)\right\}$$

$$\Rightarrow \quad \frac{dA}{dx} = \frac{1}{2}\left\{\sqrt{4r^2 - x^2} - \frac{x^2}{\sqrt{4r^2 - x^2}}\right\}$$

$$= \frac{1}{2}\left\{\frac{4r^2 - x^2 - x^2}{\sqrt{4r^2 - x^2}}\right\}$$

$$\Rightarrow \quad \frac{dA}{dx} = \frac{1}{2}\left\{\frac{4r^2 - 2x^2}{\sqrt{4r^2 - x^2}}\right\}$$

The critical numbers of x are given by $\frac{dA}{dx} = 0$

$$\Rightarrow \quad \frac{1}{2}\left\{\frac{4r^2 - 2x^2}{\sqrt{4r^2 - x^2}}\right\} = 0$$

$$\Rightarrow \quad 4r^2 - 2x^2 = 0$$

$$\Rightarrow \quad 4r^2 = 2x^2$$

$$\therefore \quad x = \sqrt{2}r$$

Now, $\qquad \frac{dA}{dx} = \frac{1}{2}\left\{\frac{4r^2 - 2x^2}{\sqrt{4r^2 - x^2}}\right\}$

Again, differentiating w.r.t. x, we get

$$\frac{d^2A}{dx^2} = \frac{1}{2}\left\{(-4x)\frac{1}{\sqrt{4r^2 - x^2}} + (4r^2 - 2x^2)\left(\frac{-1}{2}\right)(4r^2 - x^2)^{-3/2}\frac{d}{dx}(4r^2 - x^2)\right\}$$

$$\Rightarrow \quad \frac{d^2A}{dx^2} = \frac{1}{2}\left\{\frac{-4x}{\sqrt{4r^2 - x^2}} + \frac{x(4r^2 - 2x^2)}{(4r^2 - x^2)^{3/2}}\right\}$$

$$\Rightarrow \quad \left(\frac{d^2A}{dx^2}\right)_{x = \sqrt{2}r} = \frac{1}{2}\left\{\frac{-4(\sqrt{2}r)}{\sqrt{4r^2 - 2r^2}} + \frac{\sqrt{2}r(4r^2 - 4r^2)}{(4r^2 - 2r^2)^{3/2}}\right\}$$

$$= \frac{-2\sqrt{2}r}{\sqrt{2}r} = -2 < 0$$

Thus, A is maximum when $x = \sqrt{2}\,r$.

Putting $x = \sqrt{2}\,r$ in (i),

$$AC = \sqrt{4r^2 - (\sqrt{2}\,r)^2}$$

$$\therefore \quad AC = \sqrt{2}\,r$$

$$\therefore \quad BC = AC = \sqrt{2}\,r$$

Hence, A is maximum when the triangle is isosceles. **Hence Proved.**

29. If $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, find A^{-1}. Hence using A^{-1} solve the system of equations $2x - 3y + 5z = 11$, $3x + 2y - 4z = -5$, $x + y - 2z = -3$. [6]

Solution : We have,

$$A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$$

$$\therefore \quad |A| = \begin{vmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{vmatrix}$$

$$= 2(-4+4)+3(-6+4)+5(3-2)$$

$$= 2(0)+3(-2)+5(1)$$

$$= -1 \neq 0.$$

Hence, A is invertible and A^{-1} exists.

Let A_{ij} be the co-factors of elements a_{ij} in $A = [a_{ij}]$.

Then,

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 2 & -4 \\ 1 & -2 \end{vmatrix} = 0,$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 3 & -4 \\ 1 & -2 \end{vmatrix} = 2$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = 1,$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} -3 & 5 \\ 1 & -2 \end{vmatrix} = -1$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 2 & 5 \\ 1 & -2 \end{vmatrix} = -9,$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} = -5$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} -3 & 5 \\ 2 & -4 \end{vmatrix} = 2,$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 2 & 5 \\ 3 & -4 \end{vmatrix} = 23$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 2 & -3 \\ 3 & 2 \end{vmatrix} = 13$$

$$\therefore \quad \text{adj } A = \begin{bmatrix} 0 & 2 & 1 \\ -1 & -9 & -5 \\ 2 & 23 & 13 \end{bmatrix}^T = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{-1}\begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}$$

Now, the given system of equations is expressible as

$$\begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

or $\qquad AX = B$

where, $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$

Now, $\qquad AX = B$

$$\Rightarrow \quad A^{-1}AX = A^{-1}B$$

$$\therefore \quad X = A^{-1}B$$

$$\Rightarrow \quad X = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}\begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

$$\Rightarrow \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\therefore \quad x = 1,\ y = 2,\ z = 3.$ **Ans.**

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Mathematics 2017 (Outside Delhi) SET III

Time allowed : 3 hours **Maximum marks : 100**

Note : Except for the following questions, all the remaining questions have been asked in previous sets.

SECTION — B

12. The volume of a sphere is increasing at the rate of 8 cm³/s. Find the rate at which its surface area is increasing when the radius of the sphere is 12 cm. [2]

Solution : We have,

$$\text{Volume of sphere (V)} = \frac{4}{3}\pi r^3$$

where, r is the radius of sphere.

Now, differentiating V w.r.t. t, we get

$$\frac{dV}{dt} = \frac{4}{3}\pi 3r^2\frac{dr}{dt} = 8 \text{ cm}^3/\text{s}$$

$$\Rightarrow \quad 4\pi r^2\frac{dr}{dt} = 8 \text{ cm}^3/\text{s}$$

$$\Rightarrow \quad \frac{dr}{dt} = \frac{2}{\pi r^2} \text{ cm}^3/\text{s} \quad \text{....(i)}$$

and surface area of sphere (S) = $4\pi r^2$.

Then, differentiating S w.r.t. t, we get

$$\frac{dS}{dt} = 4\pi \cdot 2r\frac{dr}{dt}$$

$$\Rightarrow \quad \frac{dS}{dt} = 8\pi r \cdot \frac{2}{\pi r^2} \quad \text{[using (i)]}$$

$$\Rightarrow \quad \frac{dS}{dt} = \frac{16}{r} \text{ cm}^2/\text{s}.$$

When $r = 12$ cm,

$$\left[\frac{dS}{dt}\right]_{r=12} = \frac{16}{12} = \frac{4}{3} \text{ cm}^2/\text{s}.$$

Hence, the surface area is increasing at the rate of $\frac{4}{3}$ cm^2/s when radius of sphere is 12 cm. **Ans.**

SECTION — C

20. Solve the following linear programming problem graphically : **[4]**

Maximise Z = $7x + 10y$

subject to the constraints

$$4x + 6y \leq 240$$
$$6x + 3y \leq 240$$
$$x \geq 10$$
$$x \geq 0, y \geq 0$$

Solution : We have,

Maximise Z = $7x + 10y$

Subject to the constraints :

$$4x + 6y \leq 240$$
$$6x + 3y \leq 240$$
$$x \geq 10$$
$$x \geq 0, y \geq 0$$

Converting the given inequalities into equations, we obtain the following equations :

$$4x + 6y = 240$$
$$6x + 3y = 240$$
$$x = 10$$

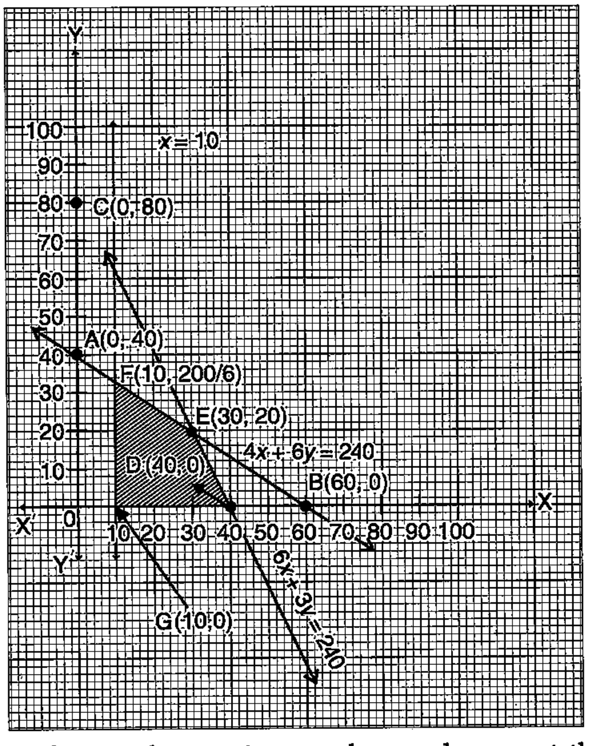
Then, $4x + 6y = 240$

x	0	60
y	40	0

$6x + 3y = 240$

x	0	40
y	80	0

and $x = 10$ is a line parallel to Y-axis.



Plotting these points on the graph, we get the shaded feasible region *i.e.*, DEFGD.

Corner points	Value of Z= $7x + 10y$
D(40, 0)	7(40) + 10(0) = 280
E (30, 20)	7(30) + 10(20) = 410
F (10, 200/6)	7(10) + 10(200/6) = 403.33
G (10, 0)	7(10) + 10(0) = 70

Clearly the maximum value of Z is 410 at (30, 20). **Ans.**

21. Find : $\int \frac{e^x dx}{(e^x - 1)^2 (e^x + 2)}$. **[4]**

Solution : Let $I = \int \frac{e^x dx}{(e^x - 1)^2 (e^x + 2)}$

Putting, $e^x = t$ and $e^x dx = dt$

$$I = \int \frac{dt}{(t-1)^2 (t+2)}$$

Let $\frac{1}{(t-1)^2 (t+2)} = \frac{A}{t-1} + \frac{B}{(t-1)^2} + \frac{C}{t+2}$

$$1 = A(t-1)(t+2) + B(t+2) + C(t-1)^2$$

Putting $t = 1$, we get

$$1 = 3B \quad \Rightarrow \quad B = 1/3$$

Putting $t = -2$, we get

$$1 = 9C \quad \Rightarrow \quad C = 1/9$$

Putting $t = 0$, we get

$$1 = -2A + 2B + C$$

$$\Rightarrow \quad 1 = -2A + \frac{2}{3} + \frac{1}{9}$$

$$\Rightarrow \quad A = -1/9$$

$$\therefore \quad I = \frac{-1}{9}\int \frac{1}{(t-1)}\,dt$$

$$\vec{d}.\vec{b} \int \frac{1}{(t-1)^2}\,dt + \frac{1}{9}\int \frac{1}{(t+2)}\,dt$$

$$\Rightarrow I = \frac{-1}{9}\log|t-1| - \frac{1}{3\cdot(t-1)} + \frac{1}{9}\log|t+2| + C$$

$$\therefore \quad I = \frac{1}{9}\log\left|\frac{e^x+2}{e^x-1}\right| - \frac{1}{3(e^x-1)} + C.$$ **Ans.**

22. If $\vec{a} = 2\hat{i} - \hat{j} - 2\hat{k}$ and $\vec{b} = 7\hat{i} + 2\hat{j} - 3\hat{k}$, then express $\vec{b}$ in the form of $\vec{b} = \vec{b_1} + \vec{b_2}$, where $\vec{b_1}$ is parallel to $\vec{a}$ and $\vec{b_2}$ is perpendicular to $\vec{a}$. **[4]**

Solution : We have,

$\vec{a} = 2\hat{i} - \hat{j} - 2\hat{k}$ and $\vec{b} = 7\hat{i} + 2\hat{j} - 3\hat{k}$,

Since, $\vec{b_1}$ is parallel to $\vec{a}$

$$\vec{b_1} = \lambda\vec{a}$$

$$\therefore \quad \vec{b_1} = 2\lambda\hat{i} - \lambda\hat{j} - 2\lambda\hat{k}$$

So, $$\vec{b_2} = \vec{b} - \vec{b_1}$$

$$= 7\hat{i} + 2\hat{j} - 3\hat{k} - (2\lambda\hat{i} - \lambda\hat{j} - 2\lambda\hat{k})$$

$$= (7-2\lambda)\hat{i} + (2+\lambda)\hat{j} + (-3+2\lambda)\hat{k}$$

Since, $\vec{b_2}$ is perpendicular to $\vec{a}$

$$\vec{a}\cdot\vec{b_2} = 0$$

$$\therefore (2\hat{i} - \hat{j} - 2\hat{k})[(7-2\lambda)\hat{i} + (2+\lambda)\hat{j} + (-3+2\lambda)\hat{k}] = 0$$

$$\Rightarrow \quad 2(7-2\lambda) - 1(2+\lambda) - 2(-3+2\lambda) = 0$$

$$\Rightarrow \quad 14 - 4\lambda - 2 - \lambda + 6 - 4\lambda = 0$$

$$\Rightarrow \quad -9\lambda + 18 = 0$$

$$\Rightarrow \quad 9\lambda = 18$$

$$\therefore \quad \lambda = 2$$

Hence, $$\vec{b} = (4\hat{i} - 2\hat{j} - 4\hat{k}) + (3\hat{i} + 4\hat{j} + \hat{k})$$

$$\vec{b} = \vec{b_1} + \vec{b_2}$$ **Ans.**

23. Find the general solution of the differential equation $\frac{dy}{dx} - y = \sin x$. **[4]**

Solution : We have,

$$\frac{dy}{dx} - y = \sin x$$

which is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

where, $P = -1$ and $Q = \sin x$

Now, $$\text{I. F.} = e^{\int P dx} = e^{\int -1 dx} = e^{-x}$$

So, the required solution is

$$ye^{-x} = \int e^{-x}\sin x\,dx + C_1 \quad ...(i)$$

Let $$I = \int e^{-x}\sin x\,dx \quad ...(ii)$$

$$I = \sin x\int e^{-x}dx - \int\left(\frac{d}{dx}(\sin x)\int e^{-x}\,dx\right)dx + C_2$$

$$\Rightarrow \quad I = -\sin x\, e^{-x} + \int\cos x\, e^{-x}\,dx + C_2$$

$$\Rightarrow \quad I = -\sin x\, e^{-x} + \cos x\int e^{-x}dx - \int\left(\frac{d}{dx}(\cos x)\int e^{-x}dx\right)dx + C_2$$

$$\Rightarrow \quad I = -\sin x\, e^{-x} - \cos x\cdot e^{-x} - \int\sin x\cdot e^{-x}dx + C_2$$

$$\Rightarrow \quad I = -\sin x\cdot e^{-x} - \cos x\cdot e^{-x} - I + C_2 \quad \text{[using (ii)]}$$

$$\Rightarrow 2I = -e^{-x}(\sin x + \cos x) + C_2$$

$$\Rightarrow \quad I = \frac{-1}{2}e^{-x}(\sin x + \cos x) + C_2$$

By equation (i),

$$ye^{-x} = \frac{-1}{2}e^{-x}(\sin x + \cos x) + C_1 + C_2$$

$$2y = -(\sin x + \cos x) + 2Ce^x$$

$$(\quad C_1 + C_2 = C)$$

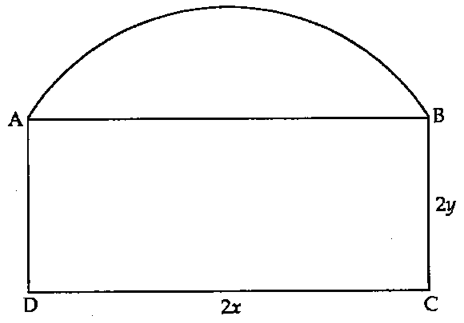
$\therefore\ 2y = 2Ce^x - \sin x - \cos x$ is the required solution. **Ans.**

SECTION – D

29. A window is in the form of a rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 m. Find the dimensions of the window to admit maximum light through the whole opening. **[6]**

Solution : Let ABCD be a window of rectangular form surmounted by a semicircle with diameter AB.

Given, Perimeter of the window (P) = 10 m



Let the length and breadth of the rectangle be $2x$ and $2y$ respectively.

Since, $\quad P = 10$ m

i.e., $\quad 2x + 4y + \pi x = 10$

$\Rightarrow \quad 4y = 10 - 2x - \pi x$

$\Rightarrow \quad 4xy = 10x - 2x^2 - \pi x^2 \quad$ ...(i)

Now, area of the window

$$A = (2x)(2y) + \frac{1}{2}\pi x^2$$

$$\Rightarrow \quad A = 4xy + \frac{1}{2}\pi x^2$$

$$\Rightarrow \quad A = 10x - 2x^2 - \pi x^2 + \frac{1}{2}\pi x^2$$

[using (i)]

$$\Rightarrow \quad A = 10x - 2x^2 - \frac{1}{2}\pi x^2$$

On differentiating A w. r. t. x, we get

$$\frac{dA}{dx} = 10 - 4x - \pi x$$

The critical numbers of x are given by $\frac{dA}{dx} = 0$

$\Rightarrow \quad 10 - 4x - \pi x = 0$

$\Rightarrow \quad -x(4 + \pi) = -10$

$\Rightarrow \quad x = \frac{10}{4+\pi}$

Now, $\quad \frac{dA}{dx} = 10 - 4x - \pi x$

differentiating Again, w.r.t. x, we get

$$\frac{d^2A}{dx^2} = -4 - \pi = -(4 + \pi) < 0$$

Thus, $\quad$ A is maximum when $x = \frac{10}{4+\pi}$ m

Now, length of the window, $2x = \frac{20}{4+\pi}$ m

and width of the window,

$$2y = \frac{10 - 2x - \pi x}{2} \quad \text{[using (i)]}$$

$$= 5 - \frac{10}{4+\pi} - \frac{\pi}{2} \cdot \frac{10}{4+\pi}$$

$$= \frac{10(4+\pi) - 20 - 10\pi}{2(4+\pi)}$$

$$= \frac{40 + 10\pi - 20 - 10\pi}{2(4+\pi)}$$

$$= \frac{10}{4+\pi} \text{ m}$$

Also, radius of the semi-circle is $\left(\frac{10}{4+\pi}\right)$m

Hence, the dimensions of the rectangular part of the window are $\frac{20}{4+\pi}$ m and $\frac{10}{4+\pi}$ m. **Ans.**

Mathematics 2017 (Delhi) SET I

Time allowed : 3 hours **Maximum marks : 100**

SECTION — A

1. **If A is a 3 × 3 invertible matrix, then what will be the value of k if det (A^{-1}) = (det A)^k.** **[1]**

Solution : Given, A is an invertible matrix.

$\therefore \quad A.\,A^{-1} = I$

$\Rightarrow \quad \det(A.A^{-1}) = \det(I)$

$\Rightarrow \quad \det(A).\det(A^{-1}) = 1 \quad [\because \det(I) = 1]$

$\Rightarrow \quad \det(A).\,(\det A)^k = 1 \quad [\because \det(A^{-1}) = (\det A)^k]$

$$(\det A)^k = \frac{1}{\det(A)}$$

$\therefore \quad (\det A)^k = (\det A)^{-1}$

on comparing both sides, we get

$k = -1$ **Ans.**

2. **Determine the value of the constant 'k' so that the function $f(x) = \begin{cases} \frac{kx}{|x|}, & \text{if } x < 0 \\ 3, & \text{if } x \geq 0 \end{cases}$ is continuous at $x = 0$.** **[1]**

Solution : Given, that the function is continuous at $x = 0$

$$\therefore \quad \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0) \quad \text{...(i)}$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{kx}{|x|}$$

$$= k \lim_{x \to 0^-} \frac{x}{-x}$$

$$= k(-1) = -k$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 3 = 3$$

From equation (i),

$$-k = 3$$

or $k = -3$ **Ans.**

3. Evaluate : $\int_2^3 3^x \, dx$. **[1]**

Solution : Let $I = \int_2^3 3^x \, dx = \left(\frac{3^x}{\log 3}\right)_2^3 + C$

where C is constant of integration

$$I = \frac{1}{\log 3}\left[3^3 - 3^2\right] + C$$

$$= \frac{1}{\log 3}(27 - 9) + C$$

$$= \frac{18}{\log 3} + C$$ **Ans.**

4. If a line makes angles 90° and 60° respectively with the positive directions of X and Y-axes, find the angle which it makes with the positive direction of Z-axis. **[1]**

Solution : We know that :

$$l^2 + m^2 + n^2 = 1 \quad ...(i)$$

and $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$

Given, $\alpha = 90°$, $\beta = 60°$

$\therefore \cos\alpha = \cos 90° = 0$ and $\cos\beta = \cos 60° = \frac{1}{2}$

From equation (i),

$$0^2 + \left(\frac{1}{2}\right)^2 + n^2 = 1$$

$$\Rightarrow n^2 = 1 - \frac{1}{4} = \frac{3}{4}$$

$$\Rightarrow \cos^2\gamma = \frac{3}{4}$$

$$\Rightarrow \cos\gamma = \pm\frac{\sqrt{3}}{2}$$

$$\Rightarrow \gamma = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) \text{ or } \cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)$$

$$\Rightarrow \gamma = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}$$ **Ans.**

SECTION – B

5. Show that all the diagonal elements of a skew symmetric matrix are zero. **[2]**

Solution : Let $A = [a_{ij}]$ be a given matrix

Since, it is skew symmetric $A' = -A$

$\therefore \quad a_{ji} = -a_{ij}$ For all i, j

$\Rightarrow \quad a_{ii} = -a_{ii}$ For all values of i

$\Rightarrow \quad 2a_{ii} = 0$ For all values of i

$\Rightarrow \quad a_{ii} = 0$ For all values of i

$\Rightarrow \quad a_{11} = a_{22} = a_{33} = ... = a_{nn} = 0$

Hence, all the diagonal elements of a skew. symmetric matrix are zero (as diagonal elements are : $a_{11}; a_{22}; ....a_{nn}$). **Hence Proved.**

6. Find $\frac{dy}{dx}$ at $x = 1, y = \frac{\pi}{4}$ if $\sin^2 y + \cos xy = K$. **[2]**

Solution : Given, $\sin^2 y + \cos xy = K$

Differentiating both sides w.r. t. x, we get

$$2\sin y \cos y \cdot \frac{dy}{dx} + \left[-\sin xy\left(x\frac{dy}{dx} + y\right)\right] = 0$$

$$\Rightarrow \sin 2y \cdot \frac{dy}{dx} - x\sin xy \frac{dy}{dx} - y\sin xy = 0$$

$$\Rightarrow \frac{dy}{dx}(\sin 2y - x\sin xy) = y\sin xy$$

$$\frac{dy}{dx} = \frac{y\sin xy}{\sin 2y - x\sin xy} \quad ...(i)$$

Given, at $x = 1, y = \frac{\pi}{4}$

From equation (i),

$$\frac{dy}{dx} = \frac{\frac{\pi}{4}\sin\frac{\pi}{4}}{\sin\frac{2\pi}{4} - 1.\sin\frac{\pi}{4}} = \frac{\frac{\pi}{4} \times \frac{1}{\sqrt{2}}}{1 - \frac{1}{\sqrt{2}}}$$

$$= \frac{\frac{\pi}{4\sqrt{2}}}{\frac{\sqrt{2}-1}{\sqrt{2}}} = \frac{\pi}{4(\sqrt{2}-1)}$$

$$= \frac{\pi\left(\sqrt{2}+1\right)}{4\left(\left(\sqrt{2}\right)^2 - 1^2\right)}$$

$$= \frac{\pi\left(\sqrt{2}+1\right)}{4}$$ **Ans.**

7. The volume of a sphere is increasing at the rate of 3 cubic centimetre per second. Find the rate of increase of its surface area, when the radius is 2 cm. **[2]**

Solution : Let V be the volume and r be the radius of sphere at any time t.

Then, $V = \frac{4}{3}\pi r^3$

Given, $\frac{dV}{dt} = 3\text{ cm}^3/\text{s}$

Differentiating V w.r.t. t, we get

$$\frac{dV}{dt} = \frac{4}{3}\pi \times 3r^2 \cdot \frac{dr}{dt}$$

$$\Rightarrow \quad 3 = 4\pi r^2 \cdot \frac{dr}{dt}$$

$$\Rightarrow \quad \frac{dr}{dt} = \frac{3}{4\pi r^2} \quad ...(i)$$

Now, let S be the surface area of the sphere,

then $S = 4\pi r^2$

Differentiating S w.r.t. t, we get

$$\frac{dS}{dt} = 8\pi r \frac{dr}{dt}$$

$$= 8\pi r \frac{3}{4\pi r^2} \quad \text{[From eq. (i)]}$$

$$= \frac{6}{r}$$

when $r = 2$

$$\left(\frac{dS}{dt}\right)_{r=2} = \frac{6}{2} = 3\text{ cm}^2/\text{sec}$$

∴ Rate of increase of surface area of the sphere is 3 square centimetre per second. **Ans.**

8. **Show that the function $f(x) = 4x^3 - 18x^2 + 27x - 7$ is always increasing on R.** **[2]**

Solution : Given, $f(x) = 4x^3 - 18x^2 + 27x - 7$

Differentiating $f(x)$ w.r.t. x, we get

$$f'(x) = 12x^2 - 36x + 27 = 3(4x^2 - 12x + 9)$$

$$= 3(2x - 3)^2 \quad \text{for any } x \in R$$

$$3 > 0 \text{ and } (2x - 3)^2 \geq 0$$

$$\therefore \quad f'(x) \geq 0$$

$\Rightarrow$ The function is always increasing on R.

Hence Proved.

9. **Find the vector equation of the line passing through the point A(1, 2, – 1) and parallel to the line $5x - 25 = 14 - 7y = 35z$.** **[2]**

Solution : Given line is $5x - 25 = 14 - 7y = 35z$

$$\Rightarrow \quad 5(x - 5) = -7(y - 2) = 35z$$

$$\Rightarrow \quad \frac{x-5}{1/5} = \frac{y-2}{-1/7} = \frac{z-0}{1/35}$$

$$\Rightarrow \quad \frac{x-5}{7} = \frac{y-2}{-5} = \frac{z-0}{1}$$

Direction ratios of this line are 7, – 5, 1.

∴ Vector equation of the line which passes through the point A (1, 2, – 1) and its direction ratio are proportional to 7, – 5, 1 is

$$\vec{r} = \hat{i} + 2\hat{j} - \hat{k} + \lambda\left(7\hat{i} - 5\hat{j} + \hat{k}\right) \quad \textbf{Ans.}$$

10. **Prove that if E and F are independent events, then the events E and F′ are also independent.** **[2]**

Solution : Since E and F are independent events :

$$P(E \cap F) = P(E) \cdot P(F) \quad ...(i)$$

$$P(E \cap F') = P(E) - P(E \cap F)$$

$$= P(E) - P(E) P(F) \quad \text{[From (i)]}$$

$$= P(E)(1 - P(F))$$

$$= P(E) P(F')$$

∴ E and F′ are also independent. **Hence Proved.**

11. **A small firm manufactures necklaces and bracelets. The total number of necklaces and bracelets that it can handle per day is at most 24. It takes one hour to make a bracelet and half an hour to make a necklace. The maximum number of hours available per day is 16. If the profit on a necklace is ₹ 100 and that on a bracelet is ₹ 300. Formulate an L.P.P. for finding how many of each should be produced daily to maximize the profit ? It is being given that at least one of each must be produced.** **[2]**

Solution : Let the manufacturer produces x pieces of necklaces and y pieces of bracelets.

Since total number of necklaces and bracelets that can be handle per day are 24.

so, $x + y \leq 24$ ...(i)

To make bracelet one needs one hour and half an hour is need to make necklace and maximum time available is 16 hours

so, $\frac{1}{2}x + y \leq 16$ ...(ii)

Now, let Z be the profit and we have to maximize it, so our LPP will be

Maximize $Z = 100x + 300y$

Subject to constraints :

$$x + y \leq 24$$

$$\frac{1}{2}x + y \leq 16$$

or $x + 2y \leq 32$

and $x, y \geq 1$ **Ans.**

12. **Find $\int \frac{dx}{x^2 + 4x + 8}$.** **[2]**

Solution : Let $I = \int \frac{dx}{x^2 + 4x + 8}$

$$= \int \frac{dx}{x^2+4x+4-4+8}$$

$$= \int \frac{dx}{(x+2)^2+(2)^2}$$

We know that,

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}+C,$$

where C is constant of integration

$$\therefore \quad I = \frac{1}{2}\tan^{-1}\left(\frac{x+2}{2}\right)+C \qquad \textbf{Ans.}$$

SECTION – C

13. Prove that $\tan\left\{\frac{\pi}{4}+\frac{1}{2}\cos^{-1}\frac{a}{b}\right\}$ $+ \tan\left\{\frac{\pi}{4}-\frac{1}{2}\cos^{-1}\frac{a}{b}\right\} = \frac{2b}{a}$ **[4]**

Solution : Let $\frac{1}{2}\cos^{-1}\frac{a}{b} = A$

$$\text{L.H.S.} = \tan\left(\frac{\pi}{4}+A\right)+\tan\left(\frac{\pi}{4}-A\right)$$

$$= \frac{\tan\frac{\pi}{4}+\tan A}{1-\tan\frac{\pi}{4}\tan A}+\frac{\tan\frac{\pi}{4}-\tan A}{1+\tan\frac{\pi}{4}\tan A}$$

$$= \frac{1+\tan A}{1-\tan A}+\frac{1-\tan A}{1+\tan A}$$

$$= \frac{(1+\tan A)^2+(1-\tan A)^2}{1-\tan^2 A}$$

$$= \frac{2+2\tan^2 A}{1-\tan^2 A}$$

$$= \frac{2(1+\tan^2 A)}{1-\tan^2 A}$$

$$= 2\times\frac{1}{\cos 2A}$$

$$= 2\times\frac{1}{\cos 2\times\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)}$$

$$= 2\times\frac{1}{\cos\left(\cos^{-1}\frac{a}{b}\right)} = \frac{2}{a/b}$$

$= \frac{2b}{a}$ = R.H.S **Hence Proved.**

14. Using properties of determinants, prove that :

$$\begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix} = 9y^2(x+y). \qquad \textbf{[4]}$$

Solution : Consider, $\begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$= \begin{vmatrix} 3x+3y & x+y & x+2y \\ 3x+3y & x & x+y \\ 3x+3y & x+2y & x \end{vmatrix}$$

$$= 3(x+y)\begin{vmatrix} 1 & x+y & x+2y \\ 1 & x & x+y \\ 1 & x+2y & x \end{vmatrix}$$

Applying $R_1 \to R_1 - R_2$ and $R_3 \to R_3 - R_2$, we get

$$= 3(x+y)\begin{vmatrix} 0 & y & y \\ 1 & x & x+y \\ 0 & 2y & -y \end{vmatrix}$$

$$= 3(x+y)[-(-y^2-2y^2)]$$

(expanded along C_1)

$$= 3(x+y)\cdot 3y^2$$

$= 9y^2(x+y)$ **Hence Proved.**

OR

Let $A = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 5 & 2 \\ 7 & 4 \end{pmatrix}$, $C = \begin{pmatrix} 2 & 5 \\ 3 & 8 \end{pmatrix}$, **find a matrix D such that CD – AB = 0.**

Solution : Let D be the matrix of order 2 × 2,

$$D = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Given, CD – AB = 0

$\therefore$ CD = AB

$$\begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2a+5c & 2b+5d \\ 3a+8c & 3b+8d \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 43 & 22 \end{bmatrix}$$

On comparing both sides, we get

$$2a + 5c = 3 \qquad ...(i)$$

$$2b + 5d = 0$$

$$\Rightarrow \quad b = \frac{-5}{2}d$$

$$3a + 8c = 43 \qquad ...(ii)$$

$$3b + 8d = 22 \qquad ...(iii)$$

Substituting $b = \frac{-5d}{2}$ in equation (iii), we get

$$3\left(\frac{-5d}{2}\right)+8d = 22$$

$$\Rightarrow \quad \frac{-15d}{2}+8d = 22$$

$$\Rightarrow \quad -15d + 16d = 44$$

$$\Rightarrow \quad d = 44$$

Also, $$b = \frac{-5}{2}d$$

$$= \frac{-5}{2} \times 44 = -110$$

From equation (i),

$$a = \frac{3-5c}{2}$$

Substituting in equation (ii), we get

$$3\left(\frac{3-5c}{2}\right)+8c = 43$$

$$\Rightarrow \quad 9 - 15c + 16c = 86$$

$$\Rightarrow \quad c = 77$$

and $$a = \frac{3-5\times 77}{2} = \frac{3-385}{2}$$

$$= \frac{-382}{2} = -191$$

$$\therefore \quad D = \begin{bmatrix} -191 & -110 \\ 77 & 44 \end{bmatrix}$$ **Ans.**

15. Differentiate the function $(\sin x)^x + \sin^{-1}\sqrt{x}$ with respect to x. **[4]**

Solution : Let $y = (\sin x)^x + \sin^{-1}\sqrt{x}$

Let, $u = (\sin x)^x$ and $v = \sin^{-1}\sqrt{x}$

$\therefore \quad y = u + v$

Differentiating y w.r.t. x, we get

$$\frac{dy}{dx} = \frac{du}{dx}+\frac{dv}{dx} \quad ...(i)$$

$$u = (\sin x)^x$$

Taking log on both sides,

$$\log u = \log (\sin x)^x = x \log \sin x$$

Differentiating w.r.t. x, we get

$$\frac{1}{u}\frac{du}{dx} = \frac{x}{\sin x}\cos x + \log \sin x$$

$$\Rightarrow \quad \frac{du}{dx} = u\,[x \cot x + \log \sin x]$$

$$= (\sin x)^x [x \cot x + \log \sin x] \quad ...(ii)$$

Also $\quad v = \sin^{-1}\sqrt{x}$

Differentiating v w.r.t. x, we get

$$\frac{dv}{dx} = \frac{1}{\sqrt{1-x}} \times \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x}\,\sqrt{1-x}} \quad ...(iii)$$

From equations (i), (ii), and (iii)

$$\frac{dy}{dx} = (\sin x)^x\,[x \cot x + \log \sin x] + \frac{1}{2\sqrt{x}\,\sqrt{1-x}}$$

$$\frac{dy}{dx} = (\sin x)^x [x \cot x + \log \sin g\ x] + \frac{1}{2\sqrt{x-x^2}}$$ **Ans.**

OR

If $x^m y^n = (x+y)^{m+n}$, prove that $\frac{d^2y}{dx^2} = 0$.

Solution : Given, $x^m y^n = (x+y)^{m+n}$

Taking log on both sides, we get

$$\log x^m y^n = \log (x+y)^{m+n}$$

$$\Rightarrow \quad m \log x + n \log y = (m+n)\log (x+y)$$

Differentiating both sides w.r.t. x, we get

$$\frac{m}{x}+\frac{n}{y}.\frac{dy}{dx} = \frac{m+n}{(x+y)}.\left[1+\frac{dy}{dx}\right]$$

$$\Rightarrow \quad \frac{m}{x}+\frac{n}{y}.\frac{dy}{dx} = \frac{m+n}{x+y}+\frac{m+n}{x+y}\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx}\left(\frac{n}{y}-\frac{m+n}{x+y}\right) = \frac{m+n}{x+y}-\frac{m}{x}$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{nx+ny-my-ny}{y(x+y)}\right) = \frac{mx+nx-mx-my}{x(x+y)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x}\left(\frac{nx-my}{nx-my}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x}$$

Again, differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{x\frac{dy}{dx}-y}{x^2}$$

$$= \frac{x \times \frac{y}{x}-y}{x^2}$$

$$= \frac{y-y}{x^2} = 0$$ **Hence Proved.**

16. Find $\int \frac{2x}{(x^2+1)(x^2+2)^2}dx$ **[4]**

Solution :

Let $$I = \int \frac{2x}{(x^2+1)(x^2+2)^2}dx$$

Put $x^2 = t$

$\Rightarrow \quad 2xdx = dt$

$$I = \int \frac{dt}{(t+1)(t+2)^2}$$

Let

$$\frac{1}{(t+1)(t+2)^2} = \frac{A}{t+1} + \frac{B}{t+2} + \frac{C}{(t+2)^2} \quad ...(A)$$

$$\Rightarrow \frac{1}{(t+1)(t+2)^2} = \frac{A(t+2)^2 + B(t+1)(t+2) + C(t+1)}{(t+1)(t+2)^2}$$

$$1 = A(t+2)^2 + B(t+1)(t+2) + C(t+1)$$

Equating coefficient of t^2, t and constant terms on both sides, we get

$$A + B = 0 \quad ...(i)$$

$$4A + 3B + C = 0 \quad ...(ii)$$

$$4A + 2B + C = 1 \quad ...(iii)$$

Subtracting equation (iii) from (ii), we get

$$B = -1$$

Substituting B = –1 in equation (i),

$$A = 1$$

Substituting the values of A and B in (ii), we get

$$4 - 3 + C = 0$$

$$\Rightarrow \quad C = -1$$

From equation (A),

$$\frac{1}{(t+1)(t+2)^2} = \frac{1}{t+1} + \left(\frac{-1}{t+2}\right) + \left(\frac{-1}{(t+2)^2}\right)$$

$$\Rightarrow \int \frac{1}{(t+1)(t+2)^2} dt$$

$$= \left(\frac{-x\cos\pi x}{\pi}\right)_0^1 - \int_0^1 -\frac{\cos\pi x}{\pi} dx - \int \frac{1}{(t+2)^2} dt$$

$$= \log(t+1) - \log(t+2) + \frac{1}{t+2} + C$$

where C is constant of integration.

$$\therefore \int \frac{2x\,dx}{(x^2+1)(x^2+2)^2} = \log\left|\frac{x^2+1}{x^2+2}\right| + \frac{1}{x^2+2} + C$$

Ans.

17. Evaluate : $\int_0^\pi \frac{x \sin x}{1+\cos^2 x} dx$ **[4]**

Solution : Let, $I = \int_0^\pi \frac{x\sin x}{1+\cos^2 x} dx \quad ...(i)$

$$= \int_0^\pi \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)}$$

$$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$I = \int_0^\pi \frac{(\pi - x).\sin x}{1+\cos^2 x} \quad ...(ii)$$

Adding equations (i) and (ii), we get

$$2I = \pi\int_0^\pi \frac{\sin x}{1+\cos^2} dx$$

Put $\cos x = t$

$\Rightarrow \quad -\sin x\,dx = dt$

when $x = 0, \quad t = 1$

and when $x = \pi$, $t = -1$

$$2I = -\pi\int_1^{-1} \frac{dt}{1+t^2}$$

$$= -\pi\int_{-1}^{1} \frac{dt}{1+t^2} \quad \left[\because \int_a^b f(x)dx = \int_b^a f(x)dx\right].$$

$$\Rightarrow 2I = \pi.\left(\tan^{-1} t\right)_{-1}^{1} = \pi[\tan^{-1}1 - \tan^{-1}(-1)]$$

$$\Rightarrow 2I = \pi\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \frac{\pi^2}{2}$$

$$\Rightarrow \quad I = \frac{\pi^2}{4}$$

Ans.

OR

Evaluate : $\int_0^{3/2} |x \sin \pi x|\, dx$.

Solution :

Let $I = \int_0^{3/2} |x\sin\pi x|\,dx.$

$$|x\sin\pi x| = \begin{cases} x\sin\pi x, & 0 \le x \le 1 \\ -x\sin\pi x, & 1 \le x \le \frac{3}{2} \end{cases}$$

$$I = \int_0^1 (x\sin\pi x)dx + \int_1^{3/2} (x\sin\pi x)dx$$

$$= \int_0^1 (x\sin\pi x)dx - \int_1^{3/2} (x\sin\pi x)\,dx$$

Applying by parts on both the integrals, we get

$$I = \left(\frac{-x\cos\pi x}{\pi}\right)_0^1 - \int_0^1 -\frac{\cos\pi x}{\pi} dx$$

$$-\left[\left(\frac{-x\cos\pi x}{\pi}\right)_1^{3/2} - \int_1^{3/2} -\frac{\cos\pi x}{\pi} dx\right]$$

$$I = \frac{1}{2}\int \frac{3-1-2v}{1+v+v^2} dv = \int \frac{dx}{x}$$

$$-\left[\left(\frac{\frac{-3}{2}\cos\frac{3\pi}{2} + \cos\pi}{\pi}\right) + \frac{1}{\pi^2}(\sin\pi x)_1^{3/2}\right]$$

$$= \frac{1}{\pi}+\frac{1}{\pi^2}(\sin\pi-\sin 0)$$
$$-\left[\frac{0-1}{\pi}+\frac{1}{\pi^2}\left(\sin\frac{3\pi}{2}-\sin\pi\right)\right]$$
$$= \frac{1}{\pi}+\frac{1}{\pi^2}(0-0)+\frac{1}{\pi}-\frac{1}{\pi^2}(-1-0)=\frac{2}{\pi}+\frac{1}{\pi^2}$$
$$\Rightarrow I = \frac{2\pi+1}{\pi^2}$$
Ans.

18. Prove that $x^2-y^2=c(x^2+y^2)^2$ is the general solution of the differential equation $(x^3-3xy^2)\,dx=(y^3-3x^2y)\,dy$, where c is a parameter. **[4]**

Solution : Given differential equation is,
$$(x^3-3xy^2)\,dx = (y^3-3x^2y)\,dy$$
$$\Rightarrow \frac{dy}{dx} = \frac{x^3-3xy^2}{y^3-3x^2y} \quad ...(i)$$
Clearly, it is a homogeneous differential equation.

Put $y = vx$
$$\Rightarrow \frac{dy}{dx} = v+x\frac{dv}{dx}$$
From equation (i),
$$v+x\frac{dv}{dx}=\frac{x^3-3x.(vx)^2}{(vx)^3-3x^2.(vx)}=\frac{x^3(1-3v^2)}{x^3(v^3-3v)}$$
$$\Rightarrow x\frac{dv}{dx}=\frac{1-3v^2}{v^3-3v}-v$$
$$=\frac{1-3v^2-v^4+3v^2}{v^3-3v}$$
$$=\frac{1-v^4}{v^3-3v}$$
$$\Rightarrow \frac{v^3-3v}{1-v^4}dv=\frac{dx}{x}$$
On integrating both sides, we get
$$\int\frac{v^3-3v}{1-v^4}dv = \int\frac{dx}{x}$$
$$\int\frac{v^3}{1-v^4}dv-3\int\frac{v}{1-v^4}dv = \int\frac{dx}{x} \quad ...(ii)$$
In the first integral (For L.H.S.)

Put $1-v^4=t$
$$\Rightarrow -4v^3dv = dt,$$
$$\Rightarrow v^3\,dv = \frac{dt}{-4}$$
In the second integral, put $v^2 = z$
$$2vdv = dz \Rightarrow vdv = \frac{dz}{2}$$
From equation (ii), we get
$$\frac{-1}{4}\int\frac{dt}{t}-\frac{3}{2}\int\frac{dz}{1-z^2}=\int\frac{dx}{x}$$
$$\Rightarrow \frac{-1}{4}\log t-\frac{3}{2}\times\frac{1}{2}\log\frac{1+z}{1-z} = \log x+\log A$$
$$\left[\because \int\frac{1}{a^2-x^2}=\frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+C\right]$$
where log A is constant of integration.
$$\frac{3}{4}\log\frac{1-z}{1+z}-\frac{1}{4}\log t=\log Ax$$
$$\Rightarrow \frac{1}{4}\log\left|\left(\frac{1-z}{1+z}\right)^3\frac{1}{t}\right| = \log Ax$$
$$\Rightarrow \log\left(\frac{\left(1-\frac{y^2}{x^2}\right)^3}{\left(1+\frac{y^2}{x^2}\right)^3\left(1-\frac{y^4}{x^4}\right)}\right)^{1/4} = \log Ax$$
$$\Rightarrow \log\left(\frac{\frac{(x^2-y^2)^3}{(x^2)^3}}{\frac{(x^2+y^2)^3}{(x^2)^3}\frac{(x^4-y^4)}{x^4}}\right)^{1/4} = \log Ax$$
$$\Rightarrow\left(\frac{(x^2-y^2)^3\,x^4}{(x^2+y^2)^3.(x^2-y^2)(x^2+y^2)}\right)^{1/4} = Ax$$
$$\Rightarrow \left(\frac{(x^2-y^2)^2\,x^4}{(x^2+y^2)^4}\right)^{1/4} = Ax$$
$$\Rightarrow \frac{(x^2-y^2)^{1/2}\,x}{(x^2+y^2)} = Ax$$
$$\Rightarrow (x^2-y^2)^{1/2}=A(x^2+y^2)$$
On squaring both sides, we get
$$x^2-y^2=A^2(x^2+y^2)^2$$
or
$$x^2-y^2=c\,(x^2+y^2)^2,$$
[where $A^2 = C$]

$x^2-y^2 = c\,(x^2+y^2)^2$ is the solution of given differential equation **Ans.**

19. Let $\vec{a}=\hat{i}+\hat{j}+\hat{k}$, $\vec{b}=\hat{i}$ and $\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$, then :

(a) Let $c_1 = 1$ and $c_2 = 2$, find c_3 which makes $\vec{a}$, $\vec{b}$ and $\vec{c}$ coplanar.

(b) If $c_2 = -1$ and $c_3 = 1$, show that no value of c_1 can make $\vec{a}, \vec{b}$ and $\vec{c}$ coplanar. [4]

Solution : Given, $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = \hat{i}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

(a) When $c_1 = 1$ and $c_2 = 2$

$$\vec{c} = \hat{i} + 2\hat{j} + c_3\hat{k}$$

We know that, $\vec{a}, \vec{b}, \vec{c}$ are coplanar if

$$[\vec{a}\ \vec{b}\ \vec{c}] = \vec{a}.(\vec{b} \times \vec{c}) = 0 \quad \text{...(i)}$$

$$\vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 1 & 2 & c_3 \end{vmatrix} = -c_3\vec{j} + 2\hat{k}$$

Given, $[\vec{a}\ \vec{b}\ \vec{c}] = 0$

$$\vec{a}.(\vec{b} \times \vec{c}) = (\vec{i} + \vec{j} + \vec{k}).(-c_3\hat{j} + 2\hat{k}) = 0$$

$\Rightarrow \quad -c_3 + 2 = 0$

$\Rightarrow \quad c_3 = 2.$ **Ans.**

(b) When $c_2 = -1$ and $c_3 = 1$

$$\vec{c} = c_1\hat{i} - \hat{j} + \hat{k}$$

$$\vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ c_1 & -1 & 1 \end{vmatrix} = -\vec{j} - \hat{k}$$

From equation (i),

$$(\hat{i} + \hat{j} + \hat{k}).(-\hat{j} - \hat{k}) = 0$$

$\Rightarrow -1 - 1 = 0$, which is not possible

$\Rightarrow$ No value of c_1 can make $\vec{a}, \vec{b}$ and $\vec{c}$ coplanar. **Hence Proved.**

20. If $\vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular vectors of equal magnitudes, show that the vector $\vec{a} + \vec{b} + \vec{c}$ is equally inclined to $\vec{a}, \vec{b}$ and $\vec{c}$. Also, find the angle which $\vec{a} + \vec{b} + \vec{c}$ makes with $\vec{a}$ or $\vec{b}$ or $\vec{c}$. [4]

Solution : Let $|\vec{a}| = |\vec{b}| = |\vec{c}| = \lambda$ ...(i)

Now, $\vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular

We have, $\vec{a}.\vec{b} = \vec{b}.\vec{c} = \vec{c}.\vec{a} = 0$ ...(ii)

$$\therefore \quad (\vec{a}+\vec{b}+\vec{b})^2 = (\vec{a}+\vec{b}+\vec{c}).(\vec{a}+\vec{b}+\vec{c})$$

$$= \vec{a}.\vec{a} + \vec{a}.\vec{b} + \vec{a}.\vec{c} + \vec{b}.\vec{a} + \vec{b}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a} + \vec{c}.\vec{b} + \vec{c}.\vec{c}$$

$$= |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 = 3\lambda^2 \text{ [using (i)]}$$

$$\Rightarrow \quad |\vec{a} + \vec{b} + \vec{c}| = \sqrt{3}\lambda$$

Suppose $\vec{a} + \vec{b} + \vec{c}$ makes angles $\theta_1, \theta_2, \theta_3$ with $\vec{a}, \vec{b}$ and $\vec{c}$ respectively, then

$$(\vec{a}+\vec{b}+\vec{c}).\vec{a}) = |\vec{a} + \vec{b} + \vec{c}||\vec{a}|\cos\theta_1$$

$$\Rightarrow \vec{a}.\vec{a} + \vec{b}.\vec{a} + \vec{c}.\vec{a} = \sqrt{3}\lambda.\lambda\cos\theta_1$$

$$\Rightarrow \quad |\vec{a}|^2 = \sqrt{3}\lambda^2\cos\theta_1,$$

[using (ii)]

$$\Rightarrow \quad \lambda^2 = \sqrt{3}\lambda^2\cos\theta_1$$

$$\Rightarrow \quad \cos\theta_1 = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad \theta_1 = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

Similarly, $\theta_2 = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

and $\theta_3 = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$\therefore \quad \theta_1 = \theta_2 = \theta_3$

Hence, $\vec{a} + \vec{b} + \vec{c}$ is equally inclined with $\vec{a}, \vec{b}$ and $\vec{c}$. **Hence Proved.**

21. The random variable X can take only the values 0, 1, 2, 3. Given that P (X = 0) = P (X = 1) = p and P(X = 2) = P(X = 3) such that $\Sigma p_i x_i^2 = 2\Sigma p_i x_i$, find the value of p. [4]

Solution: Given, P(X = 0) = P(X = 1) = p and P(X = 2) = P(X = 3)

Let P (X = 2) = P(X = 3) = k

X	0	1	2	3
P(x)	p	p	k	k

also given that $\Sigma p_i x_i^2 = 2\Sigma p_i x_i$

$\Rightarrow \quad 0 + p + 4k + 9k = 2(0 + p + 2k + 3k)$

$\Rightarrow \quad p + 13k = 2(p + 5k)$

$\Rightarrow \quad 3k = p$ ...(i)

also we know that $\Sigma p_i = 1$

$\Rightarrow \quad p + p + k + k = 1$

$\Rightarrow \quad 2p + 2k = 1$

$\Rightarrow \quad 6k + 2k = 1 \quad$ [using (i)]

$\Rightarrow \quad 8k = 1$

or $\quad k = \frac{1}{8}$

From equation (i), $p = \frac{3}{8}$ **Ans.**

22. Often it is taken that a truthful person commands, more respect in the society. A man is known to speak the truth 4 out of 5 times. He throws a die and reports that it is a six. Find the probability that it is actually a six.

Do you also agree that the value of truthfulness leads to more respect in the society ? [4]

Solution : Let E_1, E_2 and A be the events defined as follows:

E_1 = Six appears on throwing a die.

E_2 = Six does not appear on throwing a die.

and A = the man reports that it is a six

We have,

$$P(E_1) = \frac{1}{6}$$

$$P(E_2) = \frac{5}{6}$$

Now $P(A/E_1)$ = Probability that the man reports that there is a six on the die given that six has occurred on the die = 4/5 (probability that the man speaks truth)

and $P(A/E_2)$ = Probability that the man reports that there is a six on the die given that six has not occurred on the die (probability that the man does not speak truth).

$$= 1 - \frac{4}{5} = \frac{1}{5}$$

By Bayes' theorem, we have

$$P(E_1/A) = \frac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2)}$$

$$= \frac{\frac{1}{6} \times \frac{4}{5}}{\frac{1}{6} \times \frac{4}{5} + \frac{5}{6} \times \frac{1}{5}}$$

$$= \frac{4}{4+5} = \frac{4}{9}$$

Yes, truthfulness always leads to more respect in the society as truth always wins. **Ans.**

23. Solve the following L.P.P. graphically :

Minimise Z = 5x + 10y

Subject to constraints

$$x + 2y \le 120$$

$$x + y \ge 60$$

$$x - 2y \ge 0$$

and $\quad x, y \ge 0$ **[4]**

Solution : We have,

Minimise $Z = 5x + 10y$

Subject to the constraints :

$$x + 2y \le 120$$

$$x + y \ge 60$$

$$x - 2y \ge 0$$

and $\quad x, y \ge 0$

Converting the given inequalities into equations, we obtain the following equations :

$$x + 2y = 120$$

$$x + y = 60$$

$$x - 2y = 0$$

Then, $\quad x + 2y = 120$

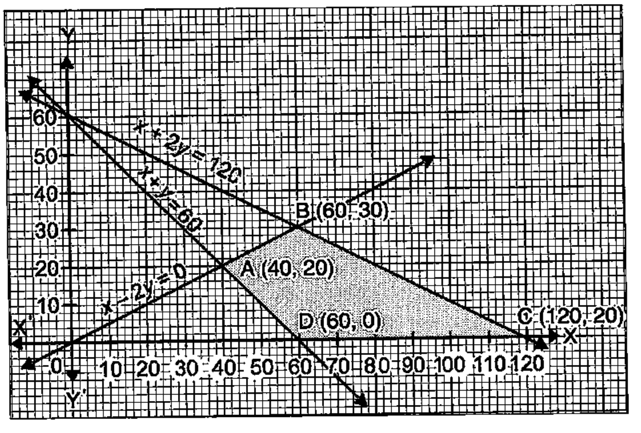
x	0	120
y	60	0

$x + y = 60$

x	0	60
y	60	0

$x - 2y = 0$

x	20	60
y	10	30



The shaded region ABCD represented by the given constraints is the feasible region. Corner points of the common shaded region are

A (40, 20), B (60, 30), C (120, 0) and D (60, 0). Value of Z at each corner point is given as :

Corner Point	$Z = 5x + 10y$
A (40, 20)	200 + 200 = 400
B (60, 30)	300 + 300 = 600
C (120, 0)	600 + 0 = 600
D (60, 0)	300 + 0 = 300 ← Minimum

Hence, minimum value of Z is 300 at (60, 0).

Ans.

SECTION – D

24. Use product $\begin{bmatrix}1 & -1 & 2\\ 0 & 2 & -3\\ 3 & -2 & 4\end{bmatrix}\begin{bmatrix}-2 & 0 & 1\\ 9 & 2 & -3\\ 6 & 1 & -2\end{bmatrix}$ to solve the system of equations $x+3z=9, -x+2y-2z=4, 2x-3y+4z=-3$. **[6]**

Solution : Consider,

$$\begin{bmatrix}1 & -1 & 2\\ 0 & 2 & -3\\ 3 & -2 & 4\end{bmatrix}\begin{bmatrix}-2 & 0 & 1\\ 9 & 2 & -3\\ 6 & 1 & -2\end{bmatrix}$$

$$= \begin{bmatrix}-2-9+12 & 0-2+2 & 1+3-4\\ 0+18-18 & 0+4-3 & 0-6+6\\ -6-18+24 & 0-4+4 & 3+6-8\end{bmatrix}$$

$$= \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = I$$

Hence, $\begin{bmatrix}1 & -1 & 2\\ 0 & 2 & -3\\ 3 & -2 & 4\end{bmatrix}^{-1} = \begin{bmatrix}-2 & 0 & 1\\ 9 & 2 & -3\\ 6 & 1 & -2\end{bmatrix}$

Now, given system of equations can be written in matrix form as follows :

$$\begin{bmatrix}1 & 0 & 3\\ -1 & 2 & -2\\ 2 & -3 & 4\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}9\\ 4\\ -3\end{bmatrix}$$

$$AX = B$$

$$\Rightarrow \quad X = A^{-1}B$$

$$\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}1 & 0 & 3\\ -1 & 2 & -2\\ 2 & -3 & 4\end{bmatrix}^{-1}\begin{bmatrix}9\\ 4\\ -3\end{bmatrix}$$

$$= \left[\begin{bmatrix}1 & -1 & 2\\ 0 & 2 & -3\\ 3 & -2 & 4\end{bmatrix}^{T}\right]^{-1}\begin{bmatrix}9\\ 4\\ -3\end{bmatrix}$$

$$= \left[\begin{bmatrix}1 & -1 & 2\\ 0 & 2 & -3\\ 3 & -2 & 4\end{bmatrix}^{-1}\right]^{T}\begin{bmatrix}9\\ 4\\ -3\end{bmatrix}$$

$$\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}-2 & 0 & 1\\ 9 & 2 & -3\\ 6 & 1 & -2\end{bmatrix}^{T}\begin{bmatrix}9\\ 4\\ -3\end{bmatrix}$$

$$= \begin{bmatrix}-2 & 9 & 6\\ 0 & 2 & 1\\ 1 & -3 & -2\end{bmatrix}\begin{bmatrix}9\\ 4\\ -3\end{bmatrix}$$

$$\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}-18+36-18\\ 0+8-3\\ 9-12+6\end{bmatrix} = \begin{bmatrix}0\\ 5\\ 3\end{bmatrix}$$

$\Rightarrow \quad x = 0, \; y = 5, z = 3$ **Ans.**

25. Consider $f : R_+ \to [-5, \infty)$, given by $f(x) = 9x^2 + 6x - 5$. Show that f is invertible with $f^{-1}(y) = \left(\frac{\sqrt{y+6}-1}{3}\right)$. **[6]**

Hence find :

(i) $f^{-1}(10)$

(ii) y if $f^{-1}(y) = \frac{4}{3}$,

where R_+ is the set of all non-negative real numbers.

Solution : For one-one :

Let $x_1, x_2 \in R_+$

$$f(x_1) = f(x_2)$$

$\Rightarrow \quad 9x_1^2 + 6x_1 - 5 = 9x_2^2 + 6x_2 - 5$

$\Rightarrow 9(x_1^2 - x_2^2) + 6(x_1 - x_2) = 0$

$\Rightarrow (x_1 - x_2)(9(x_1 + x_2) + 6) = 0$

Since $9(x_1 + x_2) + 6 > 0$ [as $x_1, x_2 \in R_+$]

$\Rightarrow \quad x_1 - x_2 = 0$ [$x_1, x_2 \in R_+$]

or $\quad x_1 = x_2$

∴ Function is one-one

For onto :

For every $y \in [-5, \infty)$ such that $f(x) = y$

$\Rightarrow \quad 9x^2 + 6x - 5 = y$

$\Rightarrow (3x)^2 + 2 \cdot (3x) \cdot 1 + 1^2 - 1^2 - 5 = y$

$\Rightarrow \quad (3x+1)^2 - 6 = y$

$\Rightarrow \quad (3x+1)^2 = y + 6$

$\Rightarrow \quad 3x + 1 = \sqrt{y+6}$

$\Rightarrow \quad x = \frac{-1+\sqrt{y+6}}{3} \in R_+$

∴ Function is onto.

Since f is both one-one and onto.

∴ Function is invertible.

$$f(x) = y$$

$$\Rightarrow \quad x = f^{-1}(y) = \frac{-1+\sqrt{y+6}}{3}$$

(i) $$f^{-1}(10) = \frac{-1+\sqrt{16}}{3} = \frac{-1+4}{3} = 1$$

Ans.

(ii) $$f^{-1}(y) = \frac{4}{3}$$

$$\Rightarrow \quad \frac{4}{3} = \frac{-1+\sqrt{y+6}}{3},$$

$$\Rightarrow \quad 4 = -1+\sqrt{y+6}$$

$$\Rightarrow \quad 5 = \sqrt{y+6}$$

Squaring on both sides,

$$25 = y + 6$$

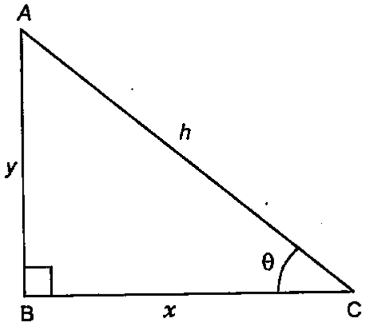
or $$y = 19$$ **Ans.**

OR

Discuss the commutativity and associativity of binary operation '∗' defined on A = Q – {1} by the rule $a * b = a - b + ab$ for all, $a, b \in$ A. Also find the identity element of ∗ in A and hence find the invertible elements of A.**

26. If the sum of lengths of the hypotenuse and a side of a right angled triangle is given, show that the area of the triangle is maximum, when the angle between them is $\frac{\pi}{3}$. [6]

Solution : Let h, x and y be the length of hypotenuse and sides of the right triangle ABC.



From ΔABC

$$h^2 = x^2 + y^2$$

$$\Rightarrow \quad y^2 = h^2 - x^2$$

If A be the area of the triangle ABC, then

$$A = \frac{1}{2}xy = \frac{1}{2}x\sqrt{h^2 - x^2}$$

$$\Rightarrow \quad A^2 = \frac{x^2}{4}(h^2 - x^2)$$

Let $$A^2 = z$$

[*also given*
$h + x = k$ (*constant*)
$h = k - x$]

$$\Rightarrow \quad z = \frac{x^2}{4}((k-x)^2 - x^2)$$

$$= \frac{x^2}{4}(k^2 - 2kx)$$

$$= \frac{k^2x^2 - 2kx^3}{4}$$

Differentiating z w.r. t. x, we get

$$\frac{dz}{dx} = \frac{2k^2x - 6kx^2}{4}$$

$$= \frac{k^2x - 3kx^2}{2}$$

Again, differentiating w.r.t. x, we get

$$\frac{d^2z}{dx^2} = \frac{k^2 - 6kx}{2}$$

For maxima or minima

Put $$\frac{dz}{dx} = 0$$

$$\Rightarrow \quad \frac{k^2x - 3kx^2}{2} = 0$$

$$\Rightarrow \quad kx.\ (k - 3x) = 0$$

$$\Rightarrow \quad k - 3x = 0, \text{ as } x \neq 0$$

or $$x = \frac{k}{3}$$

$$h = k - x = k - \frac{k}{3} = \frac{2k}{3}$$

$$y^2 = h^2 - x^2$$

$$= \frac{4k^2}{9} - \frac{k^2}{9}$$

$$= \frac{3k^2}{9} = \frac{k^2}{3}$$

$$\Rightarrow \quad y = \frac{k}{\sqrt{3}}$$

when $x = \frac{k}{3}$

$$\left(\frac{d^2z}{dx^2}\right)_{x=\frac{k}{3}} = \frac{1}{2}\left(k^2 - 6k \times \frac{k}{3}\right)$$

$$= \frac{-k^2}{2} < 0$$

∴ Area of the triangle is maximum.

From ΔABC

$$\cos\theta = \frac{x}{h} = \frac{k/3}{2k/3} = \frac{1}{2}$$

or $$\theta = \frac{\pi}{3}$$ **Hence Proved.**

27. Using integration, find the area of region bounded by the triangle whose vertices are (– 2, 1), (0, 4) and (2, 3). **[6]**

Solution : The vertices of the ΔABC are A(– 2, 1), B(0, 4) and C(2, 3).

Equation of the side AB is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$\Rightarrow \quad y - 1 = \frac{4-1}{0-(-2)}(x-(-2)) = \frac{3}{2}(x+2)$$

$$\Rightarrow \quad y = \frac{3}{2}x + 4$$

Equation of the side BC is

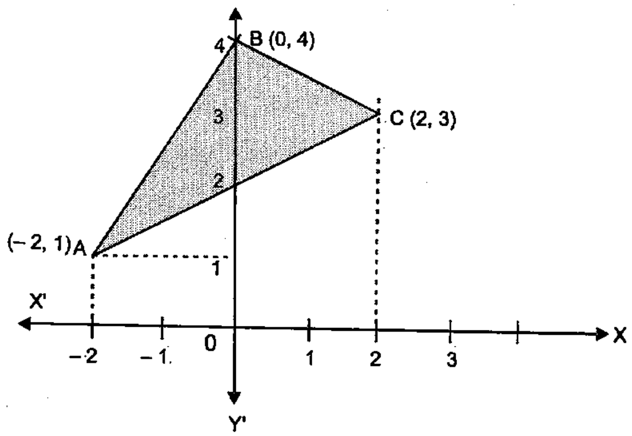
$$y - 4 = \frac{3-4}{2-0}(x-0) = \frac{-1}{2}x$$

$$\Rightarrow \quad y = \frac{-1}{2}x + 4$$

Equation of the side AC is

$$y - 1 = \frac{3-1}{2-(-2)}(x-(-2)) = \frac{1}{2}(x+2)$$

$$\Rightarrow \quad y = \frac{1}{2}x + 2$$



Required area = Shaded area

$$= \int_{-2}^{0}\left(\frac{3}{2}x+4\right)dx + \int_{0}^{2}\left(\frac{-1}{2}x+4\right)dx - \int_{-2}^{2}\left(\frac{1}{2}x+2\right)dx$$

$$= \left[\frac{3}{2}\cdot\frac{x^2}{2}+4x\right]_{-2}^{0} + \left[\frac{-1}{2}\cdot\frac{x^2}{2}+4x\right]_{0}^{2} - \left[\frac{1}{2}\cdot\frac{x^2}{2}+2x\right]_{-2}^{2}$$

$$= (0+0)-(3-8)+(-1+8)-(0+0)-(1+4)+(1-4)$$

$$= 0+5+7-0-5-3$$

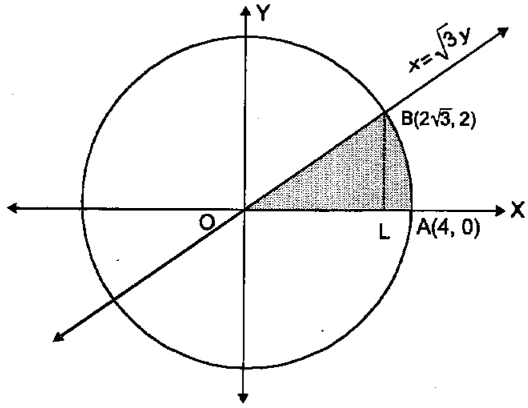
$= 4$ sq. units. **Ans.**

OR

Find the area bounded by the circle $x^2 + y^2 = 16$ and the line $\sqrt{3}y = x$ in the first quadrant, using integration.

Solution : Given, $x = \sqrt{3}y$

and $x^2 + y^2 = 16,$



$$\Rightarrow \quad (\sqrt{3}y)^2 + y^2 = 16$$

$$\Rightarrow \quad 4y^2 = 16$$

$$\Rightarrow \quad y^2 = 4$$

$$\Rightarrow \quad y = 2$$

$$\therefore \quad x = \sqrt{3}y = 2\sqrt{3}$$

$\therefore$ B$(2\sqrt{3}, 2)$ is the point of intersection in first quadrant.

Required area = Area under OBL + Area under LBA

$$= \int_{0}^{2\sqrt{3}}\frac{x}{\sqrt{3}}dx + \int_{2\sqrt{3}}^{4}\sqrt{16-x^2}\,dx$$

$$= \frac{1}{\sqrt{3}}\left(\frac{x^2}{2}\right)_{0}^{2\sqrt{3}} + \left[\frac{x}{2}\sqrt{16-x^2} + \frac{16}{2}\sin^{-1}\frac{x}{4}\right]_{2\sqrt{3}}^{4}$$

$$= \frac{1}{2\sqrt{3}}(12-0) + \left(0 + 8\sin^{-1}1\right) - \left(\frac{2\sqrt{3}}{2}\sqrt{16-12} + 8\sin^{-1}\frac{2\sqrt{3}}{4}\right)$$

$$= \frac{6}{\sqrt{3}} + 8\times\frac{\pi}{2} - 2\sqrt{3} - 8\sin^{-1}\frac{\sqrt{3}}{2}$$

$$= \frac{6\sqrt{3}}{3} + 4\pi - 2\sqrt{3} - 8\times\frac{\pi}{3}$$

$$= 2\sqrt{3} + \frac{4\pi}{3} - 2\sqrt{3}$$

$= \frac{4\pi}{3}$ sq. units. **Ans.**

28. Solve the differential equation $x\frac{dy}{dx} + y = x\cos x + \sin x$, given that $y = 1$ when $x = \frac{\pi}{2}$. **[6]**

Solution : Given differential equation is :

$$x\frac{dy}{dx} + y = x\cos x + \sin x$$

$$\Rightarrow \quad \frac{dy}{dx}+\frac{y}{x} = \cos x + \frac{\sin x}{x}$$

which is of the form $\frac{dy}{dx}+Py=Q$,

where $P = \frac{1}{x}$, $Q = \cos x + \frac{\sin x}{x}$

$$\text{I.F.} = e^{\int P dx} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$$

Required solution is

$$y \cdot \text{I.F.} = \int Q \cdot \text{I.F.} + C$$

$$y.\, x = \int\left(\cos x + \frac{\sin x}{x}\right)x\, dx + C$$

$$= \int x \cos x\, dx + \int \sin x\, dx + C$$

$$= x.\int \cos x\, dx - \int\left[\frac{d}{dx}x.\int \cos x\, dx\right]dx - \cos x + C$$

$$\Rightarrow \quad xy = x \sin x - \int \sin x\, dx - \cos x + C$$

$$\Rightarrow \quad xy = x \sin x + \cos x - \cos x + C$$

$$\Rightarrow \quad xy = x \sin x + C \quad ...(i)$$

Given, $y = 1$ when $x = \frac{\pi}{2}$

From equation (i)

$$1 \times \frac{\pi}{2} = \frac{\pi}{2}\sin\frac{\pi}{2} + C$$

$$\Rightarrow \quad \frac{\pi}{2} = \frac{\pi}{2} + C$$

$$\Rightarrow \quad C = 0$$

Substitute the value of C = 0 in (i), we get

$$xy = x \sin x$$

$\Rightarrow \quad y = \sin x$, which is the required solution.

Ans.

29. Find the equation of the plane through the line of intersection of $\vec{r}\cdot(2\hat{i}-3\hat{j}+4\hat{k}) = 1$ and $\vec{r}\cdot(\hat{i}-\hat{j})+4 = 0$ and perpendicular to the plane $\vec{r}\cdot(2\hat{i}-\hat{j}+\hat{k})+8 = 0$. Hence find whether the plane thus obtained contains the line $x - 1 = 2y - 4 = 3z - 12$. [6]

Solution : The equation of the plane passing through the line of intersection of the given planes is :

$$r\,[(2\hat{i}-3\hat{j}+4\hat{k})-1+\lambda(\hat{i}-\hat{j})]+4\lambda-1 = 0$$

$$\Rightarrow \quad \vec{r}.[(2+\lambda)\hat{i}-(3+\lambda)\hat{j}+4\hat{k}] = 1-4\lambda$$

Taking $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$, we get

$$(2+\lambda)x-(3+\lambda)y+4z = 1-4\lambda \quad ...(i)$$

$\because$ It is perpendicular to the plane

$$\vec{r}.(2\hat{i}-\hat{j}+\hat{k})+8 = 0$$

Cartesian equation of this plane is :

$$2x - y + z + 8 = 0 \quad ...(ii)$$

$\because$ Equations (i) and (ii) are perpendicular

$$\therefore \quad (2+\lambda)\,2 + (3+\lambda) + 4 = 0$$

$$\Rightarrow \quad 4 + 2\lambda + 3 + \lambda + 4 = 0$$

$$\Rightarrow \quad 11 + 3\lambda = 0$$

$$\text{or} \quad \lambda = \frac{-11}{3}$$

From equation (i),

$$\left(2-\frac{11}{3}\right)x-\left(3-\frac{11}{3}\right)y+4z = 1-4\times\frac{-11}{3}$$

$$\Rightarrow \quad \frac{-5x}{3}+\frac{2}{3}y+4z = \frac{47}{3}$$

$$\Rightarrow \quad -5x + 2y + 12z = 47$$

Required vector equation of this plane is :

$$\vec{r}.\left(-5\hat{i}+2\hat{j}+12\hat{k}\right) = 47 \quad ...(iii)$$

Now, equation of the given line is,

$$x - 1 = 2y - 4 = 3z - 12$$

$$\Rightarrow \quad \frac{x-1}{1} = 2(y-2) = 3(z-4)$$

$$\Rightarrow \quad \frac{x-1}{1} = \frac{y-2}{1/2} = \frac{z-4}{1/3}$$

$$\Rightarrow \quad \frac{x-1}{6} = \frac{y-2}{3} = \frac{z-4}{2}$$

Vector equation of this line is :

$$\vec{r} = (\hat{i}+2\hat{j}+4\hat{k})+\lambda(6\hat{i}+3\hat{j}+2\hat{k}) \quad ...(iv)$$

Obviously plane (iii) contains the line (iv) since the point $\hat{i}+2\hat{j}+4\hat{k}$ satisfy the equation of plane (iii) $\left[\text{as } (\hat{i}+2\hat{j}+4\hat{k}).\left(-5\hat{i}+2\hat{j}+12\hat{k}\right) = -5+4+48 = 47\right]$ and vector $-5\hat{i}+2\hat{j}+12\hat{k}$ is perpendicular to $6\hat{i}+3\hat{j}+2\hat{k}$ as $\left(6\hat{i}+3\hat{j}+2\hat{k}\right)\cdot\left(-5\hat{i}+2\hat{j}+12\hat{k}\right)$

$$= -30 + 6 + 24 = 0$$

$\therefore$ Plane (iii) contains the line (iv). **Ans.**

OR

Find the vector and Cartesian equations of a line passing through (1, 2, – 4) and perpendicular

to the two lines $\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7}$ and $\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$.

Solution : Cartesian equation of the line passing through (1, 2, – 4) is

$$\frac{x-1}{a} = \frac{y-2}{b} = \frac{z+4}{c} \quad ...(i)$$

Given lines are

$$\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7} \quad ...(ii)$$

and $$\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5} \quad ...(iii)$$

Let $\vec{b}_1, \vec{b}_2, \vec{b}_3$ are parallel vectors of (i), (ii) and (iii) respectively :

$$\vec{b}_1 = a\hat{i} + b\hat{j} + c\hat{k}$$
$$\vec{b}_2 = 3\hat{i} - 16\hat{j} + 7\hat{k}$$
$$\vec{b}_3 = 3\hat{i} + 8\hat{j} - 5\hat{k}$$

Given that (i) is perpendicular to both (ii) and (iii)

$\Rightarrow$ $$\vec{b}_1 \cdot \vec{b}_2 = 0$$

$$3a - 16b + 7c = 0 \quad ...(iv)$$

and $$\vec{b}_1 \cdot \vec{b}_3 = 0$$

$$3a + 8b - 5c = 0 \quad ...(v)$$

From equations (iv) and (v),

$$\frac{a}{80-56} = \frac{b}{21+15} = \frac{c}{24+48}$$

$\Rightarrow$ $$\frac{a}{24} = \frac{b}{36} = \frac{c}{72}$$

$\Rightarrow$ $$\frac{a}{2} = \frac{b}{3} = \frac{c}{6} = \lambda \text{ (say)}$$

$\Rightarrow$ $$a = 2\lambda, b = 3\lambda, c = 6\lambda$$

Putting in (i),

$$\frac{x-1}{2\lambda} = \frac{y-2}{3\lambda} = \frac{z+4}{6\lambda}$$

$\Rightarrow$ $$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6}$$

which is the cartesian equation of the line and vector equation of this line is

$$\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$$

Ans.

Mathematics 2017 (Delhi) SET II

Time allowed : 3 hours **Maximum marks : 100**

Note : Except for the following questions, all the remaining questions have been asked in previous set.

SECTION — B

12. For the curve $y = 5x - 2x^3$, if x increases at the rate of 2 units/sec, then find the rate of change of the slope of the curve when $x = 3$. [2]

Solution : Given curve is,

$$y = 5x - 2x^3$$

and $$\frac{dx}{dt} = 2 \text{ units/sec.}$$

Differentiating y w.r.t. x, we get

$$\frac{dy}{dx} = 5 - 6x^2.$$

Differentiating both sides w.r.t. t, we get

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = -12x\frac{dx}{dt}$$

when $x = 3$,

$$\left.\frac{d}{dt}\left(\frac{dy}{dx}\right)\right|_{x=3} = -12 \times 3 \times 2 = -72 \text{ units/sec.}$$

Thus, the slope of the curve is decreasing at the rate of 72 units/sec when $x = 3$. **Ans.**

SECTION — C

20. The random variable X can take only the values 0, 1, 2, 3. Given that P(2) = P(3) = p and P(0) = 2P(1). If $\Sigma p_i x_i^2 = 2\Sigma p_i x_i$, find the value of p. [4]

Solution : Given, P(2) = P(3) = p

and P(0) = 2P(1)

Let P(1) = k

X	0	1	2	3
P(x)	2k	k	p	p

also given that $\Sigma p_i x_i^2 = 2\,\Sigma p_i x_i$

$$\Rightarrow \quad 0 + k + 4p + 9p = 2(0 + k + 2p + 3p)$$
$$\Rightarrow \quad k + 13p = 2k + 10p$$
or $\quad 3p = k \quad$ ...(i)

also we know that

$$\Sigma p_i = 1$$
$$\Rightarrow \quad 2k + k + p + p = 1$$
$$\Rightarrow \quad 3k + 2p = 1$$
$$\Rightarrow \quad 9p + 2p = 1 \quad \text{[using (i)]}$$
$$\Rightarrow \quad 11p = 1$$
or $\quad p = \frac{1}{11}$. **Ans.**

21. Using vectors find the area of triangle ABC with vertices A(1, 2, 3), B(2, – 1, 4) and C (4, 5, – 1). **[4]**

Solution : Vertices of the given ΔABC are A(1, 2, 3) B(2, – 1, 4) and C(4, 5, – 1)

$$\vec{AB} = \vec{OB} - \vec{OA} = (2\hat{i} - \hat{j} + 4\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k})$$
$$= \hat{i} - 3\hat{j} + \hat{k}$$
$$\vec{AC} = \vec{OC} - \vec{OA} = (4\hat{i} + 5\hat{j} - \hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k})$$
$$= 3\hat{i} + 3\hat{j} - 4\hat{k}$$

Required area $= \frac{1}{2}|\vec{AB} \times \vec{AC}|$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 1 \end{vmatrix} = 9\hat{i} + 7\hat{j} + 12\hat{k}$$

$$|\vec{AB} \times \vec{AC}| = \sqrt{9^2 + 7^2 + 12^2}$$
$$= \sqrt{81 + 49 + 144} = \sqrt{274}$$

$\therefore$ Area of ΔABC $= \frac{1}{2}\sqrt{274}$ sq. units **Ans.**

22. Solve the following L. P. P. graphically
Maximise Z = 4x + y

Subject to following constraints :

$$x + y \le 50,$$
$$3x + y \le 90,$$
$$x \ge 10$$
$$x, y \ge 0 \quad \text{[4]}$$

Solution : We have,

Maximise $Z = 4x + y$

Subject to the constraints :

$$x + y \le 50$$
$$3x + y \le 90$$
$$x \ge 10$$
$$x, y \ge 0$$

Converting the given inequalities into equations, we obtain the following equations :

$$x + y = 50$$
$$3x + y = 90$$
$$x = 10$$

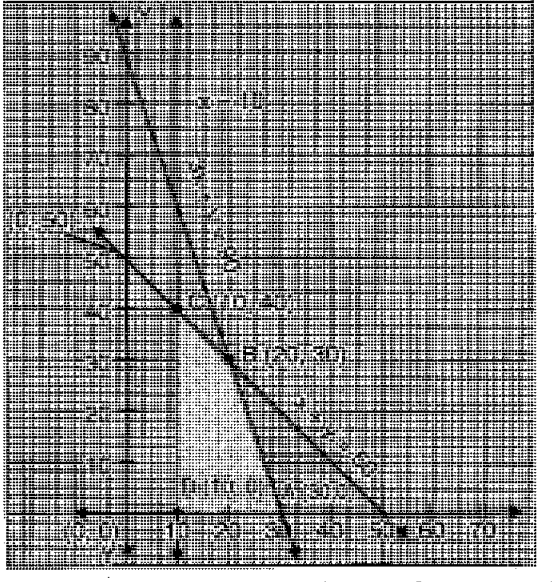
Then, $x + y = 50$

x	0	50
y	50	0

$3x + y = 90$

x	0	30
y	90	0

$x = 10$ is a line which is parallel to Y-axis



Plotting these points on the graph, we get the shaded feasible region *i.e.*, ABCD.

Corner point	Value of $Z = 4x + y$
A (30, 0)	$Z = 4 \times 30 + 0 = 120 \leftarrow$ Maximum
B (20, 30)	$Z = 4 \times 20 + 30 = 110$
C (10, 40)	$Z = 10 \times 4 + 40 = 80$
D (10, 0)	$Z = 10 \times 4 + 0 = 40$

$\therefore$ Maximum value of Z is 120 at (30, 0) **Ans.**

23. Find : $\int \frac{2x}{(x^2+1)(x^4+4)}dx$. **[4]**

Solution : Let, $\quad I = \int \frac{2x}{(x^2+1)(x^4+4)}dx$

$$= \int \frac{2x}{(x^2+1)((x^2)^2+4)}dx$$

Put $x^2 = t$,

$$\Rightarrow \quad 2xdx = dt$$

$$I = \int \frac{dt}{(t+1)(t^2+4)}$$

Let $\quad \frac{1}{(t+1)(t^2+4)} = \frac{A}{t+1} + \frac{Bt+C}{t^2+4} \quad$ ...(A)

$$\frac{1}{(t+1)(t^2+4)} = \frac{A(t^2+4) + (Bt+C)(t+1)}{(t+1)(t^2+4)}$$

$$1 = A(t^2+4) + (Bt + C)(t+1)$$

Equating coefficients of t^2, t and constant terms on both sides, we get

$$A + B = 0 \quad \Rightarrow \quad A = -B \quad ...(i)$$
$$B + C = 0 \quad \Rightarrow \quad B = -C \quad ...(ii)$$
$$4A + C = 1 \quad ...(iii)$$

From equations (i) and (ii) equations we get $A = C$.

From equation (iii)

$$4C + C = 1$$
$$5C = 1$$

or
$$C = \frac{1}{5}$$
$$A = C = \frac{1}{5}$$
$$B = -C = \frac{-1}{5}$$

From equation (A),

$$\Rightarrow \quad \frac{1}{(t+1)(t^2+4)} = \frac{1/5}{t+1} + \frac{\frac{-1}{5}t + \frac{1}{5}}{t^2+4}$$

$$\Rightarrow \int \frac{1}{(t+1)(t^2+4)}dt = \frac{1}{5}\int \frac{1}{t+1}dt - \frac{1}{5}\int \frac{t}{t^2+4}dt + \frac{1}{5}\int \frac{1}{t^2+4}dt + C$$

Where C is constant of integration.

$$I = \frac{1}{5}\int \frac{1}{t+1}dt - \frac{1}{10}\int \frac{2t}{t^2+4}dt + \frac{1}{5}\int \frac{1}{t^2+2^2}dt + C$$

$$= \frac{1}{5}\log|t+1| - \frac{1}{10}\log|t^2+4| + \frac{1}{5}\times\frac{1}{2}\tan^{-1}\frac{t}{2} + C$$

$$\Rightarrow I = \frac{1}{5}\log|x^2+1| - \frac{1}{10}\log|x^4+4| + \frac{1}{10}\tan^{-1}\frac{x^2}{2} + C$$

Ans.

SECTION — D

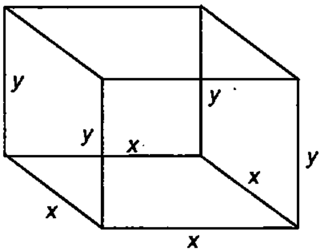
28. A metal box with a square base and vertical sides is to contain 1024 cm³. The material for the top and bottom costs ₹ 5 per cm² and the material for the sides costs ₹2.50 per cm². Find the least cost of the box. [6]

Solution : Let the length, breadth and height of the metal box be x cm, x cm and y cm respectively. It is given that,

$$x^2 y = 1024$$

$$\Rightarrow \quad y = \frac{1024}{x^2} \quad ...(i)$$

Let c be the total cost (in rupees) of material used to construct the box.



Then, $c = 5x^2 + 5x^2 + \frac{5}{2}\times 4xy$

$$= 10x^2 + 10xy$$
$$= 10x^2 + 10x \times \frac{1024}{x^2}$$
$$= 10x^2 + \frac{10240}{x}$$

Now, differentiating 'c' w.r.t. x, we get

$$\frac{dc}{dx} = 20x - \frac{10240}{x^2}$$

Again, differentiating w.r.t. x, we get

$$\frac{d^2c}{dx^2} = 20 + \frac{20480}{x^3}$$

For maxima or minima

Put $\frac{dc}{dx} = 0$

$$\Rightarrow \quad 20x - \frac{10240}{x^2} = 0$$
$$\Rightarrow \quad x^3 = 512$$
$$\Rightarrow \quad x^3 = 8^3$$
$$\Rightarrow \quad x = 8$$

when $x = 8$

$$\left(\frac{d^2c}{dx^2}\right)_{x=8} = 20 + \frac{20480}{8^3} > 0$$

Thus, the cost of the box is least when $x = 8$, putting $x = 8$ in equation (i) we obtain $y = 16$, so the dimensions of the box are $8 \times 8 \times 16$

Putting, $x = 8$ and $y = 16$ in $c = 10x^2 + 10xy$, we get

$$c = 10 \times 64 + 10 \times 8 \times 16 = 1920$$

Hence, the least cost of the box is ₹ 1920. **Ans.**

29. If A = $\begin{pmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{pmatrix}$, find A^{-1}. Using A^{-1} solve the system of equations :

$$\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 2;$$

$$\frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 5;$$

$$\frac{6}{x}+\frac{9}{y}-\frac{20}{z}=-4$$ **[6]**

Solution : Given, $A = \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}$

$$|A| = 2(120-45) - 3(-80-30) + 10(36+36)$$
$$= 2 \times 75 - 3 \times (-110) + 10 \times 72$$
$$= 150 + 330 + 720 = 1200 \neq 0$$

So, A is invertible.

$\therefore$ A^{-1} exists.

Let A_{ij} be the cofactors of elements a_{ij} in $A = [a_{ij}]$. Then,

$$A_{11} = (-1)^{1+1}\begin{bmatrix} -6 & 5 \\ 9 & -20 \end{bmatrix} = 120-45 = 75$$

$$A_{12} = (-1)^{1+2}\begin{bmatrix} 4 & 5 \\ 6 & -20 \end{bmatrix} = -(-80-30) = 110$$

$$A_{13} = (-1)^{1+3}\begin{bmatrix} 4 & -6 \\ 6 & 9 \end{bmatrix} = 36+36 = 72$$

$$A_{21} = (-1)^{2+1}\begin{bmatrix} 3 & 10 \\ 9 & -20 \end{bmatrix} = -(-60-90) = 150$$

$$A_{22} = (-1)^{2+2}\begin{bmatrix} 2 & 1 \\ -3 & -1 \end{bmatrix} = -40-60 = -100$$

$$A_{23} = (-1)^{2+3}\begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix} = -(18-18) = 0$$

$$A_{31} = (-1)^{3+1}\begin{bmatrix} 3 & 10 \\ -6 & 5 \end{bmatrix} = 15+60 = 75$$

$$A_{32} = (-1)^{3+2}\begin{bmatrix} 2 & 10 \\ 4 & 5 \end{bmatrix} = -(10-40) = 30$$

$$A_{33} = (-1)^{3+3}\begin{bmatrix} 2 & 3 \\ 4 & -6 \end{bmatrix} = -12-12 = -24$$

$$\text{Cofactor matrix of } A = \begin{bmatrix} 75 & 110 & 72 \\ 150 & -100 & 0 \\ 75 & 30 & -24 \end{bmatrix}$$

$$\text{adj } A = \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$

Given system of equations is,

$$\frac{2}{x}+\frac{3}{y}+\frac{10}{z}=2$$
$$\frac{4}{x}-\frac{6}{y}+\frac{5}{z}=5$$
$$\frac{6}{x}+\frac{9}{y}-\frac{20}{z}=-4$$

Let $\frac{1}{x}=u,\ \frac{1}{y}=v,\ \frac{1}{z}=w$

$\therefore$ The given linear equation becomes

$$2u+3v+10w = 2$$
$$4u-6v+5w = 5$$
$$6u+9v-20w = -4$$

$$A = \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}, \quad X = \begin{bmatrix} u \\ v \\ w \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 5 \\ -4 \end{bmatrix}$$

$X = A^{-1}B$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}^{-1}\begin{bmatrix} 2 \\ 5 \\ -4 \end{bmatrix}$$

$$= \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \\ -4 \end{bmatrix}$$

$$= \frac{1}{1200}\begin{bmatrix} 150+750-300 \\ 220-500-120 \\ 144+0+96 \end{bmatrix}$$

$$= \frac{1}{1200}\begin{bmatrix} 600 \\ -400 \\ 240 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/3 \\ 1/5 \end{bmatrix}$$

$u = 1/2 \Rightarrow x = 2$

$v = -1/3 \Rightarrow y = -3$

and $w = 1/5 \Rightarrow z = 5$

$\Rightarrow$ $x = 2,\ y = -3$ and $z = 5$ **Ans.**

Mathematics 2017 (Delhi) SET III

Time allowed : 3 hours **Maximum marks : 100**

Note :Except for the following questions, all the remaining questions have been asked in previous sets.

SECTION — B

12. If $y = \sin^{-1}\left(6x\sqrt{1-9x^2}\right)$, $\frac{-1}{3\sqrt{2}} < x < \frac{1}{3\sqrt{2}}$, then find $\frac{dy}{dx}$. **[2]**

Solution : Given, $y = \sin^{-1}\left(6x\sqrt{1-9x^2}\right)$

Put $x = \frac{\sin\theta}{3} \Rightarrow \theta = \sin^{-1} 3x$.

$$y = \sin^{-1}\left(6.\frac{\sin\theta}{3}\sqrt{1-\frac{9\sin^2\theta}{9}}\right)$$

$$\Rightarrow \quad y = \sin^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right)$$

$$\Rightarrow \quad y = \sin^{-1}\left(2\sin\theta\sqrt{\cos^2\theta}\right)$$

$$\Rightarrow \quad y = \sin^{-1}(2\sin\theta\cos\theta)$$

$$\Rightarrow \quad y = \sin^{-1}(\sin 2\theta) = 2\theta = 2\sin^{-1} 3x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2}{\sqrt{1-9x^2}} \times 3 = \frac{6}{\sqrt{1-9x^2}}$$ **Ans.**

SECTION — C

20. Solve the following L.P.P. graphically :

Maximise Z = 20x + 10y

Subject to the following constraints :

$$x + 2y \le 28,$$
$$3x + y \le 24,$$
$$x \ge 2$$
$$x, y \ge 0$$ **[4]**

Solution : We have

Maximise Z = 20x + 10y

Subject to the constraints :

$$x + 2y \le 28$$
$$3x + y \le 24$$
$$x \ge 2$$
$$x, y \ge 0$$

Converting the given inequalities into equations, we obtain the following equations :

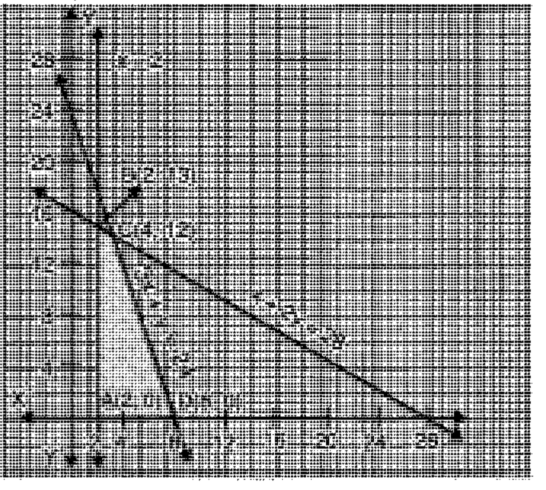
$$x + 2y = 28$$
$$3x + y = 24$$
$$x = 2$$

Then $x + 2y = 28$ $\quad$ $3x + y = 24$

x	0	28
y	14	0

x	0	8
y	24	0

$x = 2$ is a line which is parallel to y-axis.



Plotting these points on the graph, we get the shaded feasible region *i.e.*, ABCD

Corner point	Value of Z = 20x + 10y
A (2, 0)	Z = 40 + 0 = 40
B (2, 13)	Z = 40 + 130 = 170
C (4, 12)	Z = 80 + 120 = 200 ← Maximum
D (8, 0)	Z = 160 + 0 = 160

∴ Maximum value of Z is 200 at (4, 12). **Ans.**

21. Show that the family of curves for which $\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$, is given by $x^2 - y^2 = cx$. **[4]**

Solution : Given family of curve,

$$\frac{dy}{dx} = \frac{x^2+y^2}{2xy} \quad ...(i)$$

Put $y = vx$

$$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

From equation (i),

$$v + x\frac{dv}{dx} = \frac{x^2+(vx)^2}{2x(vx)} = \frac{x^2(1+v^2)}{x^2(2v)}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{1+v^2}{2v} - v$$

$$= \frac{1+v^2-2v^2}{2v} = \frac{1-v^2}{2v}$$

$$\Rightarrow \quad \frac{2v}{1-v^2}dv = \frac{dx}{x}$$

$$\Rightarrow \int \frac{2v}{1-v^2}dv = \int \frac{dx}{x}$$

$\Rightarrow -\log|1-v^2| + \log C = \log x,$

where log C is constant of integration.

$$\log\left|\frac{c}{1-\frac{y^2}{x^2}}\right| = \log x$$

$$\Rightarrow \quad \log\left|\frac{cx^2}{x^2-y^2}\right| = \log x$$

$$\Rightarrow \quad \frac{cx^2}{x^2-y^2} = x$$

$\Rightarrow \quad cx = x^2 - y^2$ **Hence Proved.**

22. Find : $\int \frac{(3\sin x - 2)\cos x}{13-\cos^2 x - 7\sin x}dx$ **[4]**

Solution : Let $I = \int \frac{(3\sin x - 2)\cos x}{13-\cos^2 x - 7\sin x}dx$

$$= \int \frac{(3\sin x - 2)\cos x}{\sin^2 x - 7\sin x + 12}dx$$

$$(\because \cos^2 x = 1 - \sin^2 x)$$

Put $\sin x = t$

$\Rightarrow \cos x\, dx = dt$

$$I = \int \frac{(3t-2)}{t^2-7t+12}dt = \int \frac{3t-2}{(t-3)(t-4)}dt$$

Let, $\frac{3t-2}{(t-3)(t-4)} = \frac{A}{t-3} + \frac{B}{t-4}$ ...(A)

$$\Rightarrow \quad \frac{3t-2}{(t-3)(t-4)} = \frac{A(t-4)+B(t-3)}{(t-3)(t-4)}$$

$$\Rightarrow \quad 3t-2 = A(t-4) + B(t-3)$$

Equating coefficient of t and constant term on both sides, we get

$A + B = 3$ ...(i)

$-4A - 3B = -2$ ...(ii)

Solving the above two equations, we get

$$A = -7, B = 10$$

From equation (A),

$$\frac{3t-2}{(t-3)(t-4)} = \frac{-7}{t-3} + \frac{10}{t-4}$$

$$\int \frac{3t-2}{(t-3)(t-4)}dt = -7\int \frac{1}{t-3}dt + 10\int \frac{1}{t-4}dt$$

$I = -7\log|t-3| + 10\log|t-4| + C,$

where C is constant of integration.

$I = 10\log|\sin x - 4| - 7\log|\sin x - 3| + C.$

Ans.

23. Solve the following equation for x :

$$\cos(\tan^{-1} x) = \sin\left(\cot^{-1}\frac{3}{4}\right)$$ **[4]**

Solution : Given equation is,

$$\cos(\tan^{-1} x) = \sin\left(\cot^{-1}\frac{3}{4}\right) \quad \text{...(i)}$$

$$\Rightarrow \quad \cos(\tan^{-1} x) = \cos\left[\frac{\pi}{2} - \cot^{-1}\frac{3}{4}\right]$$

$$\Rightarrow \quad \tan^{-1} x = \frac{\pi}{2} - \cot^{-1}\frac{3}{4}$$

$$\Rightarrow \quad \frac{\pi}{2} - \cot^{-1} x = \frac{\pi}{2} - \cot^{-1}\frac{3}{4}$$

$$\left(\because \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}\right)$$

$$\Rightarrow \quad \cot^{-1} x = \cot^{-1}\frac{3}{4}$$

or $x = \frac{3}{4}$ **Ans.**

SECTION — D

28. If $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & -2 \\ -3 & 1 & -1 \end{bmatrix}$, find A^{-1} and hence solve the system of equations $2x + y - 3z = 13$, $3x + 2y + z = 4$, $x + 2y - z = 8$ **[6]**

Solution : Given,

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 2 \\ -3 & 1 & -1 \end{bmatrix}$$

$$|A| = 2(-2-2) - 3(-1+6) + 1(1+6)$$

$$= -8 - 15 + 7 = -16 \neq 0$$

So A is invertible.

$\therefore A^{-1}$ exists

Let A_{ij} be the cofactors of elements a_{ij} in $A = [a_{ij}]$. Then,

$$A_{11} = (-1)^{1+1}\begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix} = -2 - 2 = -4$$

$$A_{12} = (-1)^{1+2}\begin{bmatrix} 1 & 2 \\ -3 & -1 \end{bmatrix} = -(-1+6) = -5$$

$$A_{13} = (-1)^{1+3}\begin{bmatrix} 1 & 2 \\ -3 & -1 \end{bmatrix} = 1 + 6 = 7$$

$$A_{21} = (-1)^{2+1}\begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix} = -(-3-1) = 4$$

$$A_{22} = (-1)^{2+2}\begin{bmatrix} 2 & 1 \\ -3 & -1 \end{bmatrix} = -2 + 3 = 1$$

$$A_{23} = (-1)^{2+3}\begin{bmatrix} 2 & 3 \\ -3 & 1 \end{bmatrix} = -(2 + 9) = -11$$

$$A_{31} = (-1)^{3+1}\begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} = 6 - 2 = 4$$

$$A_{32} = (-1)^{3+2}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = -(4 - 1) = -3$$

$$A_{33} = (-1)^{3+3}\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = 4 - 3 = 1$$

$$\therefore a_{ij} = \begin{bmatrix} -4 & -5 & 7 \\ 4 & 1 & -11 \\ 4 & -3 & 1 \end{bmatrix}$$

$$\text{adj. A} = [A_{ij}]^T = \begin{bmatrix} -4 & 4 & 4 \\ -5 & 1 & -3 \\ 7 & -11 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj. A} = \frac{-1}{16}\begin{bmatrix} -4 & 4 & 4 \\ -5 & 1 & -3 \\ 7 & -11 & 1 \end{bmatrix}$$

From the given linear equations, we have

Let $$C = \begin{bmatrix} 2 & 1 & -3 \\ 3 & 2 & 1 \\ 1 & 2 & -1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, D = \begin{bmatrix} 13 \\ 4 \\ 8 \end{bmatrix}$$

by matrix method,

$$X = C^{-1}D = (A^T)^{-1}D = (A^{-1})^T D$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{-1}{16}\begin{bmatrix} -4 & 4 & 4 \\ -5 & 1 & -3 \\ 7 & -11 & 1 \end{bmatrix}^T \begin{bmatrix} 13 \\ 4 \\ 8 \end{bmatrix}$$

$$= \frac{-1}{16}\begin{bmatrix} -4 & -5 & 7 \\ 4 & 1 & -11 \\ 4 & -3 & 1 \end{bmatrix}\begin{bmatrix} 13 \\ 4 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{-1}{16}\begin{bmatrix} -52 - 20 + 56 \\ 52 + 4 - 88 \\ 52 - 12 + 8 \end{bmatrix}$$

$$= \frac{-1}{16}\begin{bmatrix} -16 \\ -32 \\ 48 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

$\Rightarrow$ $x = 1, y = 2, z = -3.$ **Ans.**

29. Find the particular solution of the differential equation. $\tan x \cdot \frac{dy}{dx} = 2x \tan x + x^2 - y$; ($\tan x \neq 0$) given that $y = 0$ when $x = \frac{\pi}{2}$. **[6]**

Solution : Given differential equation is,

$$\tan x.\ \frac{dy}{dx} = 2x \tan x + x^2 - y$$

$$\Rightarrow \quad \frac{dy}{dx} = 2x + x^2 \cot x - y \cot x$$

$$\Rightarrow \quad \frac{dy}{dx} + y \cot x = 2x + x^2 \cot x,$$

which is of the form

$$\frac{dy}{dx} + Py = Q,$$

where $P = \cot x$, $Q = 2x + x^2 \cot x$

$$\text{I. F.} = e^{\int P\,dx} = e^{\int \cot x\,dx} = e^{\log|\sin x|} = \sin x$$

$\therefore$ Required solution is

$$y.\sin x = \int (2x + x^2 \cot x) \sin x\,dx + C$$

$$= \int 2x \sin x\,dx + \int x^2 \cot x \sin x\,dx + C$$

$$= 2\int x \sin x\,dx + \int x^2 \cos x\,dx + C$$

$$= 2\left[x.(-\cos x) - \int(-\cos x)dx\right] + x^2 .\sin x - \int 2x . \sin x\,dx + C$$

$$= -2x \cos x + 2 \sin x + x^2 \sin x - 2\left[x.(-\cos x) - \int -\cos x\,dx\right] + C$$

$$= -2x \cos x + 2 \sin x + x^2 \sin x + 2x \cos x - 2\sin x + C$$

$$\therefore \quad y \sin x = x^2 \sin x + C \quad \text{...(i)}$$

Give that $y = 0$ when $x = \frac{\pi}{2}$

$$0 = \frac{\pi^2}{4} \sin \frac{\pi}{2} + C$$

$$\Rightarrow \quad C = -\frac{\pi^2}{4}$$

From equation (i)

$$y \sin x = x^2 \sin x - \frac{\pi^2}{4}$$

$$\Rightarrow \quad (x^2 - y) \sin x = \frac{\pi^2}{4}$$ **Ans.**

Mathematics 2018

Time allowed : 3 hours **Maximum marks : 100**

SECTION – A

1. Find the value of $\tan^{-1}\sqrt{3} - \cot^{-1}(-\sqrt{3})$ **[1]**

Solution : We have,

$$\tan^{-1}\sqrt{3} - \cot^{-1}(-\sqrt{3}) = \tan^{-1}\left(\tan\frac{\pi}{3}\right) - \cot^{-1}\left(\cot\pi - \frac{\pi}{6}\right)$$

$$= \frac{\pi}{3} - \left(\pi - \frac{\pi}{6}\right)$$

$$= \frac{\pi}{3} - \frac{5\pi}{6} = \frac{2\pi - 5\pi}{6}$$

$$= \frac{-3\pi}{6} = \frac{-\pi}{2}$$ **Ans.**

2. If the matrix $A = \begin{bmatrix} 0 & a & -3 \\ 2 & 0 & -1 \\ b & 1 & 0 \end{bmatrix}$ is skew-symmetric, find the values of 'a' and 'b'. **[1]**

Solution : Given, $A = \begin{bmatrix} 0 & a & -3 \\ 2 & 0 & -1 \\ b & 1 & 0 \end{bmatrix}$

A is given to be skew-symmetric matrix

$\therefore \quad A^T = -A$

$$\therefore \quad \begin{bmatrix} 0 & 2 & b \\ a & 0 & 1 \\ -3 & -1 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & a & -3 \\ 2 & 0 & -1 \\ b & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 & b \\ a & 0 & 1 \\ -3 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -a & 3 \\ -2 & 0 & 1 \\ -b & -1 & 0 \end{bmatrix}$$

On comparing both sides we get

$$-a = 2 \text{ and } b = 3$$

$\Rightarrow \quad a = -2 \text{ and } b = 3$ **Ans.**

3. Find the magnitude of each of the two vectors $\vec{a}$ and $\vec{b}$, having the same magnitude such that the angle between them is 60° and their scalar product is $\frac{9}{2}$. **[1]**

Solution : Let $\vec{a}$ and $\vec{b}$ be two such vectors.

Now, $\quad \vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta \quad$ ...(i)

It is given that $|\vec{a}| = |\vec{b}|, \theta = 60°$ and $\vec{a}\cdot\vec{b} = \frac{9}{2}$

$\therefore$ Putting these values in equation (i)

$$\frac{9}{2} = |\vec{a}||\vec{a}|\cos 60°$$

$$\Rightarrow \quad \frac{9}{2} = |\vec{a}|^2 \cdot \frac{1}{2}$$

$$\Rightarrow \quad |\vec{a}|^2 = 9$$

$$\Rightarrow \quad |\vec{a}| = 3$$

$\therefore \quad |\vec{a}| = |\vec{b}| = 3$ **Ans.**

4. If $a * b$ denotes the larger of 'a' and 'b' and if $a \, o \, b = (a * b) + 3$, then write the value of (5) o (10), where $*$ and o are binary operations.** **[1]**

SECTION – B

5. Prove that :

$3\sin^{-1}x = \sin^{-1}(3x - 4x^3)$, $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ **[2]**

Solution : R.H.S. $= \sin^{-1}(3x - 4x^3)$

Putting $x = \sin\theta$ in R.H.S, we get

$$\text{R.H.S.} = \sin^{-1}(3\sin\theta - 4\sin^3\theta)$$

$$= \sin^{-1}(\sin 3\theta)$$

Now, $\quad -\frac{1}{2} \le x \le \frac{1}{2}$

$$\Rightarrow \quad -\frac{1}{2} \le \sin\theta \le \frac{1}{2}$$

$$\Rightarrow \quad -\frac{\pi}{6} \le \theta \le \frac{\pi}{6}$$

$$\Rightarrow \quad -\frac{\pi}{2} \le 3\theta \le \frac{\pi}{2}$$

Hence, $\sin^{-1}(\sin 3\theta) = 3\theta\left(\text{as } \frac{-\pi}{2} \le 3\theta \le \frac{\pi}{2}\right)$

$$\Rightarrow \quad \text{R.H.S.} = 3\theta$$

$$= 3\sin^{-1}x$$

$$[\because x = \sin\theta \Rightarrow \theta = \sin^{-1}x]$$

$= \text{L.H.S.}$ **Hence Proved.**

6. Given $A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$, compute A^{-1} and show that $2A^{-1} = 9I - A$. **[2]**

Solution : Given, $\quad A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$

$$\Rightarrow \quad |A| = 14 - 12 = 2$$

****Answer is not given due to the change in present syllabus**

$$\therefore \quad A^{-1} = \frac{1}{2}\begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$$

Given, $\quad \text{L.H.S.} = 2A^{-1} = 2\left(\frac{1}{2}\right)\begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$

$$\Rightarrow \quad \text{L.H.S.} = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix} \quad ...(i)$$

and $\quad \text{R.H.S.} = 9I - A = 9\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$

$$= \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$$

$$\Rightarrow \quad \text{R.H.S.} = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix} \quad ...(ii)$$

From equations (i) and (ii),

L.H.S. = R.H.S. **Hence Proved.**

7. Differentiate $\tan^{-1}\left(\frac{1+\cos x}{\sin x}\right)$ with respect to x. [2]

Solution : Let $y = \tan^{-1}\left(\frac{1+\cos x}{\sin x}\right)$

$$= \tan^{-1}\left(\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right)$$

$$= \tan^{-1}\left(\cot\frac{x}{2}\right)$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{2}-\frac{x}{2}\right)\right]$$

$$= \frac{\pi}{2}-\frac{x}{2}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{2}-\frac{x}{2}\right) = -\frac{1}{2} \quad \textbf{Ans.}$$

8. The total cost C(x) associated with the production of x units of an item is given by $C(x) = 0.005x^3 - 0.02x^2 + 30x + 5000$. Find the marginal cost when 3 units are produced, where by marginal cost we mean the instantaneous rate of change of total cost at any level of output. [2]

Solution : Cost function is given as

$$C(x) = 0.005x^3 - 0.02x^2 + 30x + 5000$$

$$\text{Marginal cost (MC)} = \frac{d}{dx}(C(x))$$

$$= 0.005(3x^2) - 0.02(2x) + 30$$

$$= 0.015x^2 - 0.04x + 30$$

When $x = 3$, $MC = 0.015(3)^2 - 0.04(3) + 30$

$$= 0.135 - 0.12 + 30$$

$$= 30.015 \quad \textbf{Ans.}$$

9. Evaluate : $\int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x}dx$

Solution : We have,

$$\int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x}dx = \int \frac{\cos^2 x - \sin^2 x + 2\sin^2 x}{\cos^2 x}dx$$

$$(\because \cos 2x = \cos^2 x - \sin^2 x)$$

$$= \int \frac{\cos^2 x + \sin^2 x}{\cos^2 x}dx$$

$$= \int \frac{1}{\cos^2 x}dx$$

$$= \int \sec^2 x\, dx$$

$$= \tan x + C \quad \textbf{Ans.}$$

10. Find the differential equation representing the family of curves $y = ae^{bx+5}$, where a and b are arbitrary constants. [2]

Solution : Given curve is

$$y = ae^{bx+5} \quad ...(i)$$

Differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx} = ae^{bx+5}\cdot\frac{d}{dx}(bx+5)$$

$$\Rightarrow \quad \frac{dy}{dx} = ae^{bx+5}\cdot b \quad ...(ii)$$

$$\Rightarrow \quad \frac{dy}{dx} = y\cdot b \quad \text{[From (i)]}$$

Differentiating again w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{dy}{dx}\cdot b$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{dy}{dx}\left(\frac{dy}{dx}\cdot\frac{1}{y}\right) \quad \left[\because b = \frac{1}{y}\frac{dy}{dx}\right]$$

$$\Rightarrow \quad y\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2 \quad \textbf{Ans.}$$

11. If θ is the angle between two vectors $\hat{i}-2\hat{j}+3\hat{k}$ and $3\hat{i}-2\hat{j}+\hat{k}$, find $\sin\theta$. [2]

Solution : Let $\vec{a} = \hat{i}-2\hat{j}+3\hat{k}$

$$\vec{b} = 3\hat{i}-2\hat{j}+\hat{k}$$

We know, $\quad \vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$$\Rightarrow \quad \cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

$$= \frac{(\hat{i}-2\hat{j}+3\hat{k})\cdot(3\hat{i}-2\hat{j}+\hat{k})}{|\hat{i}-2\hat{j}+3\hat{k}||3\hat{i}-2\hat{j}+\hat{k}|}$$

$$= \frac{1(3)+(-2)(-2)+3(1)}{\sqrt{1+4+9}\sqrt{9+4+1}}$$

$$= \frac{3+4+3}{14} = \frac{10}{14} = \frac{5}{7}$$

Now, $\sin\theta = \sqrt{1-\cos^2\theta}$

$$= \sqrt{1-\left(\frac{5}{7}\right)^2} = \sqrt{1-\frac{25}{49}}$$

$$= \sqrt{\frac{49-25}{49}}$$

$\Rightarrow \sin\theta = \sqrt{\frac{24}{49}} = \frac{2\sqrt{6}}{7}$ **Ans.**

12. A black and a red die are rolled together. Find the conditional probability of obtaining the sum 8, given that the red die resulted in a number less than 4. [2]

Solution : The sample space has 36 Outcomes. Let A be event that the sum of observations is 8.

$\therefore A = \{(2,6),(3,5),(5,3),(4,4),(6,2)\}$

$\Rightarrow n(A) = 5$

$\Rightarrow P(A) = \frac{5}{36}$

Let B be event that observation on red die is less than 4.

$\therefore B = \{(1,1),(2,1),(3,1),(4,1),(5,1),(6,1),(1,2),(2,2),(3,2),(4,2),(5,2),(6,2),(1,3),(2,3),(3,3),(4,3),(5,3),(6,3)\}$

$\therefore n(B) = 18$

$\Rightarrow P(B) = \frac{18}{36} = \frac{1}{2}$

Clearly $A \cap B = \{(5,3),(6,2)\}$

$\therefore n(A\cap B) = 2$

$\therefore P(A\cap B) = \frac{2}{36} = \frac{1}{18}$

$$P\left(\frac{A}{B}\right) = \frac{P(A\cap B)}{P(B)} = \frac{\frac{1}{18}}{\frac{1}{2}}$$

$= \frac{2}{18} = \frac{1}{9}$ **Ans.**

SECTION–C

13. Using properties of determinants, prove that :

$$\begin{vmatrix} 1 & 1 & 1+3x \\ 1+3y & 1 & 1 \\ 1 & 1+3z & 1 \end{vmatrix} = 9(3xyz + xy + yz + zx)$$ **[4]**

Solution :

$$\text{L.H.S.} = \begin{vmatrix} 1 & 1 & 1+3x \\ 1+3y & 1 & 1 \\ 1 & 1+3z & 1 \end{vmatrix}$$

Taking x, y and z common from R_1, R_2 and R_3 respectively, we get

$$= xyz\begin{vmatrix} \frac{1}{x} & \frac{1}{x} & \frac{1}{x}+3 \\ \frac{1}{y}+3 & \frac{1}{y} & \frac{1}{y} \\ \frac{1}{z} & \frac{1}{z}+3 & \frac{1}{z} \end{vmatrix}$$

On applying $R_1 \to R_1 + R_2 + R_3$, we get

$$= xyz\begin{vmatrix} \frac{1}{x}+\frac{1}{y}+\frac{1}{z}+3 & \frac{1}{x}+\frac{1}{y}+\frac{1}{z}+3 & \frac{1}{x}+\frac{1}{y}+\frac{1}{z}+3 \\ \frac{1}{y}+3 & \frac{1}{y} & \frac{1}{y} \\ \frac{1}{z} & \frac{1}{z}+3 & \frac{1}{z} \end{vmatrix}$$

Taking common $\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+3\right)$ from R_1, we get

$$= xyz\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+3\right)\begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{y}+3 & \frac{1}{y} & \frac{1}{y} \\ \frac{1}{z} & \frac{1}{z}+3 & \frac{1}{z} \end{vmatrix}$$

On applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we get

$$= xyz\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+3\right)\begin{vmatrix} 1 & 0 & 0 \\ \frac{1}{y}+3 & -3 & -3 \\ \frac{1}{z} & & \end{vmatrix}$$

On expanding along R_1, we get

$$= xyz\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+3\right)1\,(0+9)$$

$$= xy\left(\frac{yz+xz+xy+3xyz}{xyz}\right)(9)$$

$$= 9\,(3xyz + xy + yz + zx)$$

$= \text{R.H.S.}$ **Hence Proved.**

14. If $(x^2+y^2)^2 = xy$, find $\frac{dy}{dx}$. [4]

Solution : We have,

$$(x^2+y^2)^2 = xy \quad ...(i)$$

Differentiating (i) w.r.t. x, we get

$$2(x^2+y^2)\left(2x+2y.\frac{dy}{dx}\right) = x\frac{dy}{dx}+y$$

$$4x(x^2+y^2)+4y\frac{dy}{dx}(x^2+y^2) = x\frac{dy}{dx}+y$$

$$\Rightarrow \frac{dy}{dx}[4y(x^2+y^2)-x] = y-4x(x^2+y^2)$$

$$\Rightarrow \frac{dy}{dx} = \frac{y-4x(x^2+y^2)}{4y(x^2+y^2)-x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y-4x^3-4xy^2}{4x^2y+4y^3-x}$$

Ans.

OR

If $x = a(2\theta - \sin 2\theta)$ and $y = a(1-\cos 2\theta)$, find $\frac{dy}{dx}$ when $\theta = \frac{\pi}{3}$.

Solution : Given, $x = a(2\theta - \sin 2\theta)$

and $y = a(1-\cos 2\theta)$

Differentiating x and y w.r.t. θ, we get

$$\frac{dx}{d\theta} = a(2-\cos 2\theta\cdot 2) = 2a(1-\cos 2\theta)$$

and

$$\frac{dy}{d\theta} = a(\sin 2\theta\cdot 2) = 2a\sin 2\theta$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{2a\sin 2\theta}{2a(1-\cos 2\theta)} = \frac{\sin 2\theta}{1-\cos 2\theta} = \frac{2\sin\theta\cos\theta}{2\sin^2\theta} = \cot\theta$$

$$\therefore \left[\frac{dy}{dx}\right]_{\theta=\frac{\pi}{3}} = \cot\frac{\pi}{3} = \frac{1}{\sqrt{3}}$$

Ans.

15. If $y = \sin(\sin x)$, prove that :

$$\frac{d^2y}{dx^2}+\tan x\frac{dy}{dx}+y\cos^2 x = 0$$

[4]

Solution : Given, $y = \sin(\sin x)$

Differentiating y w.r.t. x, we get

$$\frac{dy}{dx} = \cos(\sin x)\cdot\cos x$$

Differentiating again w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \cos(\sin x)(-\sin x)+\cos x[-\sin(\sin x)\cdot\cos x]$$

$$= -\sin x.\cos(\sin x)-\cos^2 x\cdot\sin(\sin x)$$

Now, L.H.S. $= \frac{d^2y}{dx^2}+\tan x\frac{dy}{dx}+y\cos^2 x$

$$= -\sin x\cdot\cos(\sin x)-\cos^2 x\cdot\sin(\sin x) + \tan x[\cos(\sin x)\cdot\cos x] + \cos^2 x\cdot\sin(\sin x)$$

$$= -\sin x\cos(\sin x)+\frac{\sin x}{\cos x}[\cos(\sin x)\cdot\cos x]$$

$$= -\sin x\cdot\cos(\sin x)+\sin x\cdot\cos(\sin x)$$

$= 0 =$ R.H.S. **Hence Proved.**

16. Find the equations of the tangent and the normal to the curve $16x^2+9y^2=145$ at the point (x_1, y_1), where $x_1 = 2$ and $y_1 > 0$. **[4]**

Solution : Given curve is

$$16x^2+9y^2 = 145 \quad \text{....(i)}$$

Since (x_1, y_1) lies on equation (i),

$$\therefore 16x_1^2+9y_1^2 = 145$$

$$\Rightarrow 16(2)^2+9y_1^2 = 145 \quad [\because x_1 = 2 \text{ (given)}]$$

$$\Rightarrow 9y_1^2 = 145-64 = 81$$

$$\Rightarrow y_1 = 3 \quad [\because y_1 > 0 \text{ (given)}]$$

$\therefore$ Point of contact is (2, 3).

Differentiating equation (i) w.r.t. x, which will give us the slope of the tangent.

$$32x+18y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{32x}{18y}$$

Slope of tangent at (2, 3)

$$= \left[\frac{dy}{dx}\right]_{(2,3)} = -\frac{32(2)}{18(3)} = -\frac{64}{54} = -\frac{32}{27}$$

$\therefore$ Equation of tangent is

$$y-3 = m(x-2)$$

$$\Rightarrow y-3 = -\frac{32}{27}(x-2)$$

$$\Rightarrow 27y-81 = -32x+64$$

$$\Rightarrow 32x+27y = 145$$

The slope of the normal $= \frac{-1}{\text{Slope of tangent}}$

$$= \frac{27}{32}$$

$\therefore$ Equation of normal is

$$\Rightarrow \quad y-3 = \frac{27}{32}(x-2)$$

$$\Rightarrow \quad 32y-96 = 27x-54$$

$$\Rightarrow \quad 27x-32y = -42$$ **Ans.**

OR

Find the intervals in which the function

$$f(x) = \frac{x^4}{4} - x^3 - 5x^2 + 24x + 12 \text{ is}$$

(a) strictly increasing,
(b) strictly decreasing.

Solution : Given function is

$$f(x) = \frac{x^4}{4} - x^3 - 5x^2 + 24x + 12$$

$$\Rightarrow \quad f'(x) = \frac{4x^3}{4} - 3x^2 - 10x + 24$$

$$\Rightarrow \quad f'(x) = x^3 - 3x^2 - 10x + 24$$

For critical points, put $f'(x) = 0$

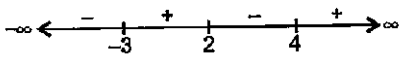
$$\therefore \quad x^3 - 3x^2 - 10x + 24 = 0$$

$$\Rightarrow \quad (x-2)(x^2 - x - 12) = 0$$

$$\Rightarrow \quad (x-2)(x-4)(x+3) = 0$$

$$\Rightarrow \quad x = 2, 4, -3$$

Therefore, we have the intervals $(-\infty, -3)$, $(-3, 2)$, $(2, 4)$ and $(4, \infty)$



Since $f'(x) > 0$ in $(-3, 2) \cup (4, \infty)$.

$\therefore$ $f(x)$ is strictly increasing in interval $(-3, 2) \cup (4, \infty)$

and $\quad f'(x) < 0$ in $(-\infty, -3) \cup (2, 4)$

$\therefore f(x)$ is strictly decreasing in $(-\infty, -3) \cup (2, 4)$.

Ans.

17. An open tank with a square base and vertical sides is to be constructed from a metal sheet so as to hold a given quantity of water. Show that the cost of material will be least when depth of the tank is half of its width. If the cost is to be borne by nearby settled lower income families, for whom water will be provided, what kind of value is hidden in this question ? [4]

Solution : Let the length, breadth and height of the open tank be x, x and y units respectively.

Then, $\quad$ Volume (V) $= x^2 y$ ...(i)

Total surface area (S) $= x^2 + 4xy$ ...(ii)

$$S = x^2 + 4x\frac{V}{x^2} \quad \text{[using (i)]}$$

$$\Rightarrow \quad S = x^2 + \frac{4V}{x}$$

$$\Rightarrow \quad \frac{dS}{dx} = 2x - \frac{4V}{x^2}$$

For critical points, put

$$\frac{dS}{dx} = 0$$

$$\Rightarrow \quad 2x - \frac{4V}{x^2} = 0$$

$$\Rightarrow \quad 2x^3 = 4V$$

$$\Rightarrow \quad 2x^3 = 4x^2 y \quad \text{[using (i)]}$$

$$\Rightarrow \quad x = 2y \quad \text{...(iii)}$$

$$\text{Now,} \quad \frac{d^2S}{dx^2} = 2 + \frac{8V}{x^3}$$

$$= 2 + \frac{8V}{8y^3} \quad \text{[using (iii)]}$$

$$= 2 + \frac{V}{y^3} > 0$$

Area is minimum, thus cost is minimum when $x = 2y$.

i.e., depth of tank is half of the width.

Value : Any relevant value. **Ans.**

18. Find $\int \frac{2\cos x}{(1 \quad \sin x)(1+\sin^2 x)} dx$ [4]

Solution : Let $I = \int \frac{2\cos x}{(1-\sin x)(1+\sin^2 x)} dx$

Put $\quad \sin x = t$

$\Rightarrow \quad \cos x\, dx = dt$

$$\therefore \quad I = \int \frac{2dt}{(1-t)(1+t^2)}$$

$$\text{Let} \quad \frac{2}{(1-t)(1+t^2)} = \frac{A}{1-t} + \frac{Bt+C}{1+t^2}$$

$$2 = A(1+t^2) + (Bt+C)(1-t)$$

$$\Rightarrow \quad 2 = (A-B)t^2 + (B-C)t + A + C$$

Put $t = 1$, we get

$$2 = 2A$$

$$\Rightarrow \quad A = 1$$

Comparing coefficients of t^2 and t on both sides,

$$A - B = 0$$

$$\Rightarrow \quad B = A$$

$$\Rightarrow \quad B = 1$$

$$\text{Also,} \quad B - C = 0$$

$$\Rightarrow \quad B = C = 1$$

$$\therefore \quad I = \int \left(\frac{1}{1-t} + \frac{t+1}{t^2+1}\right) dt$$

$$= \frac{\log(1-t)}{-1} + \frac{1}{2}\int \frac{2t}{t^2+1} dt + \int \frac{1}{t^2+1} dt$$

$$= -\log|1-t| + \frac{1}{2}\log|t^2+1| + \tan^{-1} t + C$$

$$= -\log(1-\sin x) + \frac{1}{2}\log(\sin^2 x + 1) + \tan^{-1}(\sin x) + C$$

Ans.

19. Find the particular solution of the differential equation $e^x \tan y\, dx + (2 - e^x) \sec^2 y\, dy = 0$, given that $y = \frac{\pi}{4}$ when $x = 0$. **[4]**

Solution : Given differential equation is,

$e^x \tan y\, dx + (2 - e^x) \sec^2 y\, dy = 0$

$\Rightarrow \quad e^x \tan y\, dx = (e^x - 2) \sec^2 y\, dy$

$\Rightarrow \quad \frac{e^x}{e^x - 2} dx = \frac{\sec^2 y}{\tan y} dy$

Integrating both sides, we get

$$\int \frac{e^x}{e^x - 2} dx = \int \frac{\sec^2 y}{\tan y} dy$$

$\Rightarrow \quad \log |e^x - 2| = \log |\tan y| + \log C$

$\Rightarrow \quad \log |e^x - 2| = \log |C \tan y|$

$\Rightarrow \quad e^x - 2 = C \tan y \quad$ ...(i)

Put $y = \frac{\pi}{4}$ when $x = 0$

$$e^0 - 2 = C\left(\tan\frac{\pi}{4}\right)$$

$\Rightarrow \quad -1 = C$

$\Rightarrow \quad C = -1$

$\therefore$ From equation (i),

$$e^x - 2 = -\tan y \Rightarrow y = \tan^{-1}(2 - e^x)$$

which is the required solution. **Ans.**

OR

Find the particular solution of the differential equation $\frac{dy}{dx} + 2y \tan x = \sin x$, given than $y = 0$ when $x = \frac{\pi}{3}$.

Solution : Given differential equation is,

$$\frac{dy}{dx} + (2\tan x)y = \sin x \quad \text{....(i)}$$

which is linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

where $P = 2 \tan x$, $Q = \sin x$

Integrating factor,

$$\text{I.F.} = e^{\int P.dx} = e^{\int 2\tan x.dx} = e^{2\log \sec x} = \sec^2 x$$

Solution of equation (i) is given by

$y \cdot \text{I.F.} = \int Q \cdot \text{I.F.}\, dx + C$

$\Rightarrow \quad y \sec^2 x = \int \sin x \cdot \sec^2 x \cdot dx + C$

$\Rightarrow \quad y \sec^2 x = \int \sec x \cdot \tan x \cdot dx + C$

$\Rightarrow \quad y \sec^2 x = \sec x + C$

Put $y = 0$ when $x = \frac{\pi}{3}$

$$0.\sec^2\frac{\pi}{3} = \sec\frac{\pi}{3} + C$$

$\Rightarrow \quad 0 = 2 + C$

$\Rightarrow \quad C = -2$

Hence, particular solution is

$$y \sec^2 x = \sec x - 2$$

or $\quad y = \cos x - 2\cos^2 x$ **Ans.**

20. Let $\vec{a} = 4\hat{i} + 5\hat{j} - \hat{k}$, $\vec{b} = \hat{i} - 4\hat{j} + 5\hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j} - \hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{c}$ & $\vec{b}$ and $\vec{d} \cdot \vec{a} = 21$. **[4]**

Solution : Given,

$\vec{a} = 4\hat{i} + 5\hat{j} - \hat{k}$, $\vec{b} = \hat{i} - 4\hat{j} + 5\hat{k}$, $\vec{c}\; 3\hat{i} + \hat{j} - \hat{k}$

Let, $\quad \vec{d} = x\hat{i} + y\hat{j} + z\hat{k}$

$\because \quad \vec{d}$ is perpendicular to $\vec{c}$ and $\vec{b}$

$\therefore \quad \vec{d}.\vec{b} = 0$

$x - 4y + 5z = 0 \quad$ ...(i)

and $\quad \vec{d} \cdot \vec{c} = 0$

$3x + y - z = 0 \quad$ ...(ii)

Also $\quad \vec{d}.\vec{a} = 21 \quad$ (given)

$\Rightarrow \quad 4x + 5y - z = 21 \quad$ ...(iii)

On subtracting equation (ii) from equation (iii), we get

$x + 4y = 21 \quad$ ...(iv)

On multiplying equation (iii) by 5 and adding equation (i), we get

$21x + 21y = 105$

$\Rightarrow \quad x + y = 5 \quad$ ....(v)

On subtracting equation (v) from equation (iv), we get

$$3y = 16 \Rightarrow y = \frac{16}{3}$$

From equation (v),

$$x = 5 - \frac{16}{3} = -\frac{1}{3}$$

From equation (ii),

$$z = 3x + y$$

$$= 3\left(-\frac{1}{3}\right) + \frac{16}{3} = \frac{13}{3}$$

$\therefore \quad \vec{d} = \frac{-1}{3}\hat{i} + \frac{16}{3}\hat{j} + \frac{13}{3}\hat{k}$ **Ans.**

21. Find the shortest distance between the lines

$$\vec{r} = (4\hat{i} - \hat{j}) + \lambda(\hat{i} + 2\hat{j} - 3\hat{k})$$

and $\vec{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \mu(2\hat{i} + 4\hat{j} - 5\hat{k})$, **[4]**

Solution : Given lines are

$$\vec{r} = (4\hat{i} - \hat{j}) + \lambda(\hat{i} + 2\hat{j} - 3\hat{k}) \quad \text{...(i)}$$

and $\quad \vec{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \mu(2\hat{i} + 4\hat{j} - 5\hat{k}), \quad$ ...(ii)

Comparing equation (i) with $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and equation (ii) with $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$, we get

$$\vec{a}_1 = 4\hat{i} - \hat{j}$$

$\vec{a_2} = \hat{i} - \hat{j} + 2\hat{k}$

$\vec{b_1} = \hat{i} - 2\hat{j} + 3\hat{k}$

$\vec{b_2} = 2\hat{i} + 4\hat{j} - 5\hat{k}$

Shortest distance between equation (i) and (ii) is given by

$$\text{S.D.} = \left|\frac{(\vec{a_2} - \vec{a_1}).(\vec{b_1} \times \vec{b_2})}{|\vec{b_1} \times \vec{b_2}|}\right|$$

$$\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -3 \\ 2 & 4 & -5 \end{vmatrix}$$

$$\Rightarrow \quad \vec{b_1} \times \vec{b_2} = \hat{i}(-10+12) - \hat{j}(-5+6) + \hat{k}(4-4)$$

$$= 2\hat{i} - \hat{j}.$$

$$\Rightarrow \quad |\vec{b_1} \times \vec{b_2}| = \sqrt{4+1} = \sqrt{5}$$

$$\Rightarrow \quad \text{S.D.} = \left|\frac{(-3\hat{i} + 0\hat{j} + 2\hat{k})\cdot(2\hat{i} - \hat{j})}{\sqrt{4+1}}\right| \left|-\frac{6}{\sqrt{5}}\right|$$

$$= \frac{6}{\sqrt{5}} \text{ or } \frac{6\sqrt{5}}{5} \text{ units}$$ **Ans.**

22. Suppose a girl throws a die. If she gets 1 or 2, she tosses a coin three times and notes the number of tails. If she gets 3, 4, 5 or 6, she tosses a coin once and notes whether a 'head' or 'tail' is obtained. If she obtained exactly one 'tail', what is the probability that she threw 3, 4, 5 or 6 with the dice? [4]

Solution : Let E_1 be the event that girl gets 1 or 2 on the roll and E_2 be the event that girl gets 3, 4, 5, or 6 on the roll of a die.

$$\therefore \quad P(E_1) = \frac{2}{6} = \frac{1}{3}$$

$$P(E_2) = \frac{4}{6} = \frac{2}{3}$$

Let A be the event that she gets exactly one tail. If she tossed coin 3 times and gets exactly one tail then possible outcomes are HTH, HHT, THH

$$\therefore \quad P(A/E_1) = \frac{3}{8}$$

If she tossed coin only once and exactly one tail shows

Then, $\quad P(A/E_2) = \frac{1}{2}$

$$\therefore \quad P(E_2/A) = \frac{P(E_2)\cdot P(A/E_2)}{P(E_1)\cdot P(A/E_1) + P(E_2)\cdot P(A/E_2)}$$

$$= \frac{\frac{2}{3} \times \frac{1}{2}}{\frac{1}{3} \times \frac{3}{8} + \frac{2}{3} \times \frac{1}{2}} = \frac{\frac{1}{3}}{\frac{1}{8} + \frac{1}{3}} = \frac{\frac{1}{3}}{\frac{11}{24}}$$

$$= \frac{8}{11}$$ **Ans.**

23. Two numbers are selected at random (without replacement) from the first five positive integers. Let X denote the larger of the two numbers obtained. Find the mean and variance of X. [4]

Solution : First five positive integers are 1, 2, 3, 4, 5. We select two positive numbers in $5 \times 4 = 20$ ways. Out of these, two numbers are selected at random. Let X denote larger of the two selected numbers. Then, X can have values 2, 3, 4 or 5.

$P(X = 2) = P$ (larger no. is 2) $= \{(1, 2) \text{ and } (2, 1)\}$

$$= \frac{2}{20}$$

$$P(X = 3) = \frac{4}{20}$$

$$P(X = 4) = \frac{6}{20}$$

$$P(X = 5) = \frac{8}{20}$$

Thus, the probability distribution of X is

X	2	3	4	5
P(X)	$\frac{2}{20}$	$\frac{2}{20}$	$\frac{6}{20}$	$\frac{8}{20}$

$$\therefore \quad \text{Mean} = E(X) = \sum_{i=1}^{n} x_i\, p(x_i)$$

$$= 2 \times \frac{2}{20} + 3 \times \frac{4}{20} + 4 \times \frac{6}{20} + 5 \times \frac{8}{20}$$

$$= \frac{4+12+24+40}{20} = \frac{80}{20} = 4$$

$$E(X^2) = \sum_{i=1}^{n} x_i^2\, p(x_i)$$

$$= 2^2 \times \frac{2}{20} + 3^2 \times \frac{4}{20} + 4^2 \times \frac{6}{20} + 5^2 \times \frac{8}{20}$$

$$= \frac{8}{20} + \frac{36}{20} + \frac{96}{20} + \frac{200}{20}$$

$$= \frac{8+36+96+200}{20} = \frac{340}{20} = \frac{34}{2} = 17$$

$$\text{Variance} = E(X^2) - [E(X)]^2$$

$$= 17 - (4)^2 = 1$$

Therefore, mean and variance are 4 and 1 respectively. **Ans.**

SECTION - D

24. Let $A = \{x \in Z : 0 \le x \le 12\}$. Show that $R = \{(a, b) : a, b \in A, |a - b| \text{ is divisible by } 4\}$ is an equivalence relation. Find the set of all elements related to 1. Also write the equivalence class [2]. [6]

Solution : Given, $R = \{(a, b) : a, b \in A, \ |a-b| \text{ is divisible by } 4\}$

Reflexivity : For any $a \in A$

$|a-a| = 0$, which is divisible by 4

$(a, a) \in R$

So, R is reflexive.

Symmetry : Let $(a, b) \in R$

$\Rightarrow |a-b|$ is divisible by 4

$\Rightarrow |b-a|$ is divisible by 4

$[\because |a-b| = |b-a|]$

$\Rightarrow (b, a) \in R$

So, R is symmetric.

Transitive : Let $(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow |a-b|$ is divisible by 4

$\Rightarrow |a-b| = 4k$

$\therefore a-b = \pm 4k, k \in Z$ ...(i)

Also, $|b-c|$ is divisible by 4

$\Rightarrow |b-c| = 4m$

$\therefore b-c = \pm 4m, m \in Z$ ...(ii)

Adding equations (i) and (ii)

$a-b+b-c = \pm 4(k+m)$

$\Rightarrow a-c = \pm 4(k+m)$

$|a-c|$ is divisible by 4,

$\Rightarrow (a, c) \in R$

So, R is transitive.

$\Rightarrow$ R is reflexive, symmetric and transitive.

$\therefore$ R is an equivalence relation.

Let x be an element of R such that $(x, 1) \in R$

Then $|x-1|$ is divisible by 4

$x-1 = 0, 4, 8, 12, ...$

$\Rightarrow x = 1, 5, 9$ $(\because x \le 12)$

$\therefore$ Set of all elements of A which are related to 1 are $\{1, 5, 9\}$.

Equivalence class of 2 *i.e.*

$[2] = \{(a, 2) : a \in A, |a-2| \text{ is divisible by } 4\}$

$\Rightarrow |a-2| = 4k$ (k is whole number, $k \le 3$)

$\Rightarrow a = 2, 6, 10$

Therefore, equivalence class [2] is $\{2, 6, 10\}$. **Ans.**

OR

Show that the function $f : R \to R$ defined by $f(x) = \frac{x}{x^2+1}, \forall x \in R$ is neither one-one nor onto. Also, if $g : R \to R$ is defined as $g(x) = 2x-1$, find $f o g(x)$.

Solution : Given, $f(x) = \frac{x}{x^2+1}, \forall x \in R$

For one–one, $f(x) = f(y)$

$$\frac{x}{x^2+1} = \frac{y}{y^2+1}$$

$\Rightarrow xy^2 + x = yx^2 + y$

$\Rightarrow xy^2 - yx^2 = y - x$

$\Rightarrow xy(y-x) = y-x$

$\Rightarrow xy = 1$

$\Rightarrow x = \frac{1}{y}$

Since $x \ne y$, therefore, $f(x)$ is not one-one.

For onto, $f(x) = y$

$\Rightarrow \frac{x}{x^2+1} = y$

$\Rightarrow x = yx^2 + y$

$\Rightarrow x^2y + y - x = 0$

x cannot be expressed in terms of y

$\Rightarrow f(x)$ is not onto.

As $g(x) = 2x-1$

$\therefore fog(x) = f[g(x)] = f(2x-1)$

$$= \frac{2x-1}{(2x-1)^2+1} = \frac{2x-1}{4x^2-4x+2} \text{ Ans.}$$

25. **If $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, find A^{-1}. Use it to solve the system of equations :** **[6]**

$$2x - 3y + 5z = 11$$
$$3x + 2y - 4z = -5$$
$$x + y - 2z = -3$$

Solution : $|A| = \begin{vmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{vmatrix}$

$= 2(-4+4) + 3(-6+4) + 5(3-2)$

$|A| = 0 - 6 + 5 = -1 \ne 0$

$\therefore A^{-1}$ exists.

Let A_{ij} be the cofactors of elements a_{ij} in $A = [a_{ij}]$, then

$A_{11} = (-1)^{1+1} \begin{vmatrix} 2 & -4 \\ 1 & -2 \end{vmatrix} = -4 + 4 = 0$

$A_{12} = (-1)^{1+2} \begin{vmatrix} 3 & -4 \\ 1 & -2 \end{vmatrix} = -(-6+4) = 2$

$A_{13} = (-1)^{1+3} \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = 3 - 2 = 1$

$A_{21} = (-1)^{2+1} \begin{vmatrix} -3 & 5 \\ 1 & -2 \end{vmatrix} = -(6-5) = -1$

$A_{22} = (-1)^{2+2} \begin{vmatrix} 2 & 5 \\ 1 & -2 \end{vmatrix} = -4 - 5 = -9$

$$A_{23} = (-1)^{2+3}\begin{vmatrix}2 & -3\\1 & 1\end{vmatrix} = -(2+3) = -5$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix}-3 & 5\\2 & -4\end{vmatrix} = (12-10) = 2$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix}2 & 5\\3 & -4\end{vmatrix} = -(-8-15) = 23$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix}2 & -3\\3 & 2\end{vmatrix} = 4+9 = 13$$

$$\therefore \quad A_{ij} = \begin{bmatrix}0 & 2 & 1\\-1 & -9 & -5\\2 & 23 & 13\end{bmatrix}$$

$$\text{adj}(A) = [A_{ij}]^T = \begin{bmatrix}0 & -1 & 2\\2 & -9 & 23\\1 & -5 & 13\end{bmatrix}$$

$$A^{-1} = \frac{\text{adj}(A)}{|A|} = \frac{1}{(-1)}\begin{bmatrix}0 & -1 & 2\\2 & -9 & 23\\1 & -5 & 13\end{bmatrix}$$

$$= \begin{bmatrix}0 & 1 & -2\\-2 & 9 & -23\\-1 & 5 & -13\end{bmatrix}$$

Now, given system of equations can be written as

$$\begin{bmatrix}2 & -3 & -5\\3 & 2 & -4\\1 & 1 & -2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}11\\-5\\-3\end{bmatrix}$$

$$\Rightarrow \quad AX = B$$

$$\therefore \quad X = A^{-1}B$$

$$X = \begin{bmatrix}0 & 1 & -2\\-2 & 9 & -23\\-1 & 5 & -13\end{bmatrix}\begin{bmatrix}11\\-5\\-3\end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0-5+6\\-22-45+69\\-11-25+39\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix}$$

$\therefore \quad x = 1, y = 2, z = 3$ **Ans.**

OR

Using elementary row transformations, find the inverse of the matrix:

$$A = \begin{bmatrix}1 & 2 & 3\\2 & 5 & 7\\-2 & -4 & -5\end{bmatrix}.$$

$$\begin{bmatrix}1 & 2 & 3\\2 & 5 & 7\\-2 & -4 & -5\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}A$$

On applying, $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 + 2R_1$, we get

$$\begin{bmatrix}1 & 2 & 3\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\-2 & 1 & 0\\2 & 0 & 1\end{bmatrix}A$$

On applying $R_1 \to R_1 - 3R_3$ and $R_2 \to R_2 - R_3$, we get

$$\begin{bmatrix}1 & 2 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}-5 & 0 & -3\\-4 & 1 & -1\\2 & 0 & 1\end{bmatrix}A$$

On applying $R_1 \to R_1 - 2R_2$, we get

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}3 & -2 & -1\\-4 & 1 & -1\\2 & 0 & 1\end{bmatrix}A$$

$$\therefore \quad A^{-1} = \begin{bmatrix}3 & -2 & -1\\-4 & 1 & -1\\2 & 0 & 1\end{bmatrix}$$ **Ans.**

26. Using integration, find the area of the region in the first quadrant enclosed by the X-axis, the line $y = x$ and the circle $x^2 + y^2 = 32$. **[6]**

Solution : Given curve is

$$x^2 + y^2 = 32 \quad \text{... (i)}$$

$$x^2 + y^2 = \left(\sqrt{32}\right)^2 = \left(4\sqrt{2}\right)^2$$

It is a circle with centre (0, 0) and radius $4\sqrt{2}$.

Given line is $y = x$ ....(ii)

Solving equations (i) and (ii) for points of intersections,

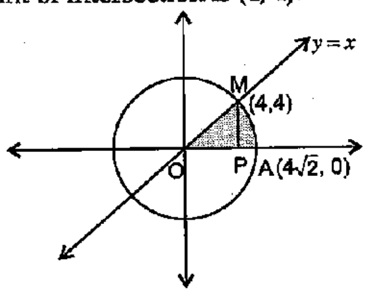
$$x^2 + x^2 = 32 \quad \text{[using (ii) in (i)]}$$

$$\Rightarrow \quad x^2 = 16$$

$$\Rightarrow \quad x = \pm 4$$

$$\Rightarrow \quad x = 4 \quad \text{(first quadrant)}$$

$\therefore$ Point of intersection is (4, 4).



Required area = Area OMA

= Area OMP + Area MPA

$$= \int_0^4 x\cdot dx + \int_4^{4\sqrt{2}} \sqrt{(4\sqrt{2})^2 - x^2}\cdot dx$$

$$= \left[\frac{x^2}{2}\right]_0^4 + \left[\frac{x}{2}\sqrt{(4\sqrt{2})^2 - x^2} + \frac{(4\sqrt{2})^2}{2}\sin^{-1}\left(\frac{x}{4\sqrt{2}}\right)\right]_4^{4\sqrt{2}}$$

$$= \frac{16}{2} + \left[\left\{\frac{4\sqrt{2}}{2}\sqrt{(4\sqrt{2})^2 - (4\sqrt{2})^2} + \frac{32}{2}\sin^{-1}1\right\} - \left\{\frac{4}{2}\sqrt{(4\sqrt{2})^2 - (4)^2} + \frac{32}{2}\sin^{-1}\frac{1}{\sqrt{2}}\right\}\right]$$

$$= 8 + \left(2\sqrt{2}(0) + 16\times\frac{\pi}{2}\right) - \left(2\times 4 + 16\times\frac{\pi}{4}\right)$$

$$= 8 + 8\pi - 8 - 4\pi$$

$= 4\pi$ sq. units **Ans.**

27. Evaluate :

$$\int_0^{\pi/4} \frac{\sin x + \cos x}{16 + 9\sin 2x} dx$$

[6]

Solution :

Let $$I = \int_0^{\pi/4} \frac{\sin x + \cos x}{16 + 9\sin 2x}.dx$$

$$= \int_0^{\pi/4} \frac{\sin x + \cos x}{16 + 9[1 - (\sin x - \cos x)^2]}\cdot dx$$

$$\Rightarrow \quad = \int_0^{\pi/4} \frac{\sin x + \cos x}{25 - 9(\sin x - \cos x)^2}\cdot dx$$

Put $\sin x - \cos x = t$

$(\cos x + \sin x)\,dx = dt$

When $x = 0, t = -1$

and $x = \frac{\pi}{4}, t = 0$

$$I = \int_{-1}^0 \frac{dt}{25 - 9t^2} = \frac{1}{9}\int_{-1}^0 \frac{dt}{\frac{25}{9} - t^2}$$

$$= \frac{1}{9}\int_{-1}^0 \frac{dt}{\left(\frac{5}{3}\right)^2 - t^2}$$

$$= \frac{1}{9}\left[\frac{1}{2\times\frac{5}{3}}\log\left|\frac{\frac{5}{3}+t}{\frac{5}{3}-t}\right|\right]_{-1}^0$$

$$= \frac{1}{9}\left[\frac{1}{\frac{10}{3}}\left(\log\left|\frac{\frac{5}{3}}{\frac{5}{3}}\right| - \log\left|\frac{\frac{2}{3}}{\frac{8}{3}}\right|\right)\right]$$

$$= \frac{1}{30}\left[\log 1 - \log\frac{1}{4}\right]$$

$$= \frac{1}{30}\,[\log 1 - (\log 1 - \log 4)]$$

$$= \frac{1}{30}\log 4 = \frac{1}{30}\log (2)^2 = \frac{2}{30}\log 2$$

$= \frac{1}{15}\log 2$ **Ans.**

OR

Evaluate $\int_1^3 (x^2 + 3x + e^x)dx$ as the limit of the sum.

Solution : We have, $\int_1^3 (x^2 + 3x + e^x).dx$

Here, $f(x) = x^2 + 3x + e^x$, $a = 1$, $b = 3$, $nh = b - a = 3 - 1 = 2$

By limit of sum, we have

$$\int_a^b f(x)\,dx \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + ... + f(a + (n-1)h)]$$

where $nh = b - a$

$$\therefore \int_1^3 (x^2 + 3x + e^x).dx$$

$$= \lim_{h\to 0} h\,[f(1) + f(1+h) + f(2+h)\,..... + f(1 + \overline{n-1}h)]$$

$$= \lim_{h\to 0} h[(1 + 3 + e) + \{(1 + h^2) + 3(1+h) + e^{1+h}\} + \{(1+2h)^2 + 3(1+2h) + e^{1+2h}\} + ...]$$

$$= \lim_{h\to 0} h\,(4 + e + (1 + h^2 + 2h + 3 + 3h + e^{1+h}) + (1 + 4h^2 + 4h + 3 + 6h + e^{1+2h}) + ...]$$

$$= \lim_{h\to 0} h\left[4 + e + (4 + h^2 + 5h + e^{1+h}) + (4 + 4h^2 + 10h + e^{1+2h}) + ...\right]$$

$$= \lim_{h\to 0} h\,[4n + e(1 + e^h + e^{2h} + ...) + h^2(1^2 + 2^2 + ...) + 5h(1 + 2 + ...)]$$

$$= \lim_{h\to 0} h\left[4n + e\left(\frac{e^{nh}-1}{e^h - 1}\right) + h^2\frac{n(n-1)(2n-1)}{6} + \frac{5h\,n(n-1)}{2}\right]$$

$$= \lim_{h \to 0}\left[4nh + e.\frac{h}{e^h - 1}(e^{nh} - 1) + \frac{nh(nh-h)(2nh-h)}{6} + \frac{5nh(nh-h)}{2}\right]$$

$$= 4(2) + e\cdot(e^2 - 1) + \frac{2(2-0)(4-0)}{6} + \frac{5(2)(2-0)}{2}$$

$$= 8 + (e^3 - e) + \frac{8}{3} + 10$$

$$= \frac{24+8+30}{3} + e^3 - e$$

$$= \frac{62}{3} + e^3 - e \qquad \textbf{Ans.}$$

28. Find the distance of the point (–1, –5, –10) from the point of intersection of the line $\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$ and the plane $\vec{r}.(\hat{i} - \hat{j} + \hat{k}) = 5$. [6]

Solution : Equation of line is

$$\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k}) \qquad ...(i)$$

Coordinates of any point on this line are

$$(2+3\lambda)\hat{i} + (-1+4\lambda)\hat{j} + (2+2\lambda)\hat{k}$$

Equation of plane is

$$\vec{r}.(\hat{i} - \hat{j} + \hat{k}) = 5 \qquad ...(ii)$$

Since, the point on line lies on the plane,

$$\vec{r}.(\hat{i} - \hat{j} + \hat{k}) = 5$$

$$[(2+3\lambda)\hat{i} + (2+4\lambda)\hat{j} + (2+2\lambda)](\hat{i} - \hat{j} + \hat{k}) = 5$$

$$\Rightarrow \quad (2+3\lambda) - (-1+4\lambda) + (2+2\lambda) = 5$$

$$\Rightarrow \quad \lambda + 5 = 5$$

$$\Rightarrow \quad \lambda = 0$$

So equation of line is

$$\vec{r} = (2\hat{i} - \hat{j} + 2\hat{k}) + 0(3\hat{i} + 4\hat{j} + 2\hat{k})$$

$$\Rightarrow \quad \vec{r} = 2\hat{i} - \hat{j} + 2\hat{k} \qquad ...(iii)$$

Let, point of intersection be (x, y, z)

$$\therefore \quad \vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \qquad ...(iv)$$

From equations (iii) and (iv)

$$x = 2, y = -1, z = 2$$

$\therefore$ Point of intersection is (2, –1, 2).
Distance between points (2, –1, 2) and (–1, –5, –10) is

$$= \sqrt{(-1-2)^2 + (-5+1)^2 + (-10-2)^2}$$

$$= \sqrt{(-3)^2 + (-4)^2 + (-12)^2}$$

$$= \sqrt{9+16+144} = \sqrt{169} = 13 \text{ units} \qquad \textbf{Ans.}$$

29. A factory manufactures two types of screws A and B, each type requiring the use of two machines, an automatic and a hand-operated. It takes 4 minutes on the automatic and 6 minutes on the hand-operated machines to manufacture a packet of screws 'A' while it takes 6 minutes on the automatic and 3 minutes on the hand-operated machine to manufacture a packet of screws 'B'. Each machine is available for at most 4 hours on any day. The manufacturer can sell a packet of screws 'A' at a profit of 70 paise and screws 'B' at a profit of ₹1. Assuming that he can sell all the screws he manufactures, how many packets of each type should the factory owner produce in a day in order to maximize his profit ? Formulate the above LPP and solve it graphically and find the maximum profit. [6]

Solution : Let the number of packets of screw 'A' manufactured in a day be x and that of screw B be y.

Therefore, $x \geq 0, y \geq 0$

Item	Number	Machine A	Machine B	Profit
Screw A	x	4 minutes	6 minutes	₹ 0.7
Screw B	y	6 minutes	3 minutes	₹ 1
Max. time available		4 hrs. = 240 min.	4 hrs. = 240 min.	

Then, the constraints are :

$$4x + 6y \leq 240 \text{ or } 2x + 3y \leq 120$$

$$6x + 3y \leq 240 \text{ or } 2x + y \leq 80$$

and total profit,

$$Z = 0.7x + y$$

So our LPP will be

$$\text{Max. } Z = 0.7x + y$$

Subject to the constraints :

$$2x + 3y \leq 120,$$

$$2x + y \leq 80,$$

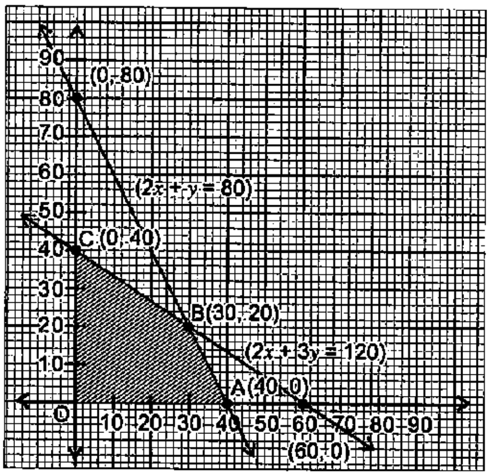
and $\quad x, y \geq 0$

Now, $2x + 3y \leq 120$ and $2x + y \leq 80$

x	0	60
y	40	0

x	0	40
y	80	0

Plotting the points on the graph, we get the feasible region OABC as shown (Shaded).



Corner points	Value of Z = 0.7x + y
C(0, 40)	0.7(0) + 40 = 40
B (30, 20)	0.7 (30) + 20 = 41 Maximum
A(40, 0)	0.7(40) + 0 = 28
O (0, 0)	0.7 (0) + 0 = 0

Hence, profit will be maximum if company produces 30 packets of screw A and 20 packets of screw B and maximum profit = ₹ 41. **Ans.**

Mathematics 2017 (Outside Delhi) SET I

Time allowed : 3 hours **Maximum marks : 100**

SECTION — A

1. If for any 2 × 2 square matrix A,

$$A(\text{adj } A) = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix},$$

then write the value of | A |. [1]

Solution : We have,

$$A(\text{adj } A) = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$$

As, $A(\text{adj } A) = |A|\, I$

$$\begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix} = 8\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

On comparing, we get

$$|A| = 8.$$ **Ans.**

2. Determine the value of 'k' for which the following function is continuous at $x = 3$.

$$f(x) = \begin{cases} \dfrac{(x+3)^2 - 36}{x-3}, & x \neq 3 \\ k, & x = 3 \end{cases}$$ [1]

Solution : Given,

$$f(x) = \begin{cases} \dfrac{(x+3)^2 - 36}{x-3}, & x \neq 3 \\ k, & x = 3 \end{cases}$$

Since $f(x)$ is continuous at $x = 3$

$$\therefore \quad \lim_{x \to 3} f(x) = f(3)$$

$$\Rightarrow \quad \lim_{x \to 3} f(x) = k$$

$$\Rightarrow \quad \lim_{x \to 3} \frac{(x+3)^2 - 36}{x-3} = k$$

$$\Rightarrow \quad \lim_{x+3 \to 6} \frac{(x+3)^2 - 6^2}{(x+3) - 6} = k$$

$$\Rightarrow \quad \lim_{x \to 3} x + 3 + 6 = k$$

$$\Rightarrow \quad 12 = k$$

Thus, $f(x)$ is continuous at $x = 3$; if $k = 12$. **Ans.**

3. Find : $\int \dfrac{\sin^2 x - \cos^2 x}{\sin x \cos x} dx$. [1]

Solution : We have,

$$\int \frac{\sin^2 x - \cos^2 x}{\sin x \cos x} dx = -2 \int \frac{\cos^2 x - \sin^2 x}{2 \sin x \cos x} dx$$

$$= -2 \int \frac{\cos 2x}{\sin 2x} dx$$

$$= -2 \int \cot 2x \, dx$$

$$= -\log |\sin 2x| + C$$

Ans.

4. Find the distance between the planes $2x - y + 2z = 5$ and $5x - 2.5y + 5z = 20$. [1]

Mathematics 2019 (Outside Delhi) SET I

Time allowed : 3 hours **Maximum marks : 100**

General Instructions :

(i) **All** questions are compulsory.

(ii) The question paper consists of **29** questions divided into four sections A, B, C and D. Section A comprises of 4 questions of **one mark** each, Section B comprises of **8** questions of **two marks** each, Section C comprises of **11** questions of **four marks** each and Section D comprises of **6** questions of **six marks** each.

(iii) All questions in Section A are to be answered in one word, one sentence or as per the exact requirement of the question.

(iv) There is no overall choice. However, internal choice has been provided in 1 question of Section A, 3 questions of Section B, 3 questions of Section C and 3 questions of Section D. You have to attempt only one of the alternatives in all such questions.

(v) Use of calculators is **not** permitted. You may ask for logarithmic tables, if required.

SECTION–A

1. If A is a square matrix satisfying A′ A = I, write the value of | A |. **[1]**

Solution : Given, A′A= I

Then $|A'A| = |I|$

$\Rightarrow |A'||A| = |I|$

$\Rightarrow |A||A| = |I| \quad [\because |A'| = |A|]$

$\Rightarrow |A|^2 = 1 \quad [\because |I| = 1]$

$\Rightarrow |A| = \pm 1$ **Ans.**

2. If $y = x\,|x|$, find $\frac{dy}{dx}$ for $x < 0$. **[1]**

Solution : If $y = x\,|x|$

Then,

$$y = \begin{Bmatrix} -x^2 & x<0 \\ x^2 & x>0 \end{Bmatrix}$$

$\Rightarrow \frac{dy}{dx} = -2x$ when $x < 0$ **Ans.**

3. Find the order and degree (if defined) of the differential equation

$$\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2 = 2x^2\log\left(\frac{d^2y}{dx^2}\right).$$ **[1]**

Solution :

$$\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2 = 2x^2\log\left(\frac{d^2y}{dx^2}\right)$$

Order of this equation is 2.

Degree of this equation is not defined. **Ans.**

4. Find the direction consines of a line which makes equal angles with the coordinate axes. [1]

Solution : Let the direction cosines of the line make an angle α with each of the coordinate axes and direction cosines be l, m and n.

$\therefore l = \cos\alpha, \quad m = \cos\alpha$ and $n = \cos\alpha$

$l^2+m^2+n^2 = 1$

$\therefore \cos^2\alpha+\cos^2\alpha+\cos^2\alpha = 1$

$\Rightarrow 3\cos^2\alpha = 1$

$\Rightarrow \cos^2\alpha = \frac{1}{3}$

$\Rightarrow \cos\alpha = \pm\frac{1}{\sqrt{3}}$

The direction cosines are $\left(\pm\frac{1}{\sqrt{3}},\pm\frac{1}{\sqrt{3}},\pm\frac{1}{\sqrt{3}}\right)$ **Ans.**

OR

A line passes through the point with position vector $2\hat{i}-\hat{j}+4\hat{k}$ and is in the direction of the vector $\hat{i}+\hat{j}-2\hat{k}$. Find the equation of the line in cartesian form.

Solution :

The line passes through a point (2, – 1, 4) and has direction ratios proportional to (1, 1, – 2).

Cartesian equation of the line

$$= \frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$$

$$= \frac{x-2}{1} = \frac{y+1}{1} = \frac{z-4}{-2}$$ **Ans.**

SECTION–B

5. Examine whether the operation ∗ defined on R, the set of all real numbers, by $a * b = \sqrt{a^2+b^2}$ is a binary operation or not, and if it is a binary operation, find whether it is associative or not.** **[2]**

** Answer is not given due to the change in present syllabus.

6. If $A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$, show that (A – 2I) (A – 3I) = 0. [2]

Solution : Given,

$$A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} \text{ and } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A - 2I = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}$$

$$A - 3I = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$$

Then, (A – 2I) (A – 3I)

$$= \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$ **Hence Proved**

7. Find : $\int \sqrt{3 - 2x - x^2}\, dx.$ [2]

Solution : Let $I = \int \sqrt{3 - 2x - x^2}\, dx$

$$= \int \sqrt{-(x^2 + 2x - 3)}\, dx$$

$$= \int \sqrt{-(x^2 + 1 + 2x - 3 - 1)}\, dx$$

$$= \int \sqrt{-[(x+1)^2 - 4]}\, dx$$

$$= \int \sqrt{4 - (x+1)^2}\, dx$$

$$= \int \sqrt{2^2 - (x+1)^2}\, dx$$

$$\left[\because \int \sqrt{a^2 - x^2}\, dx = x\sqrt{\frac{a^2 - x^2}{2}} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c\right]$$

$$= \frac{1}{2}(x+1)\sqrt{4 - (x+1)^2} + \frac{4}{2}\sin^{-1}\left(\frac{x+1}{2}\right) + c$$

$$\therefore \quad I = \int \sqrt{2^2 - (x+1)^2}\, dx$$

$$= \frac{1}{2}(x+1)\sqrt{3 - 2x - x^2} + 2\sin^{-1}\frac{(x+1)}{2} + c$$ **Ans.**

8. **Find :**

$$\int \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x}\, dx.$$ [2]

Solution :

Let $$I = \int \frac{\sin^3 x + \cos^3 x}{\sin^2 x.\cos^2 x}\, dx$$

$$= \int \frac{\sin^3 x\, dx}{\sin^2 x.\cos^2 x} + \int \frac{\cos^3 x}{\sin^2 x.\cos^2 x}\, dx$$

$$= \int \frac{\sin x}{\cos^2 x}\, dx + \int \frac{\cos x}{\sin^2 x}\, dx$$

$$= \int \tan x.\sec x\, dx + \int \cot x.\text{cosec}\, x\, dx$$

$$= \sec x - \text{cosec}\, x + c$$

$$\left[\because \int \tan x.\sec x\, dx = \sec x \text{ and } \int \cot x.\text{cosec}\, x\, dx = -\text{cosec}\, x\right]$$

Ans.

OR

Find : $\int \frac{x - 3}{(x-1)^3} e^x\, dx$

Solution :

Let, $$I = \int \frac{x - 3}{(x-1)^3} e^x\, dx$$

$$= \int \frac{x - 2 - 1}{(x-1)^3} e^x\, dx$$

$$= \int \frac{x - 1 - 2}{(x-1)^3} e^x\, dx$$

$$= \int \left[\frac{x - 1}{(x-1)^3} - \frac{2}{(x-1)^3}\right] e^x\, dx$$

$$= \int \left[\frac{1}{(x-1)^2} - \frac{2}{(x-1)^3}\right] e^x\, dx \quad ...(i)$$

Here, $\frac{d}{dx}\left[\frac{1}{(x-1)^2}\right] = \frac{d}{dx}(x-1)^{-2} = -2(x-1)^{-3}$

Then, $f'(x) = \frac{-2}{(x-1)^3}$

We know that,

$$\int [f(x) + f'(x)]\, e^x\, dx = e^x f(x) + c$$

Then from equation (i), we have

$$\int \left[\frac{1}{(x-1)^2} - \frac{2}{(x-1)^3}\right] e^x\, dx = \frac{e^x}{(x-1)^2} + c$$ **Ans.**

9. Find the differential equation of the family of curves $y = Ae^{2x} + Be^{-2x}$, where A and B are arbitrary constants. [2]

Solution : Given, $y = Ae^{2x} + Be^{-2x}$ (i)

On differentiating equation (i) w.r.t. x, we get

$$\frac{dy}{dx} = 2Ae^{2x} - 2Be^{-2x} \quad ...(ii)$$

Again, differentiating equation (ii) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = 4Ae^{2x} - 4Be^{-2x}$$
$$= 4(Ae^{2x} + Be^{-2x})$$
$$= 4y$$

$\therefore \frac{d^2y}{dx^2} - 4y = 0$ is the required differential equation **Ans.**

10. If $|\vec{a}| = 2$, $|\vec{b}| = 7$ and $\vec{a} \times \vec{b} = 3\hat{i} + 2\hat{j} + 6\hat{k}$, find the angle between $\vec{a}$ and $\vec{b}$. [2]

Solution : Given,

$$|\vec{a}| = 2, |\vec{b}| = 7 \text{ and } (\vec{a} \times \vec{b}) = 3\hat{i} + 2\hat{j} + 6\hat{k}$$

$\therefore$ Angle between $\vec{a}$ and $\vec{b}$ is given by

$$\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|} \quad ...(i)$$

Then, $|\vec{a} \times \vec{b}| = \sqrt{(3)^2 + (2)^2 + (6)^2}$

$$= \sqrt{49} = 7$$

$\therefore \quad \sin\theta = \frac{7}{2 \times 7} = \frac{1}{2}$

$\Rightarrow \quad \sin\theta = \sin\frac{\pi}{6}$

$\Rightarrow \quad \theta = \frac{\pi}{6}$ **Ans.**

OR

Find the volume of a cuboid whose edges are given by $-3\hat{i} + 7\hat{j} + 5\hat{k}$, $-5\hat{i} + 7\hat{j} - 3\hat{k}$ and $7\hat{i} - 5\hat{j} - 3\hat{k}$.

Solution :

If a, b, c are edges of a cuboid.

Then, volume of cuboid $= \vec{a}.(\vec{b} \times \vec{c})$

Here,

$$\vec{a} = -3\hat{i} + 7\hat{j} + 5\hat{k}$$
$$\vec{b} = -5\hat{i} + 7\hat{j} - 3\hat{k}$$
$$\vec{c} = 7\hat{i} + 5\hat{j} - 3\hat{k}$$

Then,

$$\vec{a}.(\vec{b} \times \vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & 7 & 5 \\ -5 & 7 & -3 \\ 7 & -5 & -3 \end{vmatrix}$$

$$= -3(-21 - 15) - 7(15 + 21) + 5(25 - 49)$$
$$= -3 \times (-36) - 7 \times 36 + 5 \times (-24)$$
$$= 108 - 252 - 120$$
$$= -264 \text{ cubic units}$$ **Ans.**

11. If P(not A) = 0·7, P(B) = 0·7 and P(B/A) = 0·5, then find P(A/B). [2]

Solution :

Given, P(not A) = 0.7

P (B) = 0.7

and P (B/A) = 0.5

We know that

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

$$0.5 = \frac{P(A \cap B)}{0.3}$$

$[\because P(\text{not } A) = 0.7 \text{ then } P(A) = 1 - 0.7 = 0.3]$

$\Rightarrow \quad P(A \cap B) = 0.15$

Also we know,

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{0.15}{0.7} = \frac{15}{7}$$ **Ans.**

12. A coin is tossed 5 times. What is the probability of getting (i) 3 heads, (ii) at most 3 heads? [2]

Solution :

Here, $n = 5$, $p = \frac{1}{2}$ and $q = \frac{1}{2}$

We know that,

$$p(x) = {}^nC_x p^x q^{n-x}$$
$$= {}^5C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{5-x}$$

(i) For 3 heads, $x = 3$

Then, the probability of getting 3 heads is

$$P(3) = {}^5C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{3}\right)^{5-3}$$
$$= {}^5C_3 \left(\frac{1}{2}\right)^3 \times \left(\frac{1}{2}\right)^2$$
$$= {}^5C_3 \times \left(\frac{1}{2}\right)^5$$

$$= \frac{5 \times 4 \times !3}{2 \times !3}\left(\frac{1}{2}\right)^5$$

$$= 10\left(\frac{1}{2}\right)^5 = \frac{5}{16}$$ **Ans.**

(ii) Probability of getting at most 3 heads is

$P(x \leq 3) = P(0) + P(1) + P(2) + P(3)$

Then, $P(0) = {}^5C_0\left(\frac{1}{2}\right)^5$

$$= \frac{1}{32}$$

$$P(1) = {}^5C_1\left(\frac{1}{2}\right)^5 = \frac{5}{32}$$

$$P(2) = {}^5C_2\left(\frac{1}{2}\right)^5 = \frac{5}{16}$$

$$P(3) = {}^5C_3\left(\frac{1}{2}\right)^5 = \frac{5}{16}$$

$\therefore$ $$P(x \leq 3) = \frac{1}{32}+\frac{5}{32}+\frac{5}{16}+\frac{5}{16}$$

$$= \frac{26}{32} = \frac{13}{16}$$ **Ans.**

OR

Find the probabilty distribution of X, the number of heads in a simultaneous toss of two coins.

Solution :

If we toss two coins simultaneously then sample space is given by (HH, HT, TH, TT)

Then probability distribution is,

X(No. of heads)	0	1	2
P(x)	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Ans.

SECTION–C

13. Check whether the relation R defined on the set A = {1, 2, 3, 4, 5, 6} as R = {(a, b) : b = a + 1} is reflexive, symmetric or transitive. [4]

Solution : Here, $R = \{(a, b) : b = a + 1\}$

$\therefore$ $R = \{(a, a+1) : a, a+1 \in (1, 2, 3, 4, 5, 6)\}$

$\Rightarrow$ $R = \{(1,2,) (2,3), (3,4), (4,5), (5,6)\}$

(i) R is not reflexive as $(a, a) \notin R \forall a$

(ii) R is not symmetric as $(1, 2) \in R$ but $(2, 1) \notin R$

(iii) R is not transitive as $(1, 2) \in R$, $(2, 3) \in R$ but $(1, 3) \notin R$

OR

Let $f : N \to Y$ be a function defined as

$$f(x) = 4x + 3,$$

where $Y = \{y \in N : y = 4x + 3$, for some $x \in N\}$. Show that f is invertible. Find its inverse.

Solution : Consider an arbitrary element of Y. By the definition of y, $y = 4x + 3$, for some x in the domain N.

This shows that $x = \frac{y-3}{4}$

Define $g : Y \to N$ by $g(y) = \frac{y-3}{4}$

Now, $gof(x) = g(f(x)) = g(4x+3) = \frac{4x+3-3}{4} = x$

and $fog(y) = f(g(y)) = f\left(\frac{y-3}{4}\right) = \frac{4(y-3)}{4} + 3$

$$= y - 3 + 3 = y$$

This shows that $gof = I_N$ and $fog = I_y$ which implies that f is invertible and g is the inverse of f.

Hence Proved.

$\therefore$ Inverse of $f = g(y) = \frac{y-3}{4}$ **Ans.**

14. Find the value of $\sin\left(\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{2}{3}\right)$. [4]

Solution :

$\sin\left(\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{2}{3}\right)$

$$= \sin\left(\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{2}{3}\right)$$

$$= \sin\left(\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{2}{3}\right)$$

$$\left(\because \cos^{-1}\frac{4}{5} = \tan^{-1}\frac{3}{4}\right)$$

$$= \sin\left(\tan^{-1}\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\times\frac{2}{3}}\right)$$

$$\left[\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}\right]$$

$$= \sin\left(\tan^{-1}\frac{\frac{9+8}{12}}{1-\frac{6}{12}}\right)$$

$$= \sin\left(\tan^{-1}\frac{\frac{17}{12}}{\frac{6}{12}}\right)$$

$$= \sin(\tan^{-1}17/16)$$

$$= \sin\left(\tan^{-1}\frac{17}{6}\right)$$

$$= \sin\left(\sin^{-1}\frac{17}{\sqrt{325}}\right)$$

$$\left(\because \tan^{-1}\frac{17}{6} = \sin^{-1}\frac{17}{\sqrt{325}}\right)$$

$$= \frac{17}{\sqrt{325}}$$ **Ans.**

15. Using properties of determinants, show that

$$\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix} = 3(a+b+c)(ab+bc+ca)$$ **[4]**

Solution :

$$\text{L.H.S.} = \begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix}$$

On applying operation $C_1 \to C_1 + C_2 + C_3$, we have

$$= \begin{vmatrix} 3a-a+b-a+c & -a+b & -a+c \\ -b+a+3b-b+c & 3b & -b+c \\ -c+a-c+b+3c & -c+b & 3c \end{vmatrix}$$

$$= \begin{vmatrix} a+b+c & -a+b & -a+c \\ a+b+c & 3b & -b+c \\ a+b+c & -c+b & 3c \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & -a+b & -a+c \\ 1 & 3b & -b+c \\ 1 & -c+b & 3c \end{vmatrix}$$

Again, applying operations $R_1 \to R_1 - R_3$ and $R_2 \to R_2 - R_3$, we get

$$= (a+b+c)\begin{vmatrix} 0 & -a+c & -a-2c \\ 0 & 2b+c & -b-2c \\ 1 & -c+b & 3c \end{vmatrix}$$

On applying $R_2 \to R_2 - R_1$, we get

$$= (a+b+c)\begin{vmatrix} 0 & -a+c & -a-2c \\ 0 & 2b+a & -b+a \\ 1 & -c+b & 3c \end{vmatrix}$$

Expanding about C_1, we get

$$= (a+b+c)\begin{vmatrix} -a+c & -(a+2c) \\ 2b+a & a-b \end{vmatrix}$$

$= (a+b+c)\,[(c-a)(a-b) + (a+2c)(2b+a)]$

$= (a+b+c)\,[ac - bc - a^2 + ab + 2ab + a^2 + 4bc + 2ac]$

$= (a+b+c)\,[3ac + 3bc + 3ab]$

$= (a+b+c) \times 3\,(ac+bc+ab)$

$= 3(a+b+c)(ac+bc+ab) = \text{R. H. S.}$

Hence Proved.

16. If $x\sqrt{1+y} + y\sqrt{1+x} = 0$ and $x \neq y$, prove that $\frac{dy}{dx} = -\frac{1}{(x+1)^2}$. **[4]**

Solution :

Given, $x\sqrt{1+y} + y\sqrt{1+x} = 0$

$\Rightarrow \quad x\sqrt{1+y} = -y\sqrt{1+x}$

On squaring both sides, we get

$\Rightarrow \quad x^2(1+y) = y^2(1+x)$

$\Rightarrow \quad x^2 + x^2y = y^2 + y^2x$

$\Rightarrow \quad x^2 + x^2y - y^2 - y^2x = 0$

$\Rightarrow \quad x^2 - y^2 + x^2y - y^2x = 0$

$\Rightarrow \quad x^2 - y^2 + x^2y - y^2x = 0$

$\Rightarrow (x-y)(x+y) + xy(x-y) = 0$

$\Rightarrow \quad (x-y)[x+y+xy] = 0$

$\Rightarrow \quad x + y + xy = 0 \qquad [\because x \neq y]$

$\Rightarrow \quad y = -\frac{x}{1+x} \qquad$ ...(i)

On differentiating equation (i) w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(1+x)\frac{d(-x)}{dx} - (-x)\frac{d}{dx}(1+x)}{(1+x)^2}$$

$$= \frac{(1+x)(-1) + x(1)}{(1+x)^2}$$

$$= \frac{-1-x+x}{(1+x)^2}$$

$$= \frac{-1}{(1+x)^2}$$

$\therefore \quad$ L.H.S = R.H.S. **Hence Proved.**

OR

If $(\cos x)^y = (\sin y)^x$, find $\frac{dy}{dx}$.

Solution :

Given, $(\cos x)^y = (\sin y)^x$

On taking log on both sides, we get

$y \log(\cos x) = x \log(\sin y) \qquad$ ..(i)

On differentiating equation (i) w.r.t. x,

$$\Rightarrow y \times \frac{d}{dx}(\log \cos x) + \log \cos x \frac{dy}{dx}$$

$$= \log \sin y \frac{dx}{dx} + x\frac{d}{dx}(\log \sin y)$$

$$\Rightarrow \quad y \times \frac{1}{\cos x}(-\sin x) + \log \cos x \frac{dy}{dx}$$

$$= \log \sin y + x.\frac{\cos y}{\sin y}.\frac{dy}{dx}$$

$$\Rightarrow \quad -y\tan x + \frac{dy}{dx}\log\cos x = \log\sin y + x\cot y\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx}(\log\cos x - x\cot y) = \log\sin y + y\tan x$$

$$\therefore \quad \frac{dy}{dx} = \frac{\log\sin y + y\tan x}{\log\cos x - x\cot y}$$ **Ans.**

17. If $(x-a)^2 + (y-b)^2 = c^2$, for some $c > 0$, prove that $\dfrac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}{\dfrac{d^2y}{dx^2}}$ is a constant independent of a and b. **[4]**

Solution :

If $(x-a)^2 + (y-b)^2 = c^2, \quad c > 0$ ...(i)

On differentiating equation (i) w.r.t. x, we get

$$2(x-a) + 2(y-b)\frac{dy}{dx} = 0$$

$$\Rightarrow \quad x - a + (y-b)\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{(x-a)}{(y-b)} \quad \text{...(ii)}$$

Again, differentiating equation (ii) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = -\left[\frac{(y-b)\dfrac{d}{dx}(x-a) - (x-a)\dfrac{d}{dx}(y-b)}{(y-b)^2}\right]$$

$$= -\left[\frac{(y-b) - (x-a)\,dy/dx}{(y-b)^2}\right]$$

$$= -\left[\frac{(y-b) - (x-a)\,dy/dx}{(y-b)^2}\right]$$

$$= -\left[\frac{(y-b) + \dfrac{(x-a)(x-a)}{(y-b)}}{(y-b)^2}\right]$$

$$\left[\text{from equation (ii)}, \frac{dy}{dx} = -\frac{(x-a)}{(y-b)}\right]$$

$$= -\left[\frac{(y-b)^2 + \dfrac{(x-a)^2}{(y-b)}}{(y-b)^2}\right]$$

$$= -\left[\frac{(y-b)^2 + (x-a)^2}{(y-b)^3}\right]$$

$$= -\left[\frac{c^2}{(y-b)^3}\right]$$

$[\because \text{From equation (i)}, (x-a)^2 + (y-b)^2 = c^2]$

Now, $$\frac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}{\dfrac{d^2y}{dx^2}}$$

$$= \frac{-\left[1 + \dfrac{(x-a)^2}{(y-b)^2}\right]^{3/2}}{\dfrac{c^2}{(y-b)^3}}$$

$$= -\frac{\left[(y-b)^2 + (x-a)^2\right]^{3/2} \times (y-b)^3}{(y-b)^{2\times3/2} \times c^2}$$

$$= -\frac{\left[(y-b)^2 + (x-a)^2\right]^{3/2} \times (y-b)^3}{(y-b)^3 \times c^2}$$

$$= -\frac{\left[(y-b)^2 + (x-a)^2\right]^{3/2}}{c^2}$$

$$= -\frac{-c^{2\times3/2}}{c^2}$$

$$\left[\because c^2 = (y-b)^2 + (x-a)^2\right]$$

$$= \frac{-c^3}{c^2} = -c = \text{constant}$$

It shows that $\dfrac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}{\dfrac{d^2y}{dx^2}}$ is independent of a and b

Hence Proved.

18. Find the equation of the normal to the curve $x^2 = 4y$ which passes through the point $(-1, 4)$. [4]

Solution :

Suppose the normal at $P(x_1, y_1)$ on the parabola $x^2 = 4y$ passes through $(-1, 4)$

Since, $P(x_1, y_1)$ lies on $x^2 = 4y$

$$\therefore \quad x_1^2 = 4y_1 \quad \text{...(i)}$$

The equation of curve is $x^2 = 4y$

Differentiating with respect to x, we have

$$2x = 4\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x}{2}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{x_1}{2}$$

The equation (x_1, y_1) of normal at $P(x_1, y_1)$ is

$$y - y_1 = \frac{-1}{\frac{dy}{dx}}(x - x_1)$$

$$y - y_1 = \frac{-2}{x_1}(x - x_1) \quad \text{...(ii)}$$

$\because$ It passes through $(-1, 4)$,

$\therefore$ Putting $x = -1$ and $y = y$, we get

$$4 - y_1 = \frac{-2}{x_1}(-1 - x_1)$$

$$\Rightarrow \quad 4 - y_1 = \frac{2}{x_1}(1 + x_1)$$

$$\Rightarrow \quad 4x_1 - x_1 y_1 = 2 + 2x_1$$

$$\Rightarrow \quad 2x_1 = 2 + x_1 y_1$$

$$\Rightarrow \quad \frac{2x_1 - 2}{x_1} = y_1 \quad \text{....(iii)}$$

Eliminating y_1 from equation (i), we have

$$x_1^2 = 4\left(\frac{2x_1 - 2}{x_1}\right)$$

$$x_1^3 = 8x_1 - 8$$

$$\Rightarrow \quad x_1 = 2$$

Putting $x_1 = 2$ in (iii), we get $y_1 = 1$

Putting values of x_1, y_1 in (ii), we get

$$y - 1 = -1(x - 2)$$

$$x + y - 3 = 0$$

Which is the required equation of normal to the given curve. **Ans.**

19. Find : $\displaystyle\int \frac{x^2 + x + 1}{(x+2)(x^2+1)} dx.$ **[4]**

Solution :

Let, $$I = \int \frac{x^2 + x + 1}{(x+2)(x^2+1)} dx$$

By partial fractions

$$\frac{x^2 + x + 1}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{Bx + C}{x^2+1}$$

$$x^2 + x + 1 = A(x^2+1) + Bx + C(x+2)$$

$$= Ax^2 + A + Bx^2 + 2Bx + Cx + 2C$$

$$x^2 + x + 1 = x^2(A + B) + x(2B + C) + A + 2C \quad \text{...(i)}$$

On comparing coefficients of equation (i), we get

$$1 = A + B$$

and $$1 = 2B + C$$

On putting $x = -2$ in (i), we get

$$(-2)^2 + (-2) + 1 = (-2)^2 A + A$$

$$3 = 4A + A$$

$$\frac{3}{5} = A$$

Then, $$1 = \frac{3}{5} + B$$

$$B = \frac{2}{5} \text{ and } C = \frac{1}{5}$$

Hence, $$\frac{x^2 + x + 1}{(x+2)(x^2+1)} = \frac{3}{5(x+2)} + \frac{\frac{2}{5}x + \frac{1}{5}}{x^2+1}$$

Then,

$$\int \frac{(x^2 + x + 1)dx}{(x+2)(x^2+1)} = \frac{3}{5}\int \frac{1}{x+2}dx + \frac{2}{5}\int \frac{x}{x^2+1}dx + \frac{1}{5}\int \frac{1}{(x^2+1)}dx$$

$$= \frac{3}{5}\log|x+2| + \frac{1}{5}\int \frac{2x}{x^2+1} + \frac{1}{5}\int \frac{dx}{x^2+1}$$

$$= \frac{3}{5}\log|x+2| + \frac{1}{5}\log|x^2+1| + \frac{1}{5}\tan^{-1}x + c$$

Ans.

20. Prove that : $\displaystyle\int_0^a f(x)dx = \int_0^a f(a-x)dx$ **and hence evaluate** $\displaystyle\int_0^{\pi/2} \frac{x}{\sin x + \cos x} dx.$ **[4]**

Solution :

$$\text{R.H.S.} = \int_0^a f(a-x)\, dx$$

Let $$a - x = v$$

$$-1 = \frac{dv}{dx}, \text{ for } x = 0, v = a$$

$$x = a, v = 0$$

Then,

$$\text{R.H.S.} = \int_a^0 f(v)(-dv)$$

$$= -\int_a^0 f(v)(dv)$$

$$= \int_0^a f(v)(dv)$$

$$\left[\because \int_a^b f(x)dx = -\int_b^a f(x)dx\right]$$

Now, replacing v by x,

$$= \int_0^a f(x)\,dx$$

Hence, $\int_0^a f(x)\,dx = \int_0^a f(a-x)dx$ **Hence Proved.**

Now, Let $I = \int_0^{\pi/2} \frac{x}{\sin x + \cos x}dx$ ...(i)

$$I = \int_a^{\pi/2} \frac{\frac{\pi}{2} - x}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)}dx$$

$$\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$I = \int_0^{\pi/2} \frac{\frac{\pi}{2} - x}{\cos x + \sin x}dx \quad \text{(ii)}$$

Adding equations (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{x + \frac{\pi}{2} - x}{\sin x + \cos x}dx$$

$$2I = \int_0^{\pi/2} \frac{\frac{\pi}{2}}{a\sin x + \cos x}dx$$

$$2I = \frac{\pi}{2}\int_0^{\pi/2} \frac{1}{\sin x + \cos x}dx$$

$$2I = \frac{\pi}{2}\int_0^{\pi/2} \frac{1}{\frac{2\tan x/2}{1+\tan^2 x/2} + \frac{1-\tan^2 x/2}{1+\tan^2 x/2}}dx$$

$$2I = \frac{\pi}{2}\int_0^{\pi/2} \frac{1+\tan^2\frac{x}{2}}{2\tan\frac{x}{2} + 1 - \tan^2\frac{x}{2}}dx$$

$$2I = \frac{\pi}{2}\int_0^{\pi/2} \frac{\sec^2 x/2}{-\tan^2 x/2 + 2\tan x/2 + 1}dx$$

Let, $\tan\frac{x}{2} = t$

Then, $\frac{d}{dx}\left(\tan\frac{x}{2}\right) = \frac{d}{dx}(t)$

$$\frac{1}{2}\sec^2\frac{x}{2}\,dx = dt$$

$$\sec^2\frac{x}{a}dx = 2dt$$

Also, $x = 0$

$\Rightarrow \quad t = \tan 0° = 0$

and $x = \pi/2$

$\Rightarrow \quad t = \tan\pi/4 = 1$

$$\therefore \quad 2I = \frac{\pi}{2}\int_0^1 \frac{2dt}{-t^2 + 2t + 1}$$

$$= \pi\int_0^1 \frac{dt}{-t^2 + 2t + 1}$$

$$= \pi\int_0^1 \frac{dt}{-(t^2 - 2t - 1)}$$

$$= \pi\int_0^1 \frac{dt}{-\left[(t-1)^2 - 2\right]}$$

$$= \pi\int_0^1 \frac{dt}{(\sqrt{2})^2 - (t-1)^2}$$

$$= \pi \times \frac{1}{2\sqrt{2}}\left[\log\left|\frac{\sqrt{2}+t-1}{\sqrt{2}-t+1}\right|\right]_0^1$$

$$= \pi \times \frac{1}{2\sqrt{2}}\left[\log 1 - \log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right]$$

$$= \frac{-\pi}{2\sqrt{2}}\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right) = \frac{\pi}{2\sqrt{2}}\log\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$

$$\Rightarrow \quad 2I = \frac{\pi}{2\sqrt{2}}\log\left\{\frac{(\sqrt{2}+1)^2}{(\sqrt{2}-1)(\sqrt{2}+1)}\right\}$$

$$\Rightarrow \quad 2I = \frac{\pi}{2\sqrt{2}}\log(\sqrt{2}+1)^2$$

$$\Rightarrow \quad I = \frac{\pi}{4\sqrt{2}}\log(\sqrt{2}+1)^2$$

$$\Rightarrow \quad I = \frac{2\pi}{4\sqrt{2}}\log(\sqrt{2}+1)$$

$$\Rightarrow \quad I = \frac{\pi}{2\sqrt{2}}\log(\sqrt{2}+1)$$ **Ans.**

21. Solve the differential equation :

$$x\frac{dy}{dx} = y - x\tan\left(\frac{y}{x}\right)$$ **[4]**

Solution :

Given, $x\frac{dy}{dx} = y - x\tan\left(\frac{y}{x}\right)$

Let, $y = vx$

Then, $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore \quad x\frac{dy}{dx} = y - x\tan\left(\frac{y}{x}\right)$

$$\Rightarrow \quad x\left(v+x\frac{dv}{dx}\right)=vx-x\tan\left(\frac{y}{x}\right)$$

$$\Rightarrow \quad x\left(v+x\frac{dv}{dx}\right)=x(v-\tan v)$$

$$\Rightarrow \quad xv+x^2\frac{dv}{dx}=xv - x\tan v$$

$$\Rightarrow \quad x^2\frac{dv}{dx}=-x\tan v$$

$$\Rightarrow \quad x\frac{dv}{dx}=-\tan v$$

$$\Rightarrow \quad \frac{dv}{\tan v}=-\frac{dx}{x}$$

Then $\int \cot v\, dv=-\int\frac{dx}{x}$

$$\log \sin v=-\log x+\log c$$

$$\Rightarrow \quad \log \sin v=\log\frac{c}{x}$$

$$\Rightarrow \quad \sin\frac{y}{x}=\frac{c}{x}$$

$$\Rightarrow \quad x\sin\frac{y}{x}=c$$

Ans.

OR

Solve the differential equation :

$$\frac{dy}{dx}=-\left[\frac{x+y\cos x}{1+\sin x}\right]$$

Solution :

$$\frac{dy}{dx}=-\left[\frac{x+y\cos x}{1+\sin x}\right]$$

$$\Rightarrow \quad \frac{dy}{dx}=-\frac{x}{1+\sin x}-\frac{y\cos x}{1+\sin x}$$

$$\Rightarrow \quad \frac{dy}{dx}+\frac{y\cos x}{1+\sin x}=\frac{-x}{1+\sin x}$$

Here , $P=\frac{\cos x}{1+\sin x}$ and $Q=\frac{-x}{1+\sin x}$

Then, $\quad IF = e^{\int p dx}$

$$= e^{\int\frac{\cos x}{1+\sin x}dx}$$

$$= e^{\log|1+\sin x|}$$

$$= 1+\sin x$$

Then, $\quad y\times IF=\int Q\times IF\ dx+c$

$$y\,(1+\sin x)=\int\frac{-x}{1+\sin x}\times(1+\sin x)dx$$

$$=\int -x\,dx$$

$$y\,(1+\sin x)=\frac{-x^2}{2}+c$$

Ans.

22. The scalar product of the vector $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ with a unit vector along the sum of the vectors $\vec{b}=2\hat{i}+4\hat{j}-5\hat{k}$ and $\vec{c}=\lambda\hat{i}+2\hat{j}+3\hat{k}$ is equal to 1. Find the value of λ and hence find the unit vector along $\vec{b}+\vec{c}$. **[4]**

Solution :

Here, $\quad \vec{a}=\hat{i}+\hat{j}+\hat{k}$, $\hat{n}$ is unit vector

$$\vec{b}=2\hat{i}+4\hat{j}-5\hat{k}$$

$$\vec{c}=\lambda\hat{i}+2\hat{j}+3\hat{k}$$

$$\vec{b}+\vec{c}=(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}$$

Then, $\quad \hat{n}=\frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{(2+\lambda)^2+36+4}}$

Given, $\quad \vec{a}.\hat{n}=1$

$$(\hat{i}+\hat{j}+\hat{k}).\left(\frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{(2+\lambda)^2+40}}\right)=1$$

$$\Rightarrow \quad (2+\lambda)+6-2=\sqrt{(2+\lambda)^2+40}$$

$$\left[\because \hat{i}\cdot\hat{i}=1,\ \hat{j}\cdot\hat{j}=1,\ \hat{k}\cdot\hat{k}=1\right]$$

$$\Rightarrow \quad (2+\lambda)+4=\sqrt{(2+\lambda)^2+40}$$

$$\Rightarrow \quad \lambda+6=\sqrt{(2+\lambda)^2+40}$$

On squaring both sides, we get

$$(6+\lambda)^2=(2+\lambda)^2+40$$

$$\Rightarrow \quad 36+\lambda^2+12\lambda=4+\lambda^2+4\lambda+40$$

$$\Rightarrow 36+12\lambda-4-4\lambda-40=0$$

$$\Rightarrow \quad 8\lambda-8=0$$

$$\Rightarrow \quad \lambda=1$$

Then, $\quad \vec{b}+\vec{c}=(2+\lambda)\hat{i}+6\hat{j}+-2\hat{k}$

$$=3\hat{i}+6\hat{j}-2\hat{k}$$

unit vector along $(\vec{b}+\vec{c})=\frac{3\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{9+36+4}}$

$$=\frac{3\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{49}}$$

$$=\frac{3\hat{i}+6\hat{j}-2\hat{k}}{7}$$

Ans.

23. If the lines $\frac{x-1}{-3}=\frac{y-2}{2\lambda}=\frac{z-3}{2}$ and $\frac{x-1}{3\lambda}=\frac{y-1}{2}=\frac{z-6}{-5}$ are perpendicular, find the value of λ. Hence find whether the lines are intersecting or not. **[4]**

Solution :

The equation of the given lines are,

$$\frac{x-1}{-3}=\frac{y-2}{2\lambda}=\frac{z-3}{2} \text{ and } \frac{x-1}{3\lambda}=\frac{y-1}{2}=\frac{z-6}{-5}$$

If these lines are perpendicular, then

$$a_1a_2+b_1b_2+c_1c_2=0$$

$$\Rightarrow \quad -3\times 3\lambda+2\lambda\times 2+2\times(-5) = 0$$

$$\Rightarrow \quad -9\lambda+4\lambda-10 = 0$$

$$\Rightarrow \quad -5\lambda = 10$$

$$\Rightarrow \quad \lambda = -2$$

Now, the lines are

$$\frac{x-1}{-3}=\frac{y-2}{-4}=\frac{z-3}{2} \text{ and } \frac{x-1}{-6}=\frac{y-1}{2}=\frac{z-6}{-5}$$

The co-ordinates of any point on first line are given by :

$$\frac{x-1}{-3}=\frac{y-2}{-4}=\frac{z-3}{2}=\alpha$$

or $\quad x-1 = -3\alpha \Rightarrow x=-3\alpha+1$

$y-2 = -4\alpha \Rightarrow y=-4\alpha+2$

$z-3 = 2\alpha \Rightarrow z=2\alpha+3$

So, coordinates of any point on this line are, $(-3\alpha+1, 4\alpha+2, 2\alpha+3)$.

The coordinates of any point on second line are given by :

$$\frac{x-1}{-6}=\frac{y-1}{2}=\frac{z-6}{-5}=\beta$$

or $\quad x-1 = -6\beta \Rightarrow x=-6\beta+1$

$y-1 = 2\beta \Rightarrow y=2\beta+1$

$z-6 = -5\beta \Rightarrow z=-5\beta+6$

So, co-ordinates of any point on second line are $(-6\beta+1, 2\beta+1, -5\beta+6)$.

If lines intersect then they have a common point. So, for some value of α and β, we have

$$-3\alpha+1 = -6\beta+1$$

$$\Rightarrow \quad -3\alpha = -6\beta$$

$$\Rightarrow \quad \alpha = 2\beta$$

and $\quad -4\alpha+2 = 2\beta+1$

$$\Rightarrow \quad -4\alpha+1 = 2\beta$$

On solving, we have

$$\alpha = \frac{1}{5}, \text{ and } \beta = \frac{1}{10}$$

The values of α and β do not satisfy the third equation. Hence, lines do not interecet each other. **Ans.**

SECTION–D

24. If $A=\begin{bmatrix}1 & 3 & 4\\2 & 1 & 2\\5 & 1 & 1\end{bmatrix}$, find A^{-1} **[6]**

Hence solve the system of equations

$$x+3y+4z = 8$$
$$2x+y+2z = 5$$
and $$5x+y+z = 7$$

Solution :

If $\quad A = \begin{bmatrix}1 & 3 & 4\\2 & 1 & 2\\5 & 1 & 1\end{bmatrix}$

$$|A| = 1(1-2)-3(2-10)+4(2-5)$$
$$= 11\neq 0$$

Then cofactors of A are

$A_{11}=-1, A_{12}=8, A_{13}=-3, A_{21}=1, A_{22}=-19,$
$A_{23}=14, A_{31}=2, A_{32}=6$ and $A_{33}=-5$

$$\text{Then, adj } A = \begin{bmatrix}A_{11} & A_{12} & A_{13}\\A_{21} & A_{22} & A_{23}\\A_{31} & A_{32} & A^{33}\end{bmatrix}^T$$

$$= \begin{bmatrix}-1 & 8 & -3\\1 & -19 & 14\\2 & 6 & -5\end{bmatrix}^T$$

$$\text{adj } A = \begin{bmatrix}-1 & 1 & 2\\8 & -19 & 6\\-3 & 14 & -5\end{bmatrix}$$

Now, $\quad A^{-1} = \frac{1}{|A|}\text{adj}A$

$$= \frac{1}{11}\begin{bmatrix}-1 & 1 & 2\\8 & -19 & 6\\-3 & 14 & -5\end{bmatrix} \quad ...(ii) \textbf{ Ans.}$$

Given system of equations are

$$x+3y+4z = 8$$
$$2x+y+2z = 5$$
$$5x+y+z = 7$$

Let, $A = \begin{bmatrix}1 & 3 & 4\\2 & 1 & 2\\5 & 1 & 1\end{bmatrix}$ $B = \begin{bmatrix}8\\5\\7\end{bmatrix}$ and $X = \begin{bmatrix}x\\y\\z\end{bmatrix}$

Then, $AX = B$

$$\begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 2 \\ 5 & 1 & 1 \end{bmatrix} X = \begin{bmatrix} 8 \\ 5 \\ 7 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 2 \\ 5 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 8 \\ 5 \\ 7 \end{bmatrix}$$

$$= \frac{1}{11}\begin{bmatrix} -1 & 1 & 2 \\ 8 & -19 & 6 \\ -3 & 14 & -5 \end{bmatrix}\begin{bmatrix} 8 \\ 5 \\ 7 \end{bmatrix}$$

[Using (ii) equation]

$$= \frac{1}{11}\begin{bmatrix} -8+5+14 \\ 64-95+42 \\ -24+70-35 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 11 \\ 11 \\ 11 \end{bmatrix}$$

$$X = \frac{1}{11}\begin{bmatrix} 11 \\ 11 \\ 11 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Hence, $x = 1$, $y = 1$, $z = 1$ **Ans.**

OR

Find the inverse of the following matrix, using elementary transformation :

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

Solution :

Given, $A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

We know that,

$$AA^{-1} = I$$

$$\therefore \quad A = IA$$

$$\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

On applying $R_3 \to R_3 + 3R_1$, we get

$$\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 6 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_3 \to R_3 - R_2$, we get

$$\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix} A$$

Interchanging $R_1 \leftrightarrow R_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A$$

Applying $R_2 \to R_2 - 5R_1$ and $R_3 \to R_3 - 2R_1$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ -5 & 2 & -2 \end{bmatrix} A$$

Applying $R_3 \to (-1) R_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix} A$$

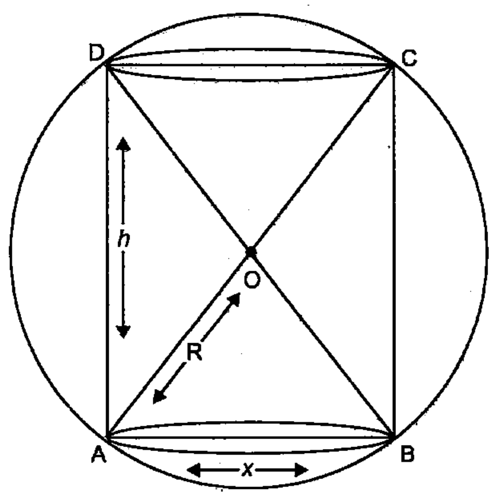
Hence, the required inverse of the matrix is

$$\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & 5 \\ 5 & -2 & 2 \end{bmatrix}$$ **Ans.**

25. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also find the maximum volume. [6]

Solution :

Let, 'x' be the diameter of the base of the cylinder and let 'h' be height of the cylinder.



In ΔABC, we have

$$(BC)^2 + (AB)^2 = (AC)^2$$

$$h^2 + x^2 = (2R)^2$$

$$x^2 = 4R^2 - h^2 \quad \text{...(i)}$$

Volume of cylinder, $V = \pi r^2 h$

$$\Rightarrow \quad V = \pi \times \left(\frac{x}{2}\right)^2 \times h$$

$$\Rightarrow \quad V = \pi \times \frac{x^2}{4} \times h$$

$$\Rightarrow \quad V = \frac{\pi(4R^2 - h^2)}{4} \times h \quad \text{[Using (i)]}$$

$$\Rightarrow \quad V = \frac{4\pi R^2 \times h}{4} - \frac{\pi h^3}{4}$$

$$\Rightarrow \quad V = \pi h R^2 - \pi \frac{h^3}{4} \quad \text{...(ii)}$$

On differentiating equation (ii) w.r.t. h, we get

$$\frac{dV}{dh} = \frac{d\left(\pi h R - \pi h^3/4\right)}{dh}$$

$$\Rightarrow \quad \frac{dV}{dh} = \pi R^2 \frac{d(h)}{dh} - \frac{\pi}{4}\frac{d(h^3)}{dh}$$

$$\Rightarrow \quad \frac{dV}{dh} = \pi R^2 - \frac{\pi}{4}(3h^2)$$

$$\Rightarrow \quad \frac{dV}{dh} = \pi R^2 - \frac{3\pi h^2}{4} \quad \text{...(iii)}$$

$$\Rightarrow \quad 0 = \pi R^2 - \frac{3\pi h^2}{4} \quad \left(\because \frac{dV}{dh} = 0\right)$$

$$\Rightarrow \quad h = \frac{2R}{\sqrt{3}}$$

Again, differentiating equation (iii) w.r.t. h, we get

$$\frac{d^2V}{dh^2} = \frac{d}{dh}\left(\pi R^2 - \frac{3\pi h^2}{4}\right)$$

$$\Rightarrow \quad \frac{d^2V}{dh^2} = 0 - \frac{3}{4}\pi \times 2h$$

$$\Rightarrow \quad \frac{d^2V}{dh^2} = -\frac{3\pi h}{2}$$

At $h = \frac{2R}{\sqrt{3}}$, we have

$$\frac{d^2V}{dh^2} = \frac{-3\pi}{2}\left(\frac{2R}{\sqrt{3}}\right) = -\sqrt{3}\,\pi R$$

$$\frac{d^2V}{dh^2} < 0$$

Hence, $h = \frac{2R}{\sqrt{3}}$ is a point of maxima.

So, V is maximum when $h = \frac{2R}{\sqrt{3}}$

Hence, the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. **Hence Proved.**

From (i), we have

$$x^2 = 4R^2 - h^2$$

$$\Rightarrow \quad x^2 = 4R^2 - \left(\frac{2R}{\sqrt{3}}\right)^2$$

$$\Rightarrow \quad x^2 = \frac{8}{3}R^2$$

$\therefore$ Maximum Volume of cylinder

$$= \pi\left(\frac{x}{2}\right)^2 \times h$$

$$= \pi \times \frac{x^2}{4} \times h$$

$$= \frac{\pi}{4}x^2 h$$

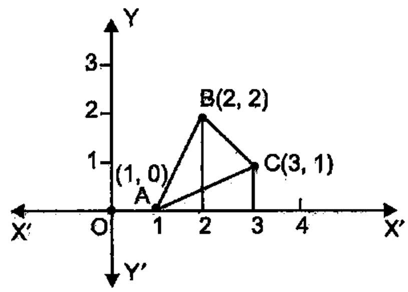
$$= \frac{\pi}{4} \times \frac{8R^2}{3} \times \frac{2R}{\sqrt{3}}$$

$$= \frac{4\pi R^3}{3\sqrt{3}} \quad \textbf{Ans.}$$

26. Using method of integration, find the area of the triangle whose vertices are (1, 0), (2, 2) and (3, 1). **[6]**

Solution :

A(1, 0), B(2, 2) and C(3, 1)



Let A(1, 0), B(2, 2) and C(3, 1) be the vertices of a triangle ABC.

Area of ΔABC = Area of ΔABD + Area of trapezium BDEC – Area of ΔAEC

Now, Equation of side AB,

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$y - 0 = 2(x - 1)$$

$$y = 2(x - 1) \quad \text{...(i)}$$

Equation of line BC,

$$y - 2 = \frac{1-2}{3-2}(x - 2)$$

$$y - 2 = \frac{-1}{1}(x - 2)$$

$$y - 2 = -(x-2)$$
$$y = 2-(x-2)$$
$$y = 4 - x \qquad ...(ii)$$

Equation of line AC,

$$y - 0 = \frac{1-0}{3-1}(x-1)$$
$$y - 0 = \frac{1}{2}\ (x-1)$$
$$y = \frac{1}{2}(x-1) \qquad ...(iii)$$

Hence, area of ΔABC

$$= \int_1^2 2(x-1)dx + \int_2^3 (4-x)dx - \int_1^3 \frac{x-1}{2}dx$$
$$= 2\left[\frac{x^2}{2} - x\right]_1^2 + \left[4x - \frac{x^2}{2}\right]_2^3 - \frac{1}{2}\left[\frac{x^2}{2} - x\right]_1^3$$
$$= 2\left[\left(\frac{2^2}{2}-2\right) - \left(\frac{1}{2}-1\right) + \left[\left(4\times 3 - \frac{3^2}{2}\right)\right.\right.$$
$$\left.\left. - \left(4\times 2 - \frac{2^2}{2}\right)\right] - \frac{1}{2}\left[\left(\frac{3^2}{2}-3\right) - \left(\frac{1}{2}-1\right)\right]$$
$$= 2\left(\frac{1}{2}\right) + \frac{3}{2} - 1 = \frac{3}{2} \text{ sq. units.}$$ **Ans.**

OR

Using method of integration, find the area of the region enclosed between two circles $x^2 + y^2 = 4$ and $(x-2)^2 + y^2 = 4$.

Solution :

Equations of the given circles are,

$$x^2 + y^2 = 4 \qquad ...(i)$$
$$(x-2)^2 + y^2 = 4 \qquad ...(ii)$$

Equation (i) is a circle with centre O at the origin and radius 2. Equation (ii) is a circle with centre C (2, 0) and radius 2.

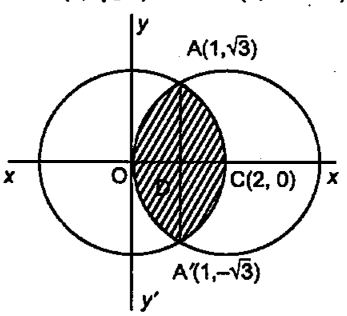
Solving equation (i) and (ii) we have

$$(x-2)^2 + y^2 = x^2 + y^2$$

or $\quad x^2 - 4x + 4 + y^2 = x^2 + y^2$

or $x = 1$ which gives $y = \pm\sqrt{3}$

Thus, the points of intersection of the given circles are A $(1, \sqrt{3})$ and A′$(1, -\sqrt{3})$



Required area of the enclosed region OACA′O between circles

$= 2$ [area of region ODCAO]

$= 2$ [area of region ODAO + area of region OCAD]

$$= 2\left[\int_0^1 y\,dx + \int_1^2 y\,dx\right]$$
$$= 2\left[\int_0^1 \sqrt{4-(x-2)^2}\,dx + \int_1^2 \sqrt{4-x^2}\,dx\right] \text{ [from (i)]}$$
$$= 2\left[\frac{1}{2}(x-2)\sqrt{4-(x-2)^2} + \frac{1}{2}\times 4\sin^{-1}\left(\frac{x-2}{2}\right)\right]_0^1$$
$$+ 2\left[\frac{1}{2}x\sqrt{4-x^2} + \frac{1}{2}\times 4\sin^{-1}\frac{x}{2}\right]_1^2$$
$$= \left[(x-2)\sqrt{4-(x-2)^2} + 4\sin^{-1}\left(\frac{x-2}{2}\right)\right]_1^2$$
$$+ \left[x\sqrt{4-x^2} + 4\sin^{-1}\frac{x}{2}\right]_1^2$$
$$= \left[-\sqrt{3} + 4\sin^{-1}\left(\frac{-1}{2}\right) - 4\sin^{1}(-1)\right]$$
$$+ \left[4\sin^{-1}1 - \sqrt{3} - 4\sin^{-1}\frac{1}{2}\right]$$
$$= \left[\left(-\sqrt{3} - 4\times\frac{\pi}{6}\right) + 4\times\frac{\pi}{2}\right] + \left[4\times\frac{\pi}{2} - \sqrt{3} - 4\times\frac{\pi}{6}\right]$$
$$= \frac{8\pi}{3} - 2\sqrt{3}$$ **Ans.**

27. Find the vector and cartesian equations of the plane passing through the points having position vectors $\hat{i} + \hat{j} - 2\hat{k}$, $2\hat{i} - \hat{j} + \hat{k}$ and $\hat{i} + 2\hat{j} + \hat{k}$. Write the equation of a plane passing through a point (2, 3, 7) and parallel to the plane obtained above. Hence, find the distance between the two parallel planes. [6]

Solution : Let A, B,C be the points with position vectors $\hat{i} + \hat{j} - 2\hat{k}, 2\hat{i} - \hat{j} + \hat{k}$ and $\hat{i} + 2\hat{j} + \hat{k}$ respectively.

Then, $\quad \vec{AB}$ = P. V. of B – P. V. of A

$$= \left(2\hat{i} - \hat{j} + \hat{k}\right) - \left(\hat{i} + \hat{j} - 2\hat{k}\right)$$
$$= \hat{i} - 2\hat{j} + 3\hat{k}$$

and $\quad \vec{BC}$ = P. V. of C – P.V. of B

$$= \left(\hat{i} + 2\hat{j} + \hat{k}\right) - \left(2\hat{i} - \hat{j} + \hat{k}\right)$$
$$= -\hat{i} + 3\hat{j} + 0\hat{k}$$

A vector normal to the plane containing points A, B & C is

$$\vec{n} = \vec{AB} \times \vec{AC}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ -1 & 3 & 0 \end{vmatrix}$$

$$= -9\hat{i} - 3\hat{j} + \hat{k}$$

The required plane passes through the point having position vector $\vec{a} = \hat{i} + \hat{j} - 2\hat{k}$ and is normal to the vector $-9\hat{i} - 3\hat{j} + 2\hat{k}$. So, its vector equation is,

$$\left(\vec{r} - \vec{a}\right) . \vec{n} = 0$$

$$\Rightarrow \quad \vec{r} . \vec{n} - \vec{a} . \vec{n} = 0$$

$$\Rightarrow \quad \vec{r} . \vec{n} = \vec{a} . \vec{n}$$

$$\Rightarrow \vec{r} . (-9\hat{i} - 3\hat{j} + \hat{k}) = (\hat{i} + \hat{j} + 2\hat{k}) . \left(-9\hat{i} - 3\hat{j} + \hat{k}\right)$$

$$\Rightarrow \vec{r} . (-9\hat{i} - 3\hat{j} + \hat{k}) = -9 - 3 - 2$$

$$\Rightarrow \hat{r} . (-9\hat{i} - 3\hat{j} + \hat{k}) = -14$$

This is the required vector equation of the plane
The cartesian equation of plane is given by

$$(x\hat{i} + y\hat{j} + z\hat{k}) . (-9\hat{i} - 3\hat{j} + \hat{k}) = -14$$

$$-9x - 3y + z = -14$$

$$9x + 3y - z = 14$$

Direction ratios of this plane are $(9, 3, -1)$

Then the equation of plane parallel to the above plane and passing through (2,3,7) is

$$= a(x - x_1) + b(y - y_1) + c(z - z_1)$$

$$= 9(x - 2) + 3(y - 3) - 1(z - 7)$$

$$\Rightarrow \quad 9x + 3y - z - 20 = 0$$

This is the required parallel plane.

Then, Distance between $9x + 3y - z + 14 = 0$ and $9x + 3y - z - 20 = 0$

Let $P(x_1, y_1, z_1)$ be any point on $9x + 3y - z + 14 = 0$

Then,

$$9x_1 + 3y_1 - z_1 - 14 = 0$$

Let d be the distance between planes. Then,

d = length of perpendicular from $P(x_1, y_1, z_1)$ to $9x + 3y - z - 20 = 0$

$$d = \left|\frac{9x_1 + 3y_1 - z_1 - 20}{\sqrt{(9)^2 + (3)^2 + (-1)^2}}\right|$$

$$= \left|\frac{+14 - 20}{\sqrt{91}}\right|$$

$$= \frac{6}{\sqrt{91}} \text{ units.}$$ **Ans.**

OR

Find the equation of the line pasing through (2, –1, 2) and (5, 3, 4) and of the plane passing through (2, 0, 3), (1, 1, 5) and (3, 2, 4). Also, find their point of intersection.

Solution :

We know that the equation of line passing through points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

So equation of line is given by

$$\Rightarrow \quad \frac{x - 2}{3} = \frac{y + 1}{4} = \frac{z - 2}{2}$$

Now, equation of plane passing through points (2, 0,3), (1,1,5) and (3,2,4) is given by

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

$$\begin{vmatrix} x - 2 & y - 0 & z - 3 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow (x - 2)(-3) - y(-1 - 2) + (z - 3)(-2 - 1) = 0$$

$$\Rightarrow -3x + 6 + 3y - 3z + 9 = 0$$

$$\Rightarrow -x + y - z + 5 = 0 \quad \text{...(i)}$$

This is the required equation of plane.

Now,

Let $$\frac{x - 2}{3} = \frac{y + 1}{4} = \frac{z - 2}{2} = k$$

$$\therefore \quad x = 3k + 2$$

$$y = 4k - 1$$

$$z = 2k + 2$$

On putting these values in equation (i), we have

$$-(3k + 2) + 4k - 1 - (2k + 2) + 5 = 0$$

$$\Rightarrow -3k - 2 + 4k - 1 - 2k - 2 + 5 = 0$$

$$\Rightarrow \quad -2k = 0$$

$$\Rightarrow \quad k = 0$$

Then, intersection points are

$$x = 3k + 2 = 2,$$

$$y = 4k - 1 = -1,$$

$$z = 2k + 2 = 2$$

$\therefore$ Point of interaction is (2, –1, 2) **Ans.**

28. There are three coins. One is a two-headed coin, another is a biased coin that comes up heads 75% of the time and the third is an unbiased coin. One of the three coins is chosen at random and tossed. If it shows heads, what is the probability that it is the two-headed coin? [6]

Solution : Given, there are three coins.

Let, E_1 = coin is two headed

E_2 = biased coin

E_3 = unbiased coin

A = shows only head

Here, $P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$

Then $P\left(\frac{A}{E_1}\right) = 1$

$\left(\frac{A}{E_2}\right) = \frac{75}{100} = \frac{3}{4}$ (given)

$P\left(\frac{A}{E_3}\right) = \frac{1}{2}$

Now, Probability of two headed coin

$P(E_1/A)$

$$= \frac{P(E_1)P(A/E_1)}{P(E_1)\times P(A/E_1) + P(E_2)\times P(A/E_2) + P(E_3)\times P(A/E_3)}$$

$$= \frac{\frac{1}{3}\times 1}{\frac{1}{3}\times 1 + \frac{1}{3}\times\frac{3}{4} + \frac{1}{3}\times\frac{1}{2}} = \frac{1}{\frac{4+3+2}{4}} = \frac{4}{9}$$ **Ans.**

29. A company produces two types of goods, A and B, that require gold and silver. Each unit of type A requries 3 g of silver and 1 g of gold while that of type B requires 1 g of silver and 2 g of gold. The company can use at the most of 9 g of silver and 8 of gold. If each unit of type A brings a profit of ₹ 40 and that of type B ₹ 50, find the number of units of each type that the company should produce to maximize profit. Formulate the above LPP and solve it graphically and also find the maximum profit [6]

Solution :

There are two types of goods, A and B and let units of type A be x and units of type B be y.

	A	B
Gold	1	2
Silver	3	1
Profit	40	50

Then, Total profit of goods

$P = 40x + 50y, x \geq 0, y \geq 0$

Hence, the mathematical formulation of the problem is as follows :

Maximise $P = 40x + 50y$ ...(i)

Subject to the constraints :

$x + 2y \leq 8$ ...(ii)

$3x + y \leq 9$ ...(iii)

$x \geq 0, y \geq 0$

To solve this LPP, we draw the lines

$x + 2y = 8$

$3x + y = 9$

$x = 0$ and $y = 0$

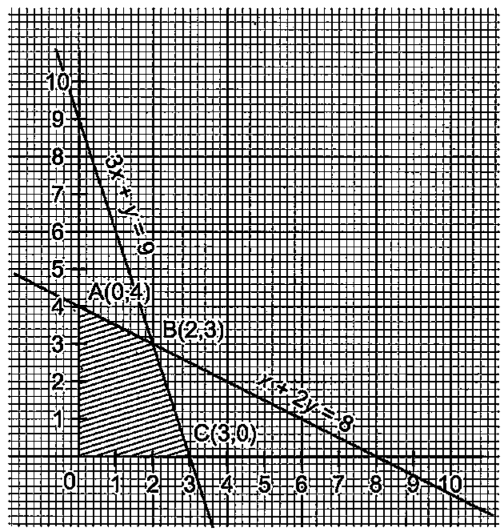
$x + 2y = 8$

x	0	8
y	4	0

and $3x + y = 9$

x	0	3
y	9	0

Plotting these points on the graph



The shaded region is the required feasible region.

Corner Points	Maximum $P = 40x + 50y$
A(0,4)	$0 + 50 \times 4 = 200$
B(2,3)	$2 \times 40 + 3 \times 50 = 230$
C(3,0)	$40 \times 3 + 0 = 120$
O (0,0)	$0 + 0 = 0$

Clearly, P is maximum at B, (2,3) and the maximum profit is ₹ 230. **Ans.**

Mathematics 2019 (Outside Delhi) SET II

Time allowed : 3 hours **Maximum marks : 100**

SECTION–A

1. Find $|AB|$, if $A = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix}$. [1]

Solution :

Given, $A = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix}$

Then, $AB = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\therefore \quad |AB| = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$ **Ans.**

2. Differentiate $e^{\sqrt{3x}}$, with respect to x. [1]

Solution :

Let $y = e^{\sqrt{3x}}$

On differentiating equation (i) w.r.t x, we get

$$\frac{dy}{dx} = e^{\sqrt{3x}} \times \sqrt{3} \times \frac{1}{2} x^{\frac{-1}{2}}$$

$$= \frac{\sqrt{3}}{2}\frac{e^{\sqrt{3x}}}{\sqrt{x}}$$ **Ans.**

SECTION–B

6. If $A = \begin{bmatrix} p & 2 \\ 2 & p \end{bmatrix}$ and $|A^3| = 125$, the find the value of p. [2]

Solution :

Given, $A = \begin{bmatrix} p & 2 \\ 2 & p \end{bmatrix}$ and $|A^3| = 125$

Now, $|A^3| = 125$

or $|A^3| = 5^3$

$|A| = 5$

Also we have $|A| = \begin{vmatrix} p & 2 \\ 2 & p \end{vmatrix}$

$\Rightarrow \quad p^2 - 4 = 5$

$\Rightarrow \quad p^2 = 9$

$\Rightarrow \quad p = \pm 3$ **Ans.**

12. Find the general solution of the differential equation $\frac{dy}{dx} = e^{x+y}$ [2]

Solution :

Given, $\frac{dy}{dx} = e^{x+y}$

$\Rightarrow \quad \frac{dy}{dx} = e^x.e^y$

$\Rightarrow \quad \frac{dy}{e^y} = e^x.dx$

$\Rightarrow \quad e^{-y}\,dy = e^x.dx$

On integrating both sides, we get

$$\int e^{-y} dy = \int e^x dx$$

$\Rightarrow e^{-y} = e^x + c$, which is the required solution. **Ans.**

SECTION–C

21. If $(a + bx)\, e^{y/x} = x$, then prove that

$$x^3 \frac{d^2y}{dx^2} = \left(x\frac{dy}{dx} - y\right)^2$$ **[4]**

Solution : We have

$(a + bx)\, e^{y/x} = x$

$\Rightarrow \quad e^{y/x} = \frac{x}{a+bx}$

$\Rightarrow \quad \frac{y}{x} = \log \frac{x}{a+bx}$

$\Rightarrow \quad y = x \log\left(\frac{x}{a+bx}\right)$

$\Rightarrow \quad y = x\,\{\log x - \log(a + bx)\}$

$\Rightarrow \quad \frac{y}{x} = \log x - \log(a + bx)$

On differentiating w.r.t. x, we get

$$\frac{x\frac{dy}{dx} - y}{x^2} = \frac{1}{x} - \frac{1}{a+bx} \times b$$

$\Rightarrow \quad \frac{1}{x^2}\left(x\frac{dy}{dx} - y\right) = \frac{1}{x} - \frac{b}{a+bx}$

$\Rightarrow \quad x\frac{dy}{dx} - y = x^2\left(\frac{1}{x} - \frac{b}{a+bx}\right)$

$$\Rightarrow \quad x\frac{dy}{dx} - y = \frac{ax}{a+bx} \quad ...(i)$$

Again, differentiating both sides w.r.t. x, we get

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} - \frac{dy}{dx} = \frac{(a+bx)a - ax\times b}{(a+bx)^2}$$

$$\Rightarrow \quad x\frac{d^2y}{dx^2} = \frac{a^2 + abx - abx}{(a+bx)^2}$$

$$\Rightarrow \quad x\frac{d^2y}{dx^2} = \frac{a^2}{(a+bx)^2}$$

On multiplying both sides by x^2, we get

$$x^3\frac{d^2y}{dx^2} = \frac{a^2x^2}{(a+bx)^2}$$

$$x^3\frac{d^2y}{dx^2} = \left(\frac{ax}{a+bx}\right)^2 \quad ...(ii)$$

From (i) and (ii), we get

$$x^3\frac{d^2y}{dx^2} = \left(x\frac{dy}{dx} - y\right)^2$$

Hence Proved.

22. The volume of a cube is increasing at the rate of 8 cm³/s. How fast is the surface area increasing when the length of its edge is 12 cm? [4]

Solution :

Let x be the length of side, V be the volume and S be the surface area of cube.

Then, $V = x^3$ and $S = 6x^2$, where x is a function of time t

Now, $$\frac{dV}{dt} = 8\text{cm}^3/\text{s (given)}$$

$$\therefore \quad 8 = \frac{dV}{dt} = \frac{d}{dt}(x^3) = 3x^2\frac{dx}{dt}$$

(By chain rule)

$$8 = 3x^2\frac{dx}{dt}$$

$$\frac{8}{3x^2} = \frac{dx}{dt} \quad ...(i)$$

Now, $$\frac{dS}{dt} = \frac{d(6x^2)}{dt}$$

$$\Rightarrow \quad \frac{dS}{dt} = 12x\frac{dx}{dt}$$

$$\Rightarrow \quad \frac{dS}{dt} = 12x\frac{8}{3x^2} \quad \text{[Using (i)]}$$

$$= \frac{32}{x}$$

Hence, when $x = 12$ cm

Then, $$\frac{dS}{dt} = \frac{32}{12} = \frac{8}{3} \text{ cm}^2/\text{s}$$ **Ans.**

23. Find the cartesian and vector equations of the plane passing through the point A(2, 5, – 3), B(– 2, – 3, 5) and C(5, 3, – 3). [4]

Solution :

We know that the general equation of the plane passing through three points (x_1, y_1, z_1) (x_2, y_2, z_2), (x_3, y_3, z_3)

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

Then the plane passing through A (2, 5, –3), B (– 2, – 3, 5,), C (5, 3, +3)

$$\begin{vmatrix} x-2 & y-5 & z+3 \\ -2-2 & -3-5 & 5+3 \\ 5-2 & 3-5 & -3+3 \end{vmatrix} = 0$$

$$\begin{vmatrix} x-2 & y-5 & z+3 \\ -4 & -8 & 8 \\ 3 & -2 & 0 \end{vmatrix} = 0$$

$$\Rightarrow \quad (x-2)(0+16) - (y-5)(0-24) + (z+3)(8+24) = 0$$

$$\Rightarrow \quad 16(x-2) + 24(y-5) + 32(z+3) = 0$$

$$\Rightarrow \quad 8[2x - 4 + 3y - 15 + 4z + 12] = 0$$

$$\Rightarrow \quad 2x + 3y + 4z - 7 = 0$$

$$\Rightarrow \quad 2x + 3y + 4z = 7$$

This is the required cartesian equation of the plane.

Now,

The required plane passes through the point A(2, 5, –3) whose position vector is $a = 2\hat{i} + 5\hat{j} - 3\hat{k}$ and is normal to the vector $\vec{n}$ given by $\vec{n} = \vec{AB} \times \vec{AC}$

$$\therefore \quad \vec{AB} = -2\hat{i} - 3\hat{j} + 5\hat{k} - (2\hat{i} + 5\hat{j} - 3\hat{k})$$

$$\vec{AB} = -4\hat{i} - 8\hat{j} + 8\hat{k} + 8\hat{k}$$

$$\vec{AC} = \left(5\hat{i} + 3\hat{j} - 3\hat{k} - \left(2\hat{i} + 5\hat{j} - 3\hat{k}\right)\right)$$

$$= 3\hat{i} - 2\hat{j}$$

Now, $\vec{n} = \vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -4 & -8 & 8 \\ 3 & -2 & 0 \end{vmatrix}$

$$= 16\hat{i} - (-24)\hat{j} + (8+24)\hat{k}$$

$$= 16\hat{i} + 24\hat{j} + 32\hat{k}$$

The vector equation of the plane is given by

$$\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$$

$\Rightarrow \vec{r}.(16\hat{i} + 24\hat{j} + 32\hat{k})$

$$= (2\hat{i} + 5\hat{j} - 3\hat{k})(16\hat{i} + 24\hat{j} + 32\hat{k})$$

$$= 32 + 120 - 96$$

$\Rightarrow \quad \vec{r}.(16\hat{i} + 24\hat{j} + 32\hat{k}) = 56$

$\Rightarrow \quad \vec{r}.(2\hat{i} + 3\hat{j} + 4\hat{k}) = 7$

This is the required vector equation. **Ans.**

SECTION–D

24. Find the point on the curve $y^2 = 4x$, which is nearest to the point (2, – 8). [6]

Solution :

Given curve is of the form, $y^2 = 4x$ and let $p(x,y)$ is a point on the curve which is nearest to the point (2, – 8).

$\because \quad y^2 = 4x$

then $\quad \frac{y^2}{4} = x$

$\therefore$ P(x, y) will be P$\left(\frac{y^2}{4}, y\right)$

Now, the distance between point A and P is given by

$$AP = \sqrt{(x-2)^2 + (y+8)^2}$$

$$= \sqrt{\left(\frac{y^2}{4} - 2\right)^2 + (y+8)^2}$$

$$= \sqrt{\frac{y^4}{16} - y^2 + 4 + y^2 + 16y + 64}$$

$$= \sqrt{\frac{y^4}{16} + 16y + 68}$$

and let $Z = AP^2 = \frac{y^4}{16} + 16y + 68 \quad$ ...(i)

Now, differentiate equation (i) w.r.t. y, we get

$$\frac{dZ}{dy} = \frac{1}{16} \times 4y^3 + 16$$

$$= \frac{y^3}{4} + 16$$

For maximum or minumim value of Z, we have

$$\frac{dZ}{dy} = 0$$

$\Rightarrow \quad \frac{y^3}{4} + 16 = 0$

$\Rightarrow \quad y^3 + 64 = 0$

$\Rightarrow \quad (y+4)(y^2 - 4y + 16) = 0$

($\because y^2 - 4y + 16 = 0$ gives imaginary value of y)

$\Rightarrow \quad y = -4$

Now, $\quad \frac{d^2Z}{dy^2} = \frac{1}{4} \times 3y^2 = \frac{3}{4}y^2$

At $y = -4$,

$$\frac{d^2z}{dy^2} = \frac{3}{4}(-4)^2$$

$$= 12 > 0$$

Thus, Z is minimum when $y = -4$

Substituting $y = -4$ in equation of curve $y^2 = 4x$.

We have $x = 4$

Hence, the point (4, –4) on the curve $y^2 = 4x$ is nearest to the point (12, – 8). **Ans.**

25. Find $\int_1^3 (x^2 + 2 + e^{2x})\, dx$ as the limit of sums. [6]

Solution :

We have

$$\int_a^b f(x)dx = \lim_{h \to 0} h\, [f(a) + f(a+h) + f(a+2h) + ...f(a+(n-1)h]$$

where $h = \frac{b-a}{n}$

(i) Here, $\quad a = 1$ and $b = 3$

$$f(x) = x^2 + 2 + e^{2x}$$

$\therefore \quad h = \frac{3-1}{n} = \frac{2}{n} \Rightarrow nh = 2$

Now, $I = \int_1^3 (x^2 + 2 + e^{2x})dx$

$$= \lim_{h \to 0} h[f(1) + f(1+h) + f(1+2h) ...f(1+(n-1)h]$$

$$I = \lim_{h\to 0} h\ [(1+2+e^2) + \{(1+h)^2 + 2 + e^{2(1+h)}\} + \{(1+2h)^2 + 2 + e^{2(1+2h)} + ...\}\ \{(1+(n-1)h)^2 + 2 + e^{2(1+(n-1)h)}\}]$$

$$I = \lim_{h\to 0} h\Big[1+2+e^2+1+h^2+2h + 2 + e^{2(1+h)} + 1 + 4h^2 + 4h + 2e^{2(1+2h)} + ...\{1+(n-1)^2h^2 + 2(n-1)h + 2 + e^{2(1+(n-1)h)}\}\Big]$$

$$I = \lim_{h\to 0} h[1+1+1+......+2+2+.....2h+4h + 2(n-1)h + ... h^2 + 4h^2 + 4h^2 + (n-1)^2h^2 + ... + e^2 + e^{2(1+h)} + e^{2(1+2h)} + ...]$$

$$I = \lim_{h\to 0} h[1+1+1+......+2+2+.....2h+4h + 2(n-1)h + h^2 + 4h^2 + (n-1)^2h^2 + ... + e^2(1+e^{2h}+e^{4h}+.....e^{2(n-1)h}\Big]$$

$$\left[\because 1+e^{2h}+e^{4h}+.....e^{2(n-1)h} = \frac{1(1-e^{2hn})}{1-e^{2n}}\right]$$

$$I = \lim_{h\to 0} h\Big[n + 2n + 2n(1+2+3+.....n-1) + h^2(1+2^2+3^2+......(n-1)^2 + \frac{e^2\times(1-e^{2hn})}{1-e^{2n}}\Big]$$

$$\left[\begin{array}{l}\because\ 1+1+1+.....n = n\\ \quad 2+2+2+n = 2n\end{array}\right]$$

$$I = \lim_{h\to 0} h\left[3n + 2h\frac{n(n-1)}{2} + \frac{h^2(n-1)n(2n-1)}{6} + e^2\frac{(1-e^{2hn})}{1-e^{2n}}\right]$$

$$\left[\because 1+2+3+4+.....n = \frac{n(n+1)}{2}\right.$$

and $$\left.1^2+2^2+3^2+n^2 = \frac{n(n+1)(2n+1)}{6}\right]$$

$$I = \lim_{h\to 0}\left[3nh + nh^2(n-1) + \frac{h^3(n-1)n(2n-1)}{6} + \frac{e^2h(1-e^{2hn})}{1-e^{2n}}\right]$$

$$I = \lim_{h\to 0}\left[nh\times 3 + nh\times h(n-1) + \frac{nh\,h^2(n-1)(2n-1)}{6} + \frac{e^2h(1-e^{2hn})}{1-e^{2n}}\right]$$

Put $nh = 2$

$$I = \lim_{h\to 0}\left[6 + 2(2-h) + \frac{(nh-h)nh(2nh-h)}{6} + \frac{e^2}{2}\times\frac{2h(1-e^4)}{(1-e^{2n})}\right]$$

$$\lim_{h\to 0}\left[6+2+(2-h)+\frac{(2-h)\,2\,(4-h)}{6} + \frac{e^2}{2}(-1)(1-e^4)\right]$$

Now,

$$I = 6 + 4 + \frac{2\times 2\times 4}{6} - \frac{e^2(1-e^4)}{2}$$

$$\Rightarrow \quad I = 10 + \frac{8}{3} - \frac{e^2}{2}(1-e^4)$$

$$\Rightarrow \quad I = \frac{38}{3} - \frac{e^2}{2}(1-e^4)$$ **Ans.**

OR

Using integration, find the area of the triangular region whose sides have the equation $y = 2x + 1$, $y = 3x + 1$ and $x = 4$.

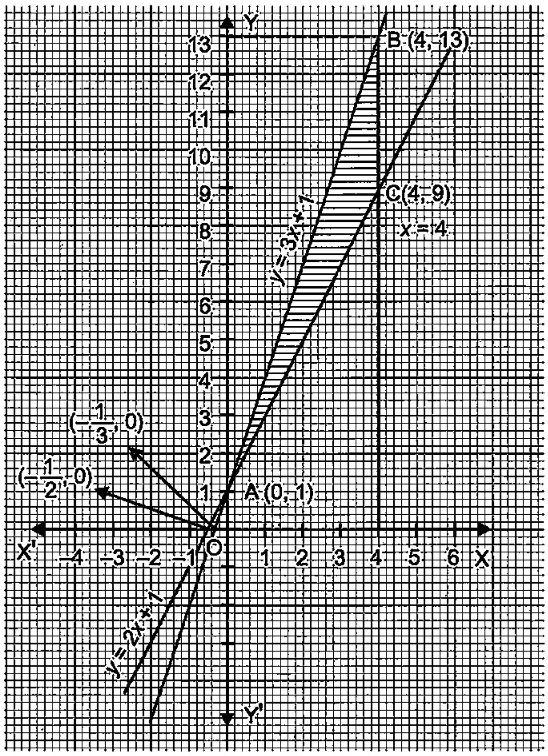
Solution :

The equations of sides of triangle are

$y = 2x + 1,$ ...(i)

$y = 3x + 1$ ...(ii)

and $x = 4$ ...(iii)



The equation $y = 2x + 1$ meets x and y axes at $\left(-\frac{1}{2}, 0\right)$ and (0,1). By joining these two points we obtain the graph of $x + 2y = 2$. Similarly, graphs of other equations are drawn.

Solving equation (i), (ii) and (iii) in pairs, we obtain the coordinates of verticies of ΔABC are A(0,1), B (4,13) and C (4,9).

Then, area of ΔABC = Area (OLBAO) – Area (OLCAO)

$$= \int_0^4 (3x+1)dx - \int_0^4 (2x+1)dx$$

$$= \int_0^4 (3x+1-2x-1)\,dx$$

$$= \int_0^4 x\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^4$$

$$= \frac{1}{2} \times 4 \times 4$$

= 8 square units. **Ans.**

Mathematics 2019 (Outside Delhi) SET III

Time allowed : 3 hours **Maximum marks : 100**

SECTION–A

1. Find the differenital equation representing the family of curves $y = ae^{2x} + 5$, where a is an arbitrary constant. [1]

Solution :

Given, $y = ae^{2x} + 5$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = a.e^{2x}.(2) + 0 \text{ [By Chain rule]}$$

$$\Rightarrow \quad \frac{dy}{dx} = 2a.e^{2x} \quad \text{...(i)}$$

Again, differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = 2\left(a.e^{2x}.2\right)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 2\frac{dy}{dx} \quad \text{[using (i)]}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 0$$

This is the required differential equation. **Ans.**

2. If $y = \cos\sqrt{3x}$, then find $\frac{dy}{dx}$. [1]

Solution :

Given, $y = \cos\left(\sqrt{3x}\right)$

On differentiating w.r.t.x, we get

$$\frac{dy}{dx} = -\sin\left(\sqrt{3x}\right) \times \sqrt{3} \times \frac{1}{2\sqrt{x}}$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{\sqrt{3}\sin\left(\sqrt{3x}\right)}{2\sqrt{x}} \quad \textbf{Ans.}$$

SECTION–B

5. Show that the points $A(-2\hat{i} + 3\hat{j} + 5\hat{k})$, $B(\hat{i} + 2\hat{j} + 3\hat{k})$ and $C(7\hat{i} - \hat{k})$ are collinear. [2]

Solution :

Given, $\vec{A} = -2\hat{i} + 3\hat{j} + 5\hat{k}$

$\vec{B} = \hat{i} + 2\hat{j} + 3\hat{k}$

$\vec{C} = 7\hat{i} + 0\hat{j} + \hat{k}$

Now, $\vec{AB}$ = position vector of B – position vector of A

$$= \hat{i} + 2\hat{j} + 3\hat{k} - \left(-2\hat{i} + 3\hat{j} + 5\hat{k}\right)$$

$$= \hat{i} + 2\hat{j} + 3\hat{k} + 2\hat{i} - 3\hat{j} - 5\hat{k}$$

$$= 3\hat{i} - \hat{j} - 2\hat{k}$$

$\vec{BC}$ = position vector of C – position vector B

$$= 7\hat{i} + 0\hat{j} - 0\hat{k} - (\hat{i} + 2\hat{j} + 3\hat{k})$$

$$= 7\hat{i} + 0\hat{j} - \hat{k} - \hat{i} - 2\hat{j} - 3\hat{k}$$

$$= 6\hat{i} - 2\hat{j} - 4\hat{k}$$

$$= 2(3\hat{i} - \hat{j} - 2\hat{k})$$

Clearly, $\vec{BC} = 2\vec{AB} \Rightarrow AB \,||\, BC$

But B is a common point to $\vec{AB}$ and $\vec{BC}$.

∴ $\vec{AB}$ and $\vec{BC}$ are collinear vectors.

Hence, points A, B and C are collinear.

Hence Proved.

OR

Find, $|\vec{a} \times \vec{b}|$ if $\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} + 5\hat{j} - 2\hat{k}$.

Solution :

Given, $\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}$

and $b = 3\hat{i} + 5\hat{j} - 2\hat{k}$

Then, $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 3 \\ 3 & 5 & -2 \end{vmatrix}$

$$= \hat{i}(-2-15) - \hat{j}(-4-9) + \hat{k}(10-3)$$

$$= -17\hat{i} + 13\hat{j} + 7\hat{k}$$

$$|\vec{a} \times \vec{b}| = \sqrt{(-17)^2 + (13)^2 + (7)^2}$$

$$= \sqrt{289 + 169 + 49} = \sqrt{507}$$ **Ans.**

6. Find : $\int \frac{x-5}{(x-3)^3} e^x dx$. **[2]**

Solution :

Let $I = \int \frac{x-5}{(x-3)^3} e^x dx$

$$I = \int \frac{(x-3)-2}{(x-3)^3} . e^x dx$$

$$= \int \frac{(x-3)-2}{(x-3)^3} . e^x dx$$

$$= \int \left[\frac{(x-3)}{(x-3)^3} - \frac{2}{(x-3)^3} \right] e^x dx$$

$$= \int \left[\frac{1}{(x-3)^2} - \frac{2}{(x-3)^3} \right] e^x dx \quad ...(i)$$

Now,

$$\frac{d}{dx}(x-3)^{-2} = -2(x-3)^{-3} = \frac{-2}{(x-3)^3}$$

and we know that

$$\int [f(x) + f'(x)] e^x dx = e^x f(x) + c$$

Here, $f(x) = \frac{1}{(x-3)^2}$ and $f'(x) = \frac{-2}{(x-3)^3}$

Then,

$$\int \frac{x-5}{(x-3)^3} e^x dx = e^x \times \frac{1}{(x-3)^2} + C = \frac{e^x}{(x-3)^2} + C$$ **Ans.**

SECTION–C

13. Solve for x :

$\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}\left(\frac{8}{31}\right)$. **[4]**

Solution :

Given,

$$\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}\frac{8}{31}$$

We know that

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\frac{x+y}{1-xy}$$

$\therefore \tan^{-1}(x+1) + \tan^{-1}(x-1)$

$$= \tan^{-1}\left[\frac{x+1+x-1}{1-[(x+1)(x-1)]}\right]$$

$$= \tan^{-1}\left[\frac{2x}{1-(x^2-1)}\right]$$

$$= \tan^{-1}\left[\frac{2x}{2-x^2}\right]$$

Now, $\tan^{-1}\frac{2x}{2-x^2} = \tan^{-1}\frac{8}{31}$

$\Rightarrow \frac{2x}{2-x^2} = \frac{8}{31}$

$\Rightarrow 62x = 16 - 8x^2$

$\Rightarrow 31x = 8 - 4x^2$

$\Rightarrow 4x^2 + 31x - 8 = 0$

$\Rightarrow 4x^2 + 32x - x - 8 = 0$

$\Rightarrow 4x(x+8) - 1(x+8) = 0$

$\Rightarrow (x+8)(4x-1) = 0$

$\Rightarrow x = -8$ and $x = \frac{1}{4}$ **Ans.**

14. If $x = ae^t(\sin t + \cos t)$ and $y = ae^t(\sin t - \cos t)$, then prove that $\frac{dy}{dx} = \frac{x+y}{x-y}$. **[4]**

Solution :

Given, $x = ae^t(\sin t + \cos t)$ ...(i)

and $y = ae^t(\sin t - \cos t)$ ...(ii)

On differentiating equation (i) with w.r.t. t, we get

$$\frac{dx}{dt} = a\frac{d}{dt}\left[e^t(\sin t + \cos t)\right] = a\frac{d}{dt}\left[e^t \sin t + e^t \cos t\right]$$

$$= a\left[e^t \cos t + \sin t\, e^t + e^t(-\sin t) + e^t \cos t)\right]$$

$$= a\left[e^t \cos t + e^t \sin t - \sin t + e^t + e^t \cos t\right]$$

$$= 2ae^t \cos t \quad ...(iii)$$

On differentiating equation (ii) w.r.t. t, we get

$$\frac{dy}{dt} = a\frac{d}{dt}\left[e^t(\sin t - \cos t)\right]$$

$$= a\frac{d}{dt}\left[e^t \sin t - e^t \cos t\right]$$

$$= a\left[e^t \cos t + \sin t\, e^t - (-e^t \sin t + \cos e^t)\right]$$

$$= a\left[e^t \cos t + e^t \sin t + \sin t\, e^t - e^t \cos t\right]$$

$$= 2ae^t \sin t \quad ...(iv)$$

On dividing equation (iv) by (iii), we get

$$\frac{dy/dt}{dx/dt} = \frac{2ae^t \sin t}{2ae^t \cos t}$$

$$\frac{dy}{dx} = \frac{\sin t}{\cos t}$$

$$\text{L.H.S.} = \frac{dy}{dx} = \tan t \quad ...(v)$$

Now, $\text{R.H.S.} = \frac{x+y}{x-y}$

$$= \frac{ae^t(\sin t + \cos t) + ae^t(\sin t - \cos t)}{ae^t(\sin t + \cos t) - ae^t(\sin t - \cos t)}$$

$$= \frac{2ae^t \sin t}{2ae^t \cos t} = \tan t \quad ...(vi)$$

From equations (v) and (vi), we have

$$\frac{dy}{dx} = \frac{x+y}{x-y}$$ **Hence Proved**

OR

Differentiate $x^{\sin x} + (\sin x)^{\cos x}$ with respect to x.

Solution :

Let $y = x^{\sin x} + (\sin x)^{\cos x}$

Let $x^{\sin x} = u \quad ...(i)$

and $(\sin x)^{\cos x} = v \quad ...(ii)$

Then,

$$y = u + v$$

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

Now, $u = x^{\sin x}$

$$\log u = \sin x \log x$$

On differentiating w.r.t. x, we get

$$\frac{1}{u}\frac{du}{dx} = \sin x \times \frac{1}{x} + \log x \cos x$$

$$\frac{du}{dx} = u\left(\frac{\sin x}{x} + \log x \cos x\right)$$

$$\frac{du}{dx} = x^{\sin x}\left(\frac{\sin x}{x} + \log x \cos x\right) \quad ...(iii)$$

Now, $v = (\sin x)^{\cos x}$

$$\log v = \cos x \log \sin x$$

On differentiating w.r.t. x, we get

$$\frac{1}{v}\frac{dv}{dx} = \cos x \frac{1}{\sin x} \times \cos x + \log \sin x(-\sin x)$$

$$\frac{1}{v}\frac{dv}{dx} = \cot x \times \cos x + \sin x \log(\sin x)$$

$$\frac{dv}{dx} = v(\cot x \cos x - \sin x \log \sin x)$$

$$\frac{dv}{dx} = (\sin x)^{\cos x}(\cot x - \cos x - \sin x \log \sin x) \quad ...(iv)$$

So, $$\frac{dy}{dx} = x^{\sin x}\left(\frac{\sin x}{x} + \log x \cos x\right) + (\sin x)^{\cos x}(\cot x \cos x - \sin x \log \sin x)$$

Ans.

15. Find :

$$\int \frac{2\cos x}{(1-\sin x)(2-\cos^2 x)}dx$$ **[4]**

Solution :

Let $I = \int \frac{2\cos x\, dx}{(1-\sin x)(1-\sin^2 x)}$

$$\Rightarrow \quad I = \int \frac{2\cos x}{(1-\sin x)(2-1+\sin^2 x)}dx$$

$$I = \int \frac{2\cos x\, dx}{(1-\sin x)(1-\sin^2 x)}$$

Now, let $\sin x = t$

$\Rightarrow \quad \cos x dx = dt$

Then, $I = \int \frac{2dt}{(1-t)(1+t^2)}$

Now, solving it by partial fraction,

$$\Rightarrow \quad \frac{2}{(1-t)(1+t^2)} = \frac{A}{(1-t)} + \frac{Bt+C}{(1+t^2)}.$$

$$\Rightarrow \quad 2 = A(1+t^2) + (Bt+C)(1-t)$$

$$\Rightarrow \quad 2 = A + At^2 + Bt - Bt^2 + C - Ct$$

$$\Rightarrow \quad 2 = t^2 + (A-B) + t(B-C) + A + C$$

Equating the coefficient of t^2, t and of constant terms of both sides, we get

$$A - B = 0$$

$$B - C = 0$$

and $A + C = 2$

On solving, we get

$$A = C = 1 \text{ and } B = 1$$

$$\therefore \int \frac{2dt}{(1-t)(1+t^2)} = \int \left[\frac{1}{(1-t)} + \left(\frac{t+1}{1+t^2} \right) \right]$$

$$I = \int \frac{1}{1-t}\,dt + \int \frac{t\,dt}{(1+t^2)} + \int \frac{1}{(1+t^2)}\,dt$$

$$= \log|1-t| + \frac{1}{2}\log|1+t^2| + \tan^{-1} t$$

$$= \log|1-\sin x| + \frac{1}{2}\log|1+\sin^2 x|$$

$$+ \tan^{-1}(\sin x) + c$$ **Ans.**

SECTION–D

24. Show that for the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$, [6]

$A^3 - 6A^2 + 5A + 11\,I = 0$. Hence, find A^{-1}.

Solution :

Given, $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1+2 & 1+2-1 & 1-3+3 \\ 1+2-6 & 1+4+3 & 1-6-9 \\ 2-1+6 & 2-2-3 & 2+3+9 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}$$

Now, $A^3 = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 4+2+2 & 4+4-1 & 4-6+3 \\ -3+8+28 & -3+16+14 & -3-24-42 \\ 7-3+28 & 7-6-14 & 7-9+42 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix}$$

Now, putting A^3, A^2 in $A^3 - 6A^2 + 5A + 11\,I$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - 6\begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}$$

$$+ 5\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} + 11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - \begin{bmatrix} 24 & 12 & 6 \\ -18 & 48 & -84 \\ 42 & -18 & 84 \end{bmatrix}$$

$$+ \begin{bmatrix} 5 & 5 & 5 \\ 5 & 10 & -15 \\ 10 & -5 & 15 \end{bmatrix} + \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 8-24 & 7-12 & 1-6 \\ -23+18 & 27-48 & -69+84 \\ 32-42 & -13+18 & 58-84 \end{bmatrix}$$

$$+ \begin{bmatrix} 5+11 & 5+0 & 5+0 \\ 5+0 & 10-11 & -15+0 \\ 10-0 & -5+0 & 15-11 \end{bmatrix}$$

$$= \begin{bmatrix} -16 & -5 & -5 \\ -5 & -21 & 15 \\ -10 & 5 & -26 \end{bmatrix} + \begin{bmatrix} 16 & 5 & 5 \\ 5 & 21 & -15 \\ 10 & -5 & 26 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$ **Hence Proved.**

We have, $A^3 - 6A^2 + 5A + 11I = 0$

Multiplying by A^{-1} on both sides, we get

$$A^{-1}(A^3 - 6A^2 + 5A + 11I) = 0.A^{-1}$$

$$\Rightarrow A^3A^{-1} - 6A^2.A^{-1} + 5A.A^{-1} + 11I.A^{-1} = 0$$

$$\Rightarrow A^2(AA^{-1}) - 6A.(AA^{-1}) + 5(AA^{-1}) + 11IA^{-1} = 0$$

$$\Rightarrow A^2I - 6AI + 5I + 11A^{-1} = 0$$

$$\left[\begin{array}{l} \because AA^{-1} = 1 \\ \text{and } 11IA^{-1} = 11A^{-1} \end{array}\right]$$

$$\Rightarrow A^2 - 6A + 5I + 11A^{-1} = 0$$

$$\Rightarrow 11A^{-1} = -A^2 + 6A - 5I$$

$$\Rightarrow A^{-1} = \frac{1}{11}(-A^2 + 6A - 5I)$$

$$A^{-1} = \frac{-(A^2 - 6A + 5I)}{11}$$

$$A^{-1} = \frac{1}{11}\left(\begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} - 6\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$A^{-1} = \frac{-1}{11}\left(\begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} - \begin{bmatrix} 6 & 6 & 6 \\ 6 & 12 & -18 \\ 12 & -6 & 18 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}\right)$$

$$A^{-1} = \frac{-1}{11}\left(\begin{bmatrix} 4-6 & 2-6 & 1-6 \\ -3-6 & 8-12 & -14+18 \\ 7+12 & -3+6 & 14+18 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}\right)$$

$$A^{-1} = \frac{-1}{11}\begin{bmatrix} -2 & -4 & -5 \\ -9 & -4 & 4 \\ -5 & 3 & -4 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$A^{-1} = \frac{-1}{11}\begin{bmatrix} -2+5 & -4+0 & -5+0 \\ -9+0 & -4+5 & 4+0 \\ -5+0 & 3+0 & -4+5 \end{bmatrix}$$

$$A^{-1} = \frac{-1}{11}\begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{-1}{11}\begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}$$ **Ans.**

OR

Using matrix method, solve the following system of equations :

$$3x - 2y + 3z = 8$$
$$2x + y - z = 1$$
$$4x - 3y + 2z = 4$$

Solution :

The given equations are

$$3x - 2y + 3z = 8$$
$$2x + y - z = 1$$
$$4x - 3y + 2z = 4$$

These equations can be written in the from $AX = B$, where

$$A = \begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$$

Now, $|A| = 3(2-3) + 2(4+4) + 3(-6-4)$

$= -17 \neq 0$

Hence, A is non-singular and so its inverse exists.

Confactors of A are

$A_{11} = -1,\ A_{12} = -8,\ A_{13} = -10$

$A_{21} = -5,\ A_{22} = -6,\ A_{23} = 1$

$A_{31} = -1,\ A_{32} = 9,\ A_{33} = 7$

$$\therefore \quad \text{adj } A = \begin{bmatrix} -1 & -8 & -10 \\ -5 & -6 & 1 \\ -1 & 9 & 7 \end{bmatrix}^T = \begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ 10 & 1 & 7 \end{bmatrix}$$

Now, $$A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{17}\begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}$$

$$\Rightarrow \quad X = A^{-1}B = \frac{-1}{17}\begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}\begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{-1}{17}\begin{bmatrix} -17 \\ -34 \\ -51 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Hence, $x = 1, y = 2$ and $z = 3$ **Ans.**

26. A bag contains 5 red and 4 black balls, a second bag contains 3 red and 6 black balls. One of the two bags is selected at random and two balls are drawn at random (without replacement) both of which are found to be red. Find the probability that the balls are drawn from the second bag. [6]

Solution :

Let E_1 be the event of choosing the bag I, E_2 be the event of choosing the bag II and A be the event of drawing a red ball.

Then,

$$P(E_1) = P(E_2) = \frac{1}{2}$$

Now, $$P\left(\frac{A}{E_1}\right) = \frac{5}{9} \times \frac{4}{8} = \frac{20}{72}$$

and $$P\left(\frac{A}{E_2}\right) = \frac{3}{9} \times \frac{2}{8} = \frac{6}{72}$$

Now, the probability of drawing a ball from Bag II, if it is given that it is red is $P\left(\frac{E_2}{A}\right)$.

Now, by Bayes' theorem, we have

$$P\left(\frac{E_2}{A}\right) = \frac{P(E_2).P(A/E_2)}{P(E_1).P\left(\frac{A}{E_1}\right) + P(E_2)P\left(\frac{A}{E_1}\right)}$$

$$= \frac{\frac{1}{2} \times \frac{6}{72}}{\frac{1}{2} \times \frac{20}{72} + \frac{1}{2} \times \frac{6}{72}}$$

$$= \frac{\frac{6}{2 \times 72}}{\frac{20}{2 \times 72} + \frac{6}{2 \times 72}}$$

$$= \frac{6}{20+6}$$

$$= \frac{6}{26} = \frac{3}{13}$$ **Ans.**

●●

Math 2019 (Delhi)

SET I

Time allowed : 3 hours **Maximum marks : 100**

SECTION–A

1. If A and B are square matrices of the same order 3, such that |A| = 2 and AB = 2I, write the value of |B|. **[1]**

Solution :

Given, $|A| = 2$ and $AB = 2I$

$\therefore$ $|A| = 2I$

$\Rightarrow$ $|A||B| = 2I$

$\Rightarrow$ $2|B| = 2I$

$\Rightarrow$ $|B| = I$ **Ans.**

2. If $f(x) = x + 1$, find $\frac{d}{dx}$ $(fof)(x)$. **[1]**

Solution :

Given, $f(x) = x + 1$

Now, $f^0 f(x) = f(f(x)$

$= f(x+1)$

$= x+1+1$

$= x+2$

$\therefore$ $\frac{d}{dx}(fof)(x) = 1$ **Ans.**

3. Find the order and the degree of the differential equation $x^2\frac{d^2y}{dx^2} = \left\{1+\left(\frac{dy}{dx}\right)^2\right\}^4$. **[1]**

Solution :

We have,

$$x^2\frac{d^2y}{dx^2} = \left\{1+\left(\frac{dy}{dx}\right)^2\right\}^4$$

$\therefore$ Order = 2 and degree = 1 **Ans.**

4. If a line makes angles 90°, 135°, 45° with x, y and z axes respectively, find its direction cosines. **[1]**

Solution :

Given, $\alpha = 90°$, $\beta = 135°$, $\gamma = 45°$

So, $l = \cos 90° = 0$

$m = \cos 135° = \cos(180° - 45°)$

$= -\cos 45° = -\frac{1}{\sqrt{2}}$

and $n = \cos 45° = \frac{1}{\sqrt{2}}$

$\therefore$ The required direction cosines are $0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$.

Ans.

OR

Find the vector equation of the line which passes through the point (3, 4, 5) and is parallel to the vector $2\hat{i} + 2\hat{j} - 3\hat{k}$. **[1]**

Solution :

Given, the line passes through the point (3, 4, 5) and parallel to the vector $2\hat{i}+2\hat{j}-3\hat{k}$.

D. R. s of the given vector are < 2, 2, – 3 >.

∴ Vector equation of line,

$$\vec{r}=(3\hat{i}+4\hat{j}+5\hat{k})+\lambda(2\hat{i}+2\hat{j}-3\hat{k})$$ **Ans.**

SECTION–B

5. Examine whether the operation* defined on R by $a*b = ab + 1$ is (i) a binary or not. (ii) if a binary operation, is it associative or not? [2]**

6. Find a matrix A such that 2A – 3B + 5C = 0, where $B=\begin{bmatrix}-2 & 2 & 0\\ 3 & 1 & 4\end{bmatrix}$ and $C=\begin{bmatrix}2 & 0 & -2\\ 7 & 1 & 6\end{bmatrix}$. [2]

Solution :

Given, $B=\begin{bmatrix}-2 & 2 & 0\\ 3 & 1 & 4\end{bmatrix}, C=\begin{bmatrix}2 & 0 & -2\\ 7 & 1 & 6\end{bmatrix}$

and $2A-3B+5C = 0$

$\Rightarrow\quad 2A = 3B-5C$

$\Rightarrow\quad 2A = 3B-5C$

$\Rightarrow\quad A = \frac{1}{2}(3B-5C)$

$\Rightarrow\quad A = \frac{1}{2}\left(3\begin{bmatrix}-2 & 2 & 0\\ 3 & 1 & 4\end{bmatrix}\right)-5\begin{bmatrix}2 & 0 & -2\\ 7 & 1 & 6\end{bmatrix}$

$\Rightarrow\quad A = \frac{1}{2}\left(\begin{bmatrix}-6 & 6 & 0\\ 9 & 3 & 12\end{bmatrix}-\begin{bmatrix}10 & 0 & -10\\ 35 & 5 & 30\end{bmatrix}\right)$

$\Rightarrow\quad A = \frac{1}{2}\begin{bmatrix}-16 & 6 & 10\\ -26 & -2 & -18\end{bmatrix}$

$\Rightarrow\quad A = \begin{bmatrix}-8 & 3 & 5\\ -13 & -1 & -9\end{bmatrix}$ **Ans.**

7. Find : $\int \frac{\sec^2 x}{\sqrt{\tan^2 x+4}}dx.$ [2]

Solution :

Let $\quad I = \int \frac{\sec^2 x\, dx}{\sqrt{\tan^2 x+4}}$

Let $\quad \tan x = t$

$\sec^2 x\, dx = dt$

$\therefore\quad I = \int \frac{dt}{\sqrt{t^2+4}}$

$\Rightarrow\quad I = \int \frac{dt}{\sqrt{t^2+2^2}}$

$\Rightarrow\quad I = \log|(t-2)+\sqrt{t^2+4}|+C$

$\Rightarrow\quad I = \log|(\tan x+2)+\sqrt{\tan^2 x+4}|+C$ **Ans.**

8. Find : $\int\sqrt{1-\sin 2x}\,dx,\ \frac{\pi}{4}<x<\frac{\pi}{2}.$ [2]

Solution :

Let $\quad I = \int\sqrt{1-\sin 2x}\,dx$

$I = \int\sqrt{(\cos x-\sin x)^2}\,dx$

$[\because 1-\sin 2A = (\cos A-\sin A)^2]$

$\Rightarrow\quad I = \int(\cos x-\sin x)dx$

$\Rightarrow\quad I = \sin x+\cos x+C$ **Ans.**

OR

Find : $\int \sin^{-1}(2x)\,dx$. [2]

Solution :

Let $\quad I = \int\sin^{-1}(2x)\,dx$

Let $\quad \sin^{-1}2x = t$

$2x = \sin t$

$2dx = \cos t\,dt$

$\therefore\quad I = 2\int t\cos t\,dt$

$\Rightarrow\quad I = 2\left[t\sin t-\int 1\cdot\sin t\,dt\right]$

[Using by parts]

$\Rightarrow\quad I = 2[t\sin t+\cos t]+C$

$\Rightarrow\quad I = 2[2x\sin^{-1}2x+\cos(\sin^{-1}2x)]+C$

$\Rightarrow\quad I = 2[2x\sin^{-1}2x+\sqrt{1-4x^2}]+C$ **Ans.**

9. Form the differential equation representing the family of cures $y = e^{2x}(a+bx)$, where 'a' and 'b' are arbitrary constants. [2]

Solution :

Given, $\quad y = e^{2x}(a+bx)$ ...(i)

Differentiating w.r.t. x, we get

$\frac{dy}{dx} = e^{2x}(b)+2(a+bx)e^{2x}$

$\frac{dy}{dx} = be^{2x}+2y$ [From (i)]

$\frac{dy}{dx}-2y = be^{2x}$ ...(ii)

Again, differentiating both sides w.r.t. x, we get

$\frac{d^2y}{dx^2}-2\frac{dy}{dx} = 2be^{2x}$

** Answer is not given due to the change in present syllabus.

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 2\left(\frac{dy}{dx} - 2y\right) \quad \text{[From (ii)]}$$

$$\frac{d^2y}{dx^2} - \frac{2dy}{dx} = \frac{2dy}{dx} - 4y$$

$$\frac{d^2y}{dx^2} - \frac{4dy}{dx} + 4y = 0$$

This is the required differential equation. **Ans.**

10. If the sum of two unit vectors is a unit vector, prove that the magnitude of their difference is $\sqrt{3}$. [2]

Solution :

Let $\vec{a}$ and $\vec{b}$ are two unit vectors.

Given, $|\vec{a}| = 1, |\vec{b}| = 1, |\vec{a} + \vec{b}| = 1$

We know that,

$$|\vec{a} + \vec{b}|^2 + |\vec{a} - \vec{b}|^2 = 2(|\vec{a}|^2 + |\vec{b}|^2)$$

$$(1)^2 + |\vec{a} - \vec{b}|^2 = 2(1^2 + 1^2)$$

$$\Rightarrow \quad (1)^2 + |\vec{a} - \vec{b}|^2 = 4$$

$$\Rightarrow \quad |\vec{a} - \vec{b}|^2 = 3$$

$$\Rightarrow \quad |\vec{a} - \vec{b}|^2 = \sqrt{3}$$ **Ans.**

OR

If $\vec{a} = 2\hat{i} + 3\hat{j} + \hat{k}$ + $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$ and $\vec{c} = -3\hat{i} + \hat{j} + 2\hat{k}$, find $[\vec{a}\ \vec{b}\ \vec{c}]$. [2]

Solution :

Given, $\vec{a} = 2\hat{i} + 3\hat{j} + \hat{k}$

$\vec{b} = \hat{i} + 2\hat{j} + 2\hat{k}$

$\vec{a} = -3\hat{i} + \hat{j} + 2\hat{k}$

Now, $[\vec{a}, \vec{b}, \vec{c}] = [\vec{a} . (\vec{b} \times \vec{c})]$

$$= \begin{vmatrix} 2 & 3 & 1 \\ 1 & -2 & 1 \\ -3 & 1 & 2 \end{vmatrix}$$

$$= 2(-4-1) - 3(2+3) + 1(1-6)$$

$$= -10 - 15 - 5 = -30$$ **Ans.**

11. A die marked 1, 2, 3 in red and 4, 5, 6 in green is tossed. Let A be the event "number is even" and B be the event "number is marked red". Find whether the events A and B are independent or not. [2]

Solution :

Given, S = { 1, 2, 3, 4, 5, 6 }

Let the two events be

A : The number is even

B : The number is red

$$P(A) = \frac{3}{6} = \frac{1}{2}, P(B) = \frac{3}{6} = \frac{1}{2}$$

and $P(A \cap B) = \frac{1}{6}$

$\therefore \quad P(A \cap B) \neq P(A) . P(B)$

Hence, A and B are not independent **Ans.**

12. A die is thrown 6 times. If "getting an odd number" is a "success", what is the probability of (i) 5 successes? (ii) atmost 5 success? [2]**

OR

The random variable X has a probability distribution P(X) of the following form. where 'k' is some number

$$P(X = x) = \begin{cases} k, & \text{if } x = 0 \\ 2k, & \text{if } x = 1 \\ 3k, & \text{if } x = 2 \\ 0, & \text{otherwise} \end{cases}$$

Determine the value of 'k'. [2]

Solution :

We know that,

$$\Sigma P(X) = 1$$

$$P_{(x=0)} + P_{(x=1)} + P_{(x=2)} + P_{(x=\text{other})} = 1$$

$$\Rightarrow \quad K + 2k + 3k + 0 = 1$$

$$\Rightarrow \quad 6K = 1$$

$$\Rightarrow \quad K = \frac{1}{6}$$ **Ans.**

SECTION–C

13. Show that the relation R on R defined as R = [(a, b) : a ≤ b], is reflexive, and transitive but not symmetric. [2]

Solution :

Reflexive :

Let $a \in R$

$\therefore \quad a \leq a$

So, $(a, a) \in R$

Hence, R is reflexive.

** Answer is not given due to the change in present syllabus.

Symmetric :

Let $(a, b) \in R$

Then $(b, a) \in R$

Then, $a \leq b$

$\Rightarrow b \leq a$ (which is not true)

$\therefore (b, a) \notin R$

Hence, R is not symmetric.

Transitive :

Let, $a, b, c \in R$, such that $(a, b) \in R$ and $(b, c) \in R$

Then, $a \leq b$

and $b \leq c$

$\Rightarrow a \leq c$

$\Rightarrow (a, c) \in R$

Hence, R is transitive.

Hence, R is reflexive and transitive but not Symmetric. **Hence Proved.**

OR

Prove that the function $f : N \to N$, defined by $f(x) = x^2 + x + 1$ is one-one but not onto.

Find inverse of $f : N \to S$, where S is range of f.

Solution :

Given, $f : N \to N$, $f(x) = x^2 + x + 1$

Let A be the set of natural number (domain).

and B be the set of natural number (co-domain).

For One-One :

Let $x_1, x_2 \in A$ such that $f(x_1) = f(x_2)$

$\Rightarrow x_1^2 + x_1 + 1 = x_2^2 + x_2 + 1$

$\Rightarrow x_1^2 - x_2^2 + x_1 - x_2 = 0$

$\Rightarrow (x_1 - x_2)[x_1 + x_2 + 1] = 0$

$\Rightarrow x_1 - x_2 = 0$ $[\because x_1 - x_2 + 1 \neq 0]$

$\Rightarrow x_1 = x_2$

$\therefore$ f is one-one.

For Onto :

Let $y = 1 \in N$ (Co domain)

$\therefore x^2 + x 1 = 1$

$\Rightarrow x^2 + x = 0$

$\Rightarrow x(x + 1) = 0$

$\therefore x = 0, -1 \in N$

Not possible since domain = N

So, f is not onto.

Hence, f is one-one but not onto. **Hence Proved**

Now, $f : N \to S : f(x) = x^2 + x + 1$

where S = range (Given)

$f : N \to S$ is onto as co-domain = range.

Hence, f is invertible.

Let $y = f(x) \Rightarrow y = x^2 + x + 1$

$$\Rightarrow y = x^2 + 2x\frac{1}{2} + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1$$

$$\Rightarrow y = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$$

$$\Rightarrow y - \frac{3}{4} = \left(x + \frac{1}{2}\right)^2$$

$$\Rightarrow \frac{4y - 3}{4} = \left(x + \frac{1}{2}\right)^2$$

$$\Rightarrow \pm\frac{\sqrt{4y - 3}}{2} = x + \frac{1}{2} = \frac{2x + 1}{2}$$

$$\therefore x = \frac{-1 \pm \sqrt{4y - 3}}{2}$$

$$\Rightarrow x = \frac{-1 + \sqrt{4y - 3}}{2} \quad (\because X \in N)$$

Ans.

14. Solve : $\tan^{-1} 4x + \tan^{-1} 6x = \frac{\pi}{4}$. **[2]**

Solution : We have

$$\tan^{-1} 4x + \tan^{-1} 6x = \pi/4$$

$$\Rightarrow \tan^{-1}\left(\frac{4x + 6x}{1 - 4x \,.\, 6x}\right) = \frac{\pi}{4}$$

$$\Rightarrow \frac{10x}{1 - 24x^2} = \tan\frac{\pi}{4}$$

$$\Rightarrow \frac{10x}{1 - 24x^2} = 1$$

$$\Rightarrow 10x = 1 - 24x^2$$

$$\Rightarrow 24x^2 + 10x - 1 = 0$$

$$\Rightarrow x = \frac{-10 \pm \sqrt{100 + 96}}{48}$$

$$\Rightarrow x = \frac{-10 \pm 14}{48}$$

$$\Rightarrow x = \frac{-1}{2}, \frac{1}{12}$$

$$\therefore x = \frac{1}{12}$$

Ans.

15. Using properties of determinants, prove that

$$\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a-1)^3.$$ **[2]**

Solution :

$$\text{L.H.S} = \begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and we get $R_3 \to R_3 - R_1$, we get

$$= \begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 1-a^2 & -a+1 & 0 \\ 3-a^2-2a & -2a+2 & 0 \end{vmatrix}$$

Taking $(1-a)$ common from R_2 & R_3, we get

$$= (1-a)^2 \begin{vmatrix} a^2+2a & 2a+1 & 1 \\ (1-a) & 1 & 0 \\ a-3 & 2 & 0 \end{vmatrix}$$

Expanding along C_3, we get

$$= (1-a)^2 [1(2+2a)-(a+3)]$$
$$= (1-a)^2 [2+2a+a-3]$$
$$= (1-a)^2 (a-1)$$
$$= (a-1)^2 (a-1)$$
$$= (a-1)^3 = \text{R.H.S.}$$

Hence Proved.

16. If $\log(x^2+y^2) = 2\tan^{-1}\left(\frac{y}{x}\right)$, show that

$$\frac{dy}{dx} = \frac{x+y}{x-y}.$$ **[2]**

Solution :

Given, $\log\left(x^2+y^2\right) = 2\tan^{1}\left(\frac{y}{x}\right)$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{x^2+y^2}\left(2x+2y\frac{dy}{dx}\right) = \frac{2}{1+\frac{y^2}{x^2}}\left(\frac{x\frac{dy}{dx}-y}{x^2}\right)$$

$$\Rightarrow \frac{x}{x^2+y^2} + \frac{y}{x^2+y^2}\cdot\frac{dy}{dx} = \frac{1}{x^2+y^2}\left(x\frac{dy}{dx}-y\right)$$

$$\Rightarrow \frac{x}{x^2+y^2} + \frac{y}{x^2+y^2}\cdot\frac{dy}{dx} = \frac{x}{x^2+y^2}\cdot\frac{dy}{dx} - \frac{y}{x^2+y^2}$$

$$\Rightarrow \frac{x}{x^2+y^2} + \frac{y}{x^2+y^2} = \frac{x}{x^2+y^2}\cdot\frac{dy}{dx} - \frac{y}{x^2+y^2}\cdot\frac{dy}{dx}$$

$$\Rightarrow \frac{x+y}{x^2+y^2} = \left(\frac{x-y}{x^2+y^2}\right)\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x+y}{x-y}.$$ **Ans.**

OR

If $x^y - y^x = a^b$, find $\frac{dy}{dx}$

Solution :

$$x^y + y^x = a^b$$

Let $u = x^y$ and $v = y^x$

Then, $u - v = a^b$

$$\frac{du}{dx} - \frac{dv}{dx} = 0 \quad ...(i)$$

Now, $u = x^y$

Taking log on both sides, we get

$$\log u = x \log x$$

Differentiating both sides w.r.t. x, we get.

$$\frac{1}{u}\cdot\frac{du}{dx} = \frac{y}{x} + \log x \cdot \frac{dy}{dx}$$

$$\Rightarrow \frac{du}{dx} = x^y\left[\frac{y}{x} + \log x.\frac{dy}{dx}\right]$$

and $v = y^x$

Taking log on both sides, we get

$$\log v = x \log y$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{v}\cdot\frac{dv}{dx} = \frac{x}{y}\cdot\frac{dy}{dx} + \log y$$

$$\Rightarrow \frac{dv}{dx} = y^x\left[\frac{x}{y}\frac{dy}{dx} + \log y\right]$$

From equation (i),

$$x^y\left[\frac{y}{x} + \log x.\frac{dy}{dx}\right] - y^x\left[\frac{x}{y}\frac{dy}{dx} + \log y\right] = 0$$

$$\Rightarrow y.x^{y-1} + x^y \log x.\frac{dy}{dx} - xy^{x-1}.\frac{dy}{dx} - y^x \log y = 0$$

$$\Rightarrow \frac{dy}{dx}\left[x^y \log x - xy^{x-1}\right] = y^x \log y - yx^{y-1}$$

$$\Rightarrow \quad \frac{dy}{dx} = \left[\frac{y^x \log y - y.xy^{-1}}{x^y \log x - xy^{n-1}}\right]$$

Ans.

17. If $y = (\sin^{-1} x)^2$, prove that

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 2 = 0 \quad [2]$$

Solution :

Given, $\quad y = (\sin^{-1}x)^2 \quad$ ...(i)

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{2\sin^1 x}{\sqrt{1-x^2}}$$

$$\Rightarrow \sqrt{1-x^2}\frac{dy}{dx} = 2\sin^{-1} x \quad \text{...(ii)}$$

Again differentiating both sides w.r.t. x, we get

$$\sqrt{1-x^2}\frac{d^2y}{dx^2} = \frac{2x}{2\sqrt{1-x^2}}\frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 2$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2}\frac{dy}{dx} - x\frac{dy}{dx} - 2 = 0$$

Hence Proved.

18. Find the equation of tangent to the curve $y = \sqrt{3x-2}$ which is parallel to the line $4x - 2y + 5 = 0$. Also, write the equation of normal to the curve at the point of contact. [2]

Solution :

Given, $\quad y = \sqrt{3x-2} \quad$ ...(i)

Differentiating both sides w.r.t.x, we get

$$\frac{dy}{dx} = \frac{2}{2\sqrt{3x-2}} = m_1$$

and equation of line, $4x - 2y + 5 = 0$

$$\text{Slope} = \frac{-\text{Coefficient of } x}{\text{Coefficient of } y} = \frac{-4}{-2} = 2 = m_2$$

Apply condition of parallel line,

$$\frac{3}{2\sqrt{3x-2}} = 2$$

$$\Rightarrow \quad 3 = 4\sqrt{3x-2}$$

$$\Rightarrow \quad 9 = 16(3x-2)$$

$$\Rightarrow \quad 9 = 48x - 32$$

$$41 = 48x$$

$$\Rightarrow \quad x = \frac{41}{48}$$

Putting the value of x in (i), we get

$$y = \sqrt{3\left(\frac{41}{48}\right) - 2} = \frac{3}{4}$$

$\therefore$ Point of contact $\left(\frac{41}{48}, \frac{3}{4}\right)$

Slope of curve = 2

So, equation of Tangent is

$$\left(y - \frac{3}{4}\right) = 2\left(x - \frac{41}{48}\right)$$

$$\Rightarrow \quad \left(\frac{4y-3}{4}\right) = 2\left(\frac{48x-41}{48}\right)$$

$$\Rightarrow \quad \frac{4y-3}{4} = \frac{48x-41}{28}$$

$$\Rightarrow \quad \frac{4y-3}{4} = \frac{48x-41}{24}$$

$$\Rightarrow \quad 4y - 3 = \frac{48x-41}{6}$$

$$\Rightarrow \quad 24y - 18 = 48x - 41$$

$$\Rightarrow \quad 48x - 24y - 23 = 0$$

Ans.

And equation of normal is

$$\left(y - \frac{3}{4}\right) = \frac{-1}{2}\left(x - \frac{41}{48}\right)$$

$$\Rightarrow \quad 4y - 3 = \frac{-1}{2}\left(\frac{48x-41}{48}\right)$$

$$\Rightarrow \quad 96(4y-3) = -48x + 41$$

$$\Rightarrow \quad 384y - 288 = -48x + 41$$

$$\Rightarrow \quad 48x + 384y - 329 = 0$$

Ans.

19. Find : $\int \frac{3x+5}{x^2+3x-18}dx.$ [2]

Solution :

Let $\quad I = \int \frac{3x+5}{x^2+3x-18}dx$

$$(3x+5) = A(2x+3) + B$$

Comparing coefficients of x,

$$3 = 2A$$

$$\therefore \quad \frac{3}{2} = A$$

Comparing contant terms,

$$5 = 3A + B$$

$$5 = \frac{9}{2} + B$$

$$B = \frac{1}{2}$$

$$\therefore \quad I = \frac{3}{2}\int \frac{2x+3}{x^2+3x-18}dx + \frac{1}{2}\int \frac{dx}{x^2+3x-18}$$

$$\text{Let } I_1 = \frac{3}{2}\int \frac{2x+3}{x^2+3x-18}dx$$

Put $x^2 + 3x - 18 = t$

$$(2x+3)\,dx = dt$$

$$I_1 = \frac{3}{2}\int \frac{dt}{t}$$

$$\Rightarrow \quad I_1 = \frac{3}{2}\log|t| + C_1$$

$$\Rightarrow \quad I_1 = \frac{3}{2}\log|x^2+3x-18| + C_1$$

$$\text{Now, } I_2 = \frac{1}{2}\int \frac{dx}{x^2+3x-18}$$

$$\Rightarrow \quad I_2 = \frac{1}{2}\int \frac{dx}{x^2 + 2x.\frac{3}{2} + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 - 18}$$

$$\Rightarrow \quad I_2 = \frac{1}{2}\left(\frac{dx}{\left(x+\frac{3}{2}\right)^2 - \left(\frac{9}{2}\right)^2}\right)$$

$$\Rightarrow \quad I_2 = \frac{1}{2} \times \frac{2}{2\times 9}\log\left|\frac{x+\frac{3}{2}-\frac{9}{2}}{x+\frac{3}{2}+\frac{9}{2}}\right| + C_2$$

$$\Rightarrow \quad I_2 = \frac{1}{18}\log\left|\frac{2x-6}{2x+12}\right| + C_2$$

$$\Rightarrow \quad I_2 = \frac{1}{18}\log\left|\frac{x-3}{x+6}\right| + C_2$$

$$\therefore \quad I = I_1 + I_2$$

$$I = \frac{3}{2}\log\left|x^2+3x-18\right| + \frac{1}{18}\log\left|\frac{x-3}{x+6}\right| + C$$

Ans.

20. Prove that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$, hence evaluate $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$. **[2]**

Solution :

To prove : $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

Proof : Let $x = a - t$

$$dx = -dt$$

Also, $x = 0 \Rightarrow t = a$

and $x = a \Rightarrow t = 0$

$$\therefore \quad \int_0^a f(x)dx = -\int_a^0 f(a-t)\,dt$$

$$\Rightarrow \quad \int_0^a f(x)dx = \int_0^a f(a-t)\,dt$$

$$\left[\int_b^a f(x)dx = -\int_b^a f(x)dx\right]$$

$$\Rightarrow \quad \int_0^a f(x)dx = \int_0^a f(a-x)\,dx$$

$$\left[\because \int_a^b f(x)dx = \int_a^b f(t)dt\right]$$

Hence Proved.

$$\text{Let} \quad I = \int_0^{\pi} \frac{x\sin x\,dx}{1+\cos^2 x} \quad \text{...(i)}$$

$$I = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)}dx$$

[Applying above property]

$$I = \int_0^{\pi} \frac{(\pi - x)\sin dx}{1+\cos^2 x} \quad \text{...(ii)}$$

$$\left[\because \begin{array}{l}\sin(\pi-\theta) = \sin\theta \\ \cos(\pi-\theta) = \cos\theta\end{array}\right]$$

Adding equation (i) and (ii), we get

$$2I = \pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}dx$$

Put $\cos x = t$

$$\sin x\,dx = dt$$

when $x = 0, t = 1$

and $x = \pi, t = -1$

$$\Rightarrow \quad 2I = -\pi\int_1^{-1} \frac{dt}{1+t^2}$$

$$\Rightarrow \quad 2I = -\pi\left[\tan^{-1}(t)\right]_1^{-1}$$

$$\Rightarrow \quad 2I = -\pi\left[\tan^{-1}(-1) - \tan^{-1}(1)\right]$$

$$\Rightarrow \quad 2I = -\pi[-\tan^{-1}(1) - \tan^{-1}(1)]$$

$$\left[\because \tan^{-1}(-x) = \tan^{-1} x\right]$$

$$\Rightarrow \quad 2I = -\pi\left[-2\tan^{-1}(1)\right]$$

$$\Rightarrow \quad 2I = -\pi\left(-\frac{2\pi}{4}\right)$$

$$\Rightarrow \quad 2I = \frac{2\pi^2}{4}$$

$$\Rightarrow \quad I = \frac{\pi^2}{4}$$

Ans.

21. Solve the differential equation $x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx$, given that $y = 0$ when $x = 1$. [2]

Solution :

Given, $\quad x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx$

$$\Rightarrow \quad x\,dy = (\sqrt{x^2+y^2} + y)dx$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\sqrt{x^2+y^2}+y}{x}$$

Put $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

So, $$v + x.\frac{dv}{dx} = \frac{\sqrt{x^2+v^2x^2}+vx}{x}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \sqrt{1+v^2} + v$$

$$\Rightarrow \quad x.\frac{xv}{dx} = \sqrt{1+v^2}$$

Integrating both sides, we get

$$\Rightarrow \quad \int\frac{dv}{\sqrt{1+v^2}} = \int\frac{dx}{x}$$

$$\log\left|v + \sqrt{1+v^2}\right| = \log x + \log c$$

$$\log\left|v + \sqrt{1+v^2}\right| = \log(xc)$$

$$v + \sqrt{1+x^2} = xc$$

$$\frac{y}{x} + \sqrt{1+\frac{y^2}{x^2}} = x\int c$$

$$\frac{y+\sqrt{x^2+y^2}}{x} = xc$$

$$y + \sqrt{x^2+y^2} = x^2c$$

Now, Given $y = 0$ when $x = 1$

$\therefore$ From equation (i),

$$0 + 1 = c$$

$$c = 1$$

Substitute the value of c in (i), we get

$$y + \sqrt{x^2+y^2} = x^2$$

Ans.

OR

Solve the differential equation $(1 + x^2)\frac{dy}{dx} + 2xy - 4x^2 = 0$, subject to the initial condition $y(0) = 0$. [2]

Solution :

Given, $\quad (1+x^2)\frac{dy}{dx} + 2xy - 4x^2 = 0$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{2xy}{1+x^2} - \frac{4x^2}{1+x^2} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{2xy}{1+x^2} = \frac{4x^2}{1+x^2}$$

Here, $\quad P = \dfrac{2xy}{1+x^2}$ and $Q = \dfrac{4x^2}{1+x^2}$

$$\text{I.F} = e^{\int p dx} = e^{\int\frac{2x}{1+x^2}dx}$$

$$\text{I.F.} = e^{\log|1+x^2|}$$

$$\text{I.F.} = 1 + x^2 \qquad \left[\because e^{\log x} = x\right]$$

Solution of given differential equation is

$$y\,(\text{I.F.}) = \int(\text{I.F} \times Q)\,dx$$

$$\Rightarrow \quad y\,(1+x^2) = \int(1+x^2)\left(\frac{4x^2}{1+x^2}\right)dx$$

$$\Rightarrow \quad y\,(1+x^2) = \frac{4x^3}{3} + C \qquad \text{...(i)}$$

Now, Puting $x = 0, y = 0$ in (i), we get

$$0 = 0 + C$$

$$\therefore \quad C = 0$$

Substitute the value of C in (i), we get

$$y(1+x^2) = \frac{4x^3}{3}$$

$$y = \frac{4x^3}{3(1+x^2)}$$ **Ans.**

22. If $\hat{i}+\hat{j}+\hat{k}$, $2\hat{i}+5\hat{j}$, $3\hat{i}+2\hat{j}-3\hat{k}$ and $\hat{i}-6\hat{j}-\hat{k}$ respectively are the position vectors of points A, B, C and D, then find the angle between the straight lines AB and CD. Find whether $\vec{AB}$ and $\vec{CD}$ are collinear or not. [2]

Solution :

$A(\hat{i}+\hat{j}+\hat{k})$, $B(2\hat{i}+5\hat{j})$, $C(3\hat{i}+2\hat{j}-3\hat{k})$ and $D(\hat{i}-6\hat{j}-\hat{k})$.

Now, $\vec{AB} = \hat{i}+4\hat{j}-\hat{k}$

$\vec{CD} = -2\hat{i}-8\hat{j}+2\hat{k}$

Let θ be the angle between $\vec{AB}$ & $\vec{CD}$

So, $$\cos\theta = \frac{\vec{AB}.\vec{CD}}{|\vec{AB}||\vec{CD}|}$$

$$\cos\theta = \frac{(\hat{i}+4\hat{j}-\hat{k}).(-2\hat{i}-8\hat{j}+2\hat{k})}{\sqrt{1+16+1}\sqrt{4+64+4}}$$

$$\cos\theta = \frac{-2-32-2}{\sqrt{18}\sqrt{72}}$$

$$\cos\theta = \frac{-36}{\sqrt{2}.6\sqrt{2}}$$

$$\cos\theta = -1$$

$$\cos\theta = \cos\pi$$

$$\theta = \pi$$ **Ans.**

23. Find the value of λ so that the line $\frac{1-x}{3}=\frac{7y-14}{\lambda}=\frac{z-3}{2}$ and $\frac{7-7x}{3\lambda}=\frac{y-5}{1}=\frac{6-z}{5}$ are at right angles. Also, find whether the lines are intersecting or not. [2]

Solution :

Equation of 1st line,

$$\frac{1-x}{3}=\frac{7y-14}{\lambda}=\frac{z-3}{2}$$

$$\Rightarrow \frac{x-1}{-3}=\frac{y-2}{\lambda/7}=\frac{z-3}{2}$$

DR's of 1st line $<-3, \lambda/7, 2>$

and equation of 2nd line,

$$\frac{7-7x}{3\lambda}=\frac{y-5}{1}=\frac{6-z}{5}$$

$$\Rightarrow \frac{x-1}{-3\lambda/7}=\frac{y-5}{1}=\frac{6-z}{-5}$$

DR's of 2nd line $\left\langle\frac{-3\lambda}{7},1,-5\right\rangle$

Since, given lines are at right angles,

$$-3\left(\frac{-3\lambda}{7}\right)+\frac{\lambda}{7}(1)+2(-5)=0$$

$$\Rightarrow \frac{9\lambda}{7}+\frac{\lambda}{7}-10=0$$

$$\Rightarrow 10\lambda-70=0$$

$$\Rightarrow 10\lambda=70$$

$$\Rightarrow \lambda=7$$

Now, equation of 1st line,

$$\frac{x-1}{-3}=\frac{y-2}{1}=\frac{z-3}{2}$$

and equation of 2nd line,

$$\frac{x-1}{-3}=\frac{y-5}{1}=\frac{z-6}{-5}$$

Here, $\vec{a_1} = \hat{i}+2\hat{j}+3\hat{k}$, $\vec{b_1} = -3\hat{i}+\hat{j}+2\hat{k}$

$\vec{a_2} = \hat{i}+5\hat{j}+6\hat{k}$, $\vec{b_2} = -3\hat{i}+\hat{j}=5\hat{k}$

$$\therefore \quad S.D = \left|\frac{(\vec{a_2}-\vec{a_1}).(\vec{b_1}\times\vec{b2})}{|\vec{b_1}\times\vec{b2}|}\right|$$

$$(\vec{a_2}-\vec{a_i}) = 0\hat{i}+3\hat{j}+3\hat{k}$$

and $$(\vec{b_1}\times\vec{b_2}) = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -3 & 1 & 2\\ -3 & 1 & -5\end{vmatrix}$$

$$= \hat{i}(-5-2)-\hat{j}(15+6)+\hat{k}(-3+3)$$

$$= -7\hat{i}-11\hat{j}$$

$$\therefore \quad S.D. = \frac{\left|3\hat{i}+3\hat{k}-(-7\hat{k}-11\hat{j})\right|}{\sqrt{49+121}}$$

$$S.D. = \left|\frac{-21-33}{\sqrt{170}}\right|$$

$$\text{S.D.} = \left|\frac{-54}{\sqrt{170}}\right| = \frac{54}{\sqrt{170}} \neq 0$$

Hence, The given two lines do not intersect each other. **Ans.**

SECTION-D

24. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix}$, find A^{-1}. Hence, solve the system of equations $x + y + z = 6, x + 2z = 7, 3x + y + z = 12$ **[6]**

Solution :

Given,

$$A = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{vmatrix}$$

$$|A| = 1(0-2) - 1(1-6) + 1(1-0)$$

$$|A| = -2 + 5 + 1$$

$$|A| = 4 \neq 0$$

$\therefore A^{-1}$ exits.

Now, cofactors of A are,

$a_{11} = 2, \quad a_{21} = 0, \quad a_{31} = 2$

$a_{12} = 5, \quad a_{22} = -2, \quad a_{32} = -1$

$a_{13} = 1, \quad a_{23} = 2, \quad a_{33} = -1$

$$\text{adj } A = \begin{vmatrix} -2 & 5 & 1 \\ 0 & -2 & 2 \\ 2 & -1 & -1 \end{vmatrix}^T$$

$$\text{adj } A = \begin{vmatrix} -2 & 0 & 2 \\ 5 & -2 & -1 \\ 1 & 2 & -1 \end{vmatrix}$$

So, $A^{-1} = \frac{1}{|A|}(\text{adj } A)$

$$A^{-1} = \frac{1}{4}\begin{bmatrix} -2 & 0 & 2 \\ 5 & -2 & -1 \\ 1 & 2 & -1 \end{bmatrix} \quad \text{...(i)}$$ **Ans.**

The given equations are :

$$x + y + z = 6,$$
$$x + 2z = 7,$$
and $$3x + y + z = 12$$

Here, $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} B = \begin{bmatrix} 6 \\ 7 \\ 12 \end{bmatrix}$

$$X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -2 & 0 & 2 \\ 5 & -2 & -1 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 6 \\ 7 \\ 12 \end{bmatrix}$$ [Using (i)]

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -12+0+24 \\ 30-14+12 \\ 6+14-12 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 12 \\ 4 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$\therefore \quad x = 3, \quad y = 1 \quad$ and $\quad z = 2$ **Ans.**

OR

Find the inverse of the following matrix using elementary operations.

$$A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

Solution :

Given, $A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

$$A = IA$$

$$\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}A$$

Applying $R_2 \to R_2 + R_1$, we get

$$\begin{bmatrix} 1 & 2 & -2 \\ 0 & 5 & -2 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}A$$

Now, Applying $R_1 \to R_1 + R_3$, we get

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 5 & -2 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}A$$

Applying $R_2 \to \frac{1}{5}R_2$, we get

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -\frac{2}{5} \\ 0 & -2 & \frac{1}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ \frac{1}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}A$$

Applying $R_3 \to R_3 + 2R_2$, we get

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -\frac{2}{5} \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ \frac{1}{5} & \frac{1}{5} & 0 \\ \frac{2}{5} & \frac{2}{5} & 1 \end{bmatrix} A$$

Applying $R_2 \to R_2 + 2R_3$, we get

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ \frac{2}{5} & \frac{1}{5} & 1 \end{bmatrix} A$$

Now, Applying $R_3 \to 5R_3$, we get

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} A$$

Applying $R_1 \to R_1 + R_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} A$$

Hence, $A^{-1} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$ **Ans.**

25. A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2 m and volume is 8 m³. If building of tank costs ₹ 70 per square metre for the base and ₹ 45 per square metre for the sides, what is the cost of least expresive tank? [6]

Solution :

Let the length and breadth of the tank be x and y metres, respectively.

Given, Volume = 8 m^3

[∵ depth of tank = 2 m]

$$\Rightarrow \quad 2xy = 8$$

$$\Rightarrow \quad y = \frac{8}{2x} = \frac{4}{x}$$

Let C be the cost of tank. Then

$$C = 70\,xy + 45\,(2 \times 2x + 2 \times 2y)$$

$$\Rightarrow \quad C = 70\,xy + 180x + 180\,y$$

$$\Rightarrow \quad C = 70\,x \times \frac{4}{x} + 180\,x + 180 \times \frac{4}{x}$$

$$\Rightarrow \quad C = 280 + 180x + \frac{720}{x} \quad \text{(i)}$$

Now, differentiating both sides w.r.t. x, we get

$$\frac{dC}{dx} = 180 - \frac{720}{x^2}$$

For maxima and minima, $\frac{dC}{dx} = 0$

$$180 - \frac{720}{x^2} = 0$$

$$\Rightarrow \quad 180 = \frac{720}{x^2}$$

$$\Rightarrow \quad x^2 = 4$$

$$\Rightarrow \quad x = \pm 2$$

$$\Rightarrow \quad x = 2$$

Again, Differentiating both sides w.r.t. x, we get

$$\frac{d^2C}{dx^2} = \frac{1440}{x^3}$$

$$\left.\frac{d^2C}{dx^2}\right|_{x=2} = \frac{1440}{8} = 180 > 0$$

∴ when $x = 2$, cost of tank is minimum

Substituting the value of x in equation (i), we get

$$C = 280 + 180 \times 2 + \frac{720}{2}$$

$$\Rightarrow \quad C = 280 + 360 + 360$$

$$\Rightarrow \quad C = 1000$$

Hence, the cost of least expensive tank is ₹ 1000.

Ans.

26. Using integration, find the area of a triangle ABC, whose vertices are A(2, 5), B(4, 7) and C(6, 2). [6]

Solution :

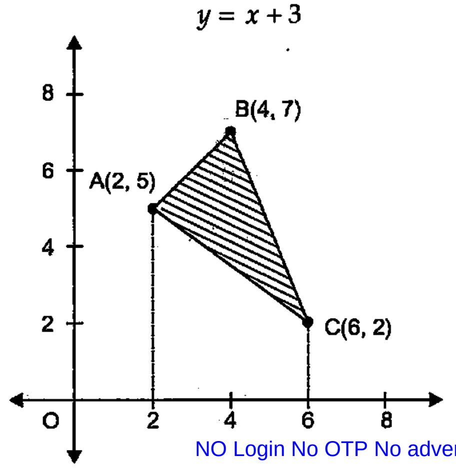
Given, A (2,5), B (4,7) and C (6,2) be the vertices of a triangle.

The equation of side AB.

$$(y - 5) = \frac{7-5}{4-2}(x - 2)$$

$$\Rightarrow \quad y - 5 = x - 2$$

$$\Rightarrow \quad y = x + 3$$



The equation of side BC,

$$(y-7)=\frac{2-7}{6-4}(x-4)$$

$$\Rightarrow \quad y-7=\frac{-5}{2}(x-4)$$

$$\Rightarrow \quad 2y-14=-5x+20$$

$$\Rightarrow \quad 2y=-5x+34$$

$$\Rightarrow \quad y=\frac{-1}{2}(5x-34)$$

The equation of sides AC,

$$(y-5)=\frac{2-5}{6-2}(x-2)$$

$$\Rightarrow \quad y-5=\frac{-3}{4}(x-2)$$

$$\Rightarrow \quad 4y-20=-3x+6$$

$$\Rightarrow \quad 4y=-3x+26$$

$$y=-\frac{1}{4}(3x-26)$$

$\therefore$ Area of ΔABC

$$=\int_2^4 y_{AB}dx+\int_4^6 y_{BC}dx-\int_2^6 y_{AC}dx$$

$$=\int_2^4 (x+3)\,dx+\int_4^6 -\frac{1}{2}(5x-34)\,dx-\int_2^6 -\frac{1}{4}(3x-26)\,dx$$

$$=\left[\frac{x^2}{2}+3x\right]_2^4-\frac{1}{2}\left[\frac{5x^2}{2}-34x\right]_4^6+\frac{1}{4}\left[\frac{3x^2}{2}-26x\right]_2^6$$

$$=\left[\left(\frac{16}{2}+12\right)-\left(\frac{4}{2}+6\right)\right]-\frac{1}{2}\left[\left(\frac{180}{2}-204\right)-\left(\frac{80}{2}-136\right)\right]+\frac{1}{4}\left[\left(\frac{108}{2}-156\right)-\left(\frac{12}{2}-52\right)\right]$$

$$=[(8+12)-(2+6)]-\frac{1}{2}[(90-204)-(40-136)]+\frac{1}{4}[(54-156)-(6-52)]$$

$$=12+\frac{1}{2}(18)-\frac{1}{4}(56)=12+9-14=7 \text{ sq. units.}$$ **Ans.**

OR

Find the area of a region lying above x-axis and included between the circle $x^2+y^2=8x$ and inside of the parabola $y^2=4x$.

Solution :

Given, equation of circle is $x^2+y^2=8x$ can be expressed as

$$(x-4)^2+y^2=16 \qquad \text{...(i)}$$

Centre is (4,0) and radius is 4

and equation of parabola is $y^2=4x$ ...(ii)

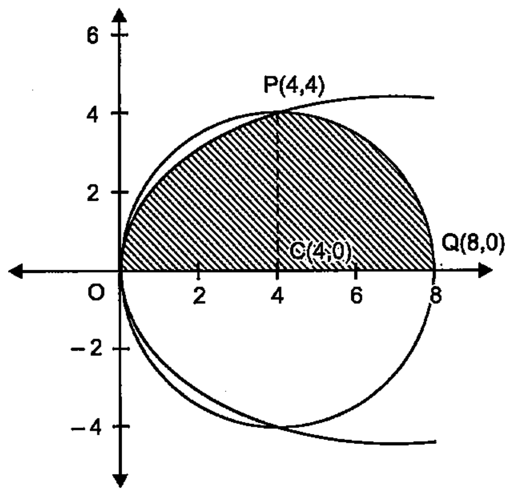
Let Required Area = I

$$\therefore I=\int_0^4 y \text{ of the parabola } dx=\int_4^8 y \text{ of the circle } dx$$

$$=\underbrace{\int_0^4 2\sqrt{x}\,dx}_{I_1}+\underbrace{\int_4^8\sqrt{4^2-(x-4)^2}\,dx}_{I_2}$$

$$I_1=2\int_0^4\sqrt{x}\,dx$$

$$I_1=2\left[\frac{x^{3/2}}{3/2}\right]_0^4$$



$$I_1=\frac{4}{3}[4^{\frac{3}{2}}-0]$$

$$I_1=\frac{4}{3}(8)=\frac{32}{3}$$

and $$I_2=\int_4^8\sqrt{4^2-(x-4)^2}\,dx$$

$$I_2=\left[\frac{x-4}{2}\sqrt{4^2-(x-4)^2}+\frac{16}{2}\sin^{-1}\left(\frac{x-4}{4}\right)\right]_4^8$$

$$I_2=\frac{1}{2}\left[\left(\frac{4}{2}(0)+16\sin^{-1}(1)\right)-(0)\right]$$

$$I_2=\frac{1}{2}\times 16\frac{\pi}{2}=4\pi$$

$$\therefore \quad I=I_1+I_2$$

$$=\left(\frac{32}{3}+4\pi\right)$$

$$=\frac{4}{3}(8+3\pi) \text{ sq. units.}$$ **Ans.**

27. Find the vector and Cartesian equation of the plane passing through the points (2, 2, –1), (3, 4, 2) and (7, 0, 6). Also find the vector equation of a plane passing through (4, 3, 1) and parallel to the plane obtained above. **[6]**

Solution :

Let A (2, 2, – 1), B (3, 4, 2,) and C (7, 0, 6)

The equation of plane passing through A(2, 2, – 1),

$$a(x-2)+b(y-2)+c(z+1) = 0 \quad ...(i)$$

Since (3, 4, 2) and (7,0,6) lies on plane

$$\therefore \quad a+2b+3c = 0 \quad ...(ii)$$

$$\text{and} \quad 5a-2b+7c = 0 \quad ...(iii)$$

Solving equation (ii) and (iii), we get

$$\frac{a}{(14+6)}=\frac{-b}{(7-15)}=\frac{c}{(-2-10)}=\text{k (say)}$$

$$\frac{a}{20}=\frac{-b}{-8}=\frac{c}{-12}=\text{k}$$

$$a=20k, \quad b=8k \quad \text{and} \quad c=-12k$$

Putting the values of a, b and c in (i), we get

$$20k(x-2)+8k(y-2)\,a-12k\,(z+1) = 0$$

$$\Rightarrow \quad 4k\,[5x-10+2y-4-3z-3] = 0$$

$$\Rightarrow \quad 5x+2y-3z-17 = 0$$

This is the required equation of the plane. **Ans.**

Now, the second plane passes through the points (4, 3, 1).

Since, this plane is parallel to the above plane,

$\therefore$ D. R.'s of the second plane be < 5, 2, – 3 >

So, equation of second plane,

$$5(x-4)+2\,(y-3)-3\,(z-1) = 0$$

$$\Rightarrow \quad 5x-20+2y-6-3z+3 = 0$$

$$\Rightarrow \quad 5x+2y-3z-23 = 0 \text{ Ans.}$$

OR

Find the vector equation of the plane that contains the lines $\vec{r} = (\hat{i}+\hat{j})+\lambda(\hat{i}+2\hat{j}+\hat{k})$ and the point (– 1, 3, – 4). Also, find the length of the perpendicular drawn from the point (2, 1, 4) to the plane, thus obtained.

Solution :

Given, equation of the given line

$$\vec{r} = (\hat{i}+\hat{j})+\lambda(\hat{i}+2\hat{j}+\hat{k})$$

The plane passes through the point (– 1, 3, – 4).

Then the equation of the plane,

$$a(x+1)+b\,(y-3)+c\,(z+4) = 0 \quad ...(i)$$

Since (1, 1) lies on the plane,

$$\therefore \quad 2a-2b+4c = 0 \quad ...(ii)$$

Also, (1, 2, –1) lies on the plane

$$\therefore \quad 2a-b+3c=0 \quad ...(iii)$$

Solving equations (ii) and (iii), we get

$$\frac{a}{(-6+4)}=\frac{-b}{(6-8)}-\frac{c}{(-2+4)}=k \text{ (say)}$$

$$\frac{a}{-2}=\frac{-b}{-2}=\frac{c}{2}=k$$

$$a=-2k, \; b=2k \text{ and } c=2k$$

Putting the values of a, b, and c in (i), we get

$$-2k(x+1)+2k(y-3)+2k\,(z+4) = 0$$

$$\Rightarrow \quad -(x+1)+(y-3)+(z+4) = 0$$

$$\Rightarrow \quad -x-1+y-3+z+4 = 0$$

$$\Rightarrow \quad -x+y+z = 0$$

$\therefore$ Vector equation of plane is,

$$\vec{r}\cdot(-\vec{i}+\vec{j}+\vec{k}) = 0 \text{ Ans.}$$

$$\text{Perpendicular distance} = \left|\frac{\vec{a}\cdot\vec{n}-d}{|\vec{n}|}\right|$$

Perpendicular distance

$$= \left|\frac{\left(2\vec{i}+\vec{j}+4\vec{k}\right)\cdot\left(-\vec{i}+\vec{j}+\vec{k}\right)-0}{\sqrt{1+1+1}}\right|$$

$$= \left|\frac{-2+1+4}{\sqrt{3}}\right|$$

$$= \frac{3}{\sqrt{3}}=\sqrt{3} \text{ units.} \quad \textbf{Ans.}$$

28. A manufacturer has three machine operators A, B and C. The first operator A produces 1% of defective items, whereas the other two operators B and C produces 5% and 7% defective items respectively. A is on the job for 50% of the time, B on the job 30% of the time and C on the job for 20% of the time. All the items are put into one stockpile and then one item is chosen at random from his and is found to be defective. What is the probability that it was produces by A? [6]

Solution :

Let H_1 be the event items produced by A

H_2 be the event items produced by B

H_3 be the event items produced by C

$$P(H_1) = \frac{50}{100}, P(H_2) = \frac{30}{100} \text{ and } P(H_3) = \frac{20}{100}$$

Let E be the event items found to be defective.

$$P\left(\frac{E}{H_1}\right) = \frac{1}{100}, P\left(\frac{E}{H_2}\right) = \frac{5}{100} \text{ and } P\left(\frac{E}{H_3}\right) = \frac{7}{100}$$

Using Bayes' theorem,

$$P\left(\frac{H_1}{E}\right) = \frac{P(H_1)P(E/H_1)}{P(H_1)P(E/H_1) + P(H_2)P(E/H_2) + P(H_3)P(E/H_3)}$$

$$= \frac{\frac{50}{100} \times \frac{1}{100}}{\frac{50}{100} \times \frac{1}{100} + \frac{30}{100} \times \frac{5}{100} + \frac{20}{100} \times \frac{7}{100}}$$

$$= \frac{50}{50 + 150 + 140} = \frac{50}{340} = \frac{5}{34}$$ **Ans.**

29. A manufacturer has employed 5 skilled men and 10 semi-skilled men and makes two models A and B of an article. The making two one item of model A requires 2 hours work by a skilled man and 2 hours work by a semi-skilled man. One item of model B requires 1 hour by a skilled man and 3 hours by a semi-skilled man. No man is expected to work more than 8 hours per day. The manufacturer's profit on an item of model A is ₹15 and on an item of model B is ₹ 10. How many of items of each model should be made per day in order to maximize daily profit? Formulate the above LPP and solve it graphically and find the maximum profit. [6]

Solution :

Let, x be the number of items of model A and y be the number of items of model B

Let Z be the required profit.

Subject to constraints :

$$2x + y \le 8 \times 5$$

$$\Rightarrow \quad 2x + y \le 40$$

$$2x + 3y \le 8 \times 10$$

$$\Rightarrow \quad 2x + 3y \le 80$$

$$x \ge 0, y \ge 0$$

Maximise $Z = 15x + 10y$

Changing the above inequalities into equations,

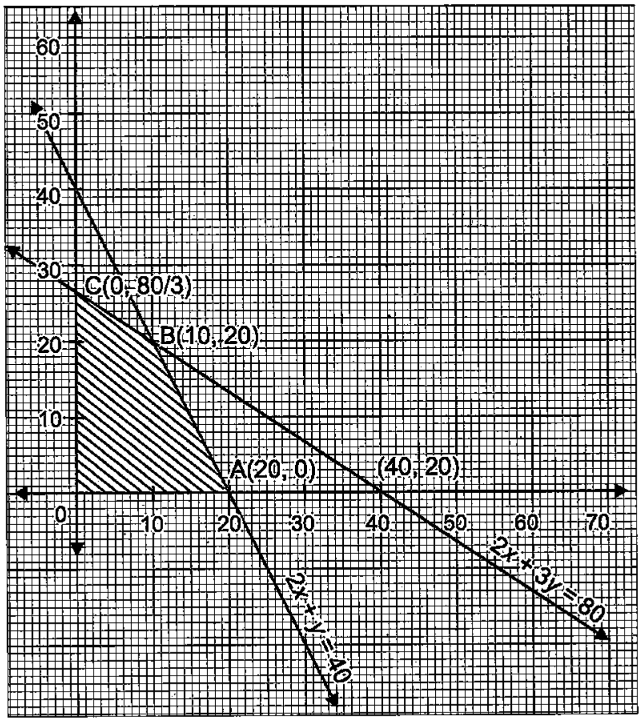
$2x + y = 40$

x	0	20
y	40	0

$2x + 3y = 80$

x	0	40
y	80/3	0

Now, ploting these points on the graph.



The shaded region is the required feasible region.

Vertices	Maximum $Z = 15x + 10y$
O (0,0)	$15 \times 0 + 10 \times 0 = 0$
A (20, 0)	$15 \times 20 + 10 \times 0 = 300$
B (10, 20)	$15 \times 10 + 10 \times 20 = 350$
C (0, 80/3)	$15 \times 0 + 10 \times 80/3 = 266.6$

Thus, the maximum profit is obtained when the manufacture produces 10 items of model A and 20 items of model B and the maximum profit ₹ 250. **Ans.**

Math 2019 (Delhi) SET II

Time allowed : 3 hours **Maximum marks : 100**

Note : Except for the following questions, all the remaining questions have been asked in previous sets.

SECTION–A

2. If $f(x) = x + 7$ and $g(x) = x - 7$, $x \in R$, then find $\frac{d}{dx}(fog)(x)$. [1]

Solution :

Given, $f(x) = x+7$

and $y(x) = x-7$

Given, $(fog)(x) = f(g(x))$

$= f(x-7)$

$= (x-7)+7$

$= x$

Now, Differentiating w.r.t, x, we get

$$\frac{d(fog)}{dx}(x) = 1$$ **Ans.**

3. Find the value of $x - y$, if

$$2\begin{bmatrix}1 & 3\\0 & x\end{bmatrix}+\begin{bmatrix}y & 0\\1 & 2\end{bmatrix}=\begin{bmatrix}5 & 6\\1 & 8\end{bmatrix}.$$ [1]

Solution :

Given, $2\begin{bmatrix}1 & 3\\0 & x\end{bmatrix}+\begin{bmatrix}y & 0\\1 & 2\end{bmatrix}=\begin{bmatrix}5 & 6\\1 & 8\end{bmatrix}$

$$\begin{bmatrix}2 & 6\\0 & 2x\end{bmatrix}+\begin{bmatrix}y & 0\\1 & 2\end{bmatrix}=\begin{bmatrix}5 & 6\\1 & 8\end{bmatrix}$$

$$\begin{bmatrix}2+y & 6\\1 & 2x+2\end{bmatrix}=\begin{bmatrix}5 & 6\\1 & 8\end{bmatrix}$$

By Definition of equality, we have

$2+y = 5$ $2x+2 = 8$

$y = 3$ $2x = 6$

$x = 3$

$\therefore$ $x-y = 3-3$

$\Rightarrow$ $x-y = 0$ **Ans.**

SECTION–B

6. If $A = \begin{bmatrix}2 & 0 & 1\\2 & 1 & 3\\1 & -1 & 0\end{bmatrix}$, then find $(A^2 - 5A)$. [1]

Solution :

Given, $A = \begin{bmatrix}2 & 0 & 1\\2 & 1 & 3\\1 & -1 & 0\end{bmatrix}$

Now, $A^2 = \begin{bmatrix}2 & 0 & 1\\2 & 1 & 3\\1 & -1 & 0\end{bmatrix}\begin{bmatrix}2 & 0 & 1\\2 & 1 & 3\\1 & -1 & 0\end{bmatrix}$

$$A^2 = \begin{bmatrix}4+0+1 & 0+0-1 & 2+0+0\\4+2+3 & 0+1-3 & 2+3+0\\2-2+0 & 0-1-0 & 1-3+0\end{bmatrix}$$

$$A^2 = \begin{bmatrix}5 & -1 & 2\\9 & -2 & 5\\0 & -1 & -2\end{bmatrix}$$

and $5A = 5\begin{bmatrix}2 & 0 & 1\\2 & -1 & 3\\1 & -1 & 0\end{bmatrix} = \begin{bmatrix}10 & 0 & 5\\10 & 5 & 15\\5 & -5 & 0\end{bmatrix}$

$$\therefore \quad A^2 - 5A = \begin{bmatrix}2 & -1 & 2\\9 & -2 & 5\\0 & -1 & -2\end{bmatrix} - \begin{bmatrix}10 & 0 & 5\\10 & 5 & 15\\5 & -5 & 0\end{bmatrix}$$

$$A^2 - 5A = \begin{bmatrix}-5 & -1 & -3\\-1 & -7 & -10\\-5 & 4 & -2\end{bmatrix}$$ **Ans.**

12. Find : $\int \frac{\tan^2 x \sec^2 x}{1-\tan^6 x}dx$. [1]

Solution :

Let $I = \int \frac{\tan^2 x \sec^2 x}{1-\tan^6 x}dx$

$\Rightarrow$ $I = \int \frac{\tan^2 x \sec^2 x}{1-(\tan^3 x)^2}dx$

Put $\tan^3 x = t$

$3\tan^2 x \sec^2 x\, dx = dt$

$\tan^2 x . \sec^2 dx = \frac{dt}{3}$

$\Rightarrow$ $I = \frac{1}{3}\int \frac{dt}{1-t^2}$

$\Rightarrow$ $I = \frac{1}{3}\times\frac{1}{2\times 1}\log\left|\frac{1+t}{1-t}\right| + C$

$\Rightarrow$ $I = \frac{1}{6}\log\left|\frac{1+\tan^3 x}{1-\tan^3 x}\right| + C$ **Ans.**

SECTION–C

13. Solve for x : $\tan^{-1}(2x) + \tan^{-1}(3x) = \frac{\pi}{4}$. **[2]**

Solution :

Given, $\tan^{-1} 2x + \tan^{-1} 3x = \frac{\pi}{4}$

$$\tan^{-1}\left[\frac{2x+3x}{1-(2x)(3x)}\right] = \frac{-\pi}{4}$$

$$\left[\because \tan^{-1}x + \tan^{-1}y = \tan^{1}\left(\frac{x+y}{1-xy}\right)\right]$$

$$\Rightarrow \tan^{-1}\left(\frac{5x}{1-6x^2}\right) = \tan\frac{\pi}{4}$$

$$\Rightarrow \left(\frac{5x}{1-6x^2}\right) = \tan\frac{\pi}{4}$$

$$\Rightarrow \frac{5x}{1-6x^2} = 1$$

$$\Rightarrow 5x = 1-6x^2$$

$$\Rightarrow 6x^2+5x-1=0$$

$$\Rightarrow 6x^2+6x-x-1=0$$

$$\Rightarrow (6x-1)(x+1)=0$$

$$\Rightarrow x = \frac{1}{6} \text{ or } x = -1$$

$$\therefore x = \frac{1}{6}$$ **Ans.**

18. Using properties of determinants, prove the following :

$$\begin{vmatrix} a+b+c & -c & -b \\ -c & a+b+c & -a \\ -b & -a & a+b+c \end{vmatrix} = 2(a+b)(b+c)(c+a).$$ **[4]**

Solution :

Taking L. H. S.

$$= \begin{vmatrix} a+b+c & -c & -b \\ -c & a+b+c & -a \\ -b & -a & a+b+c \end{vmatrix}$$

Applying $R_2 \to R_2 + R_1$ and $R_3 \to R_3 + R_1$, we get

$$= \begin{vmatrix} a+b+c & -c & -b \\ a+b & a+b & -(a+b) \\ a+c & -(a+c) & a+c \end{vmatrix}$$

Taking $(a+b)$ common from R_2 and $(a+c)$ common from R_3, we get

$$= (a+b)(a+c)\begin{vmatrix} a+b+c & -c & -b \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix}$$

Now, Applying $C_1 \to C_2 + C_2$, we get

$$= (a+b)(a+c)\begin{vmatrix} (a+b) & -c & -b \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{vmatrix}$$

Applying $C_1 \to C_2 + C_3$, we get

$$= (a+b)(a+c)\begin{vmatrix} (a+b) & -(b+c) & -b \\ 2 & 0 & -1 \\ 0 & 0 & 1 \end{vmatrix}$$

Now, Expanding along R_3, we get

$$= (a+b)(a+c)[1.\{0+2(b+c)\}]$$

$$= 2(a+b)(a+c)[2.(b+c)]$$

$$= 2(a+b)(b+c)(c+a) = \text{R.H.S.}$$

Hence Proved.

19. If $x = \cos t + \log\tan\left(\frac{t}{2}\right)$, $y = \sin t$, then find the values of $\frac{d^2y}{dt^2}$ and $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{4}$. **[4]**

Solution :

Given,

$$x = \cos t + \log\tan\left(\frac{t}{2}\right)$$

Differentiating w.r.t. t, we get

$$\frac{dx}{dt} = -\sin t + \frac{1}{\tan\left(\frac{t}{2}\right)}\cdot\frac{1}{2}\sec^2\frac{t}{2}$$

$$\Rightarrow \frac{dx}{dt} = -\sin t + \frac{\cos\left(\frac{t}{2}\right)}{2.\sin\left(\frac{t}{2}\right)}\cdot\frac{1}{\cos^2\left(\frac{t}{2}\right)}$$

$$\Rightarrow \frac{dx}{dt} = -\sin t + \frac{1}{2\sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)}$$

$$\frac{dx}{dt} = -\sin t + \frac{1}{\sin t}$$

$$\left[\because A = 2\sin\frac{A}{2}\cos A\frac{A}{2}\right]$$

$$\Rightarrow \frac{dx}{dt} = \frac{-\sin^2 t + 1}{\sin t}$$

$$\Rightarrow \frac{dx}{dt} = -\frac{\cos^2 t}{\sin t} \quad \text{...(i)}$$

and $y = \sin t$ (Given)

Differentiating w.r.t., we get

$$\frac{dy}{dt} = \cos t$$

Now, $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$\Rightarrow \quad \frac{dy}{dx} = \cos t \times \frac{\sin t}{\cos^2 t}$

$\Rightarrow \quad \frac{dy}{dx} = \tan t$

Again, Differentiating both sides w.r.t., x, we get

$$\frac{d^2y}{dx^2} = \sec^2 t \frac{dt}{dx}$$

$\Rightarrow \quad \frac{d^2y}{dx^2} = \sec^2 t \times \frac{\sin t}{\cos^2 t}$ [using (i)]

$$\Rightarrow \quad \left.\frac{d^2y}{dx^2}\right|_{t=\frac{\pi}{4}} = \sec^2 \pi/4 . \frac{\sin \pi/4}{\cos^2 \pi/4}$$

$$= \left(\sqrt{2}\right)^2 . \frac{\frac{1}{\sqrt{2}}}{\left(\frac{1}{\sqrt{2}}\right)^2}$$

$$= \frac{2 \times \frac{1}{\sqrt{2}}}{\frac{1}{2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$ **Ans.**

SECTION–D

24. Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of a radius r is $\frac{4r}{3}$. Also find the maximum volume of cone. **[6]**

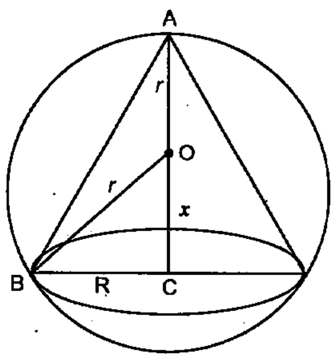
Solution :

Let R be the radius of cone. Let OA = OB = r (radius of sphere)

$$AC = r + x \quad ...(i)$$

(height of cone)

Let V be the volume of cone.



To prove : $AC = \frac{4r}{3}$

Now, $V = \frac{\pi}{3}(BC)^2 (AC)$ ...(ii)

(from the figure)

$$V = \frac{\pi}{3}(BC)^2 (r + x)$$

Now in ΔOBC,

$$(OB)^2 = (OC)^2 + (BC)^2$$

$$r^2 = x^2 + R^2$$

$$r^2 - x^2 = (BC)^2 \quad ...(iii)$$

$\therefore \quad V = \frac{\pi}{3} (r^2 - x^2)(r + x)$

$$V = \frac{\pi}{3} \underset{\text{I}}{(r + x)^2} \underset{\text{II}}{(r - x)}$$

Differentiating both sides w. r. t. x, we get

$$\frac{dV}{dx} = \frac{\pi}{3} [(r + x)^2 (-1) + (r - x). 2(r + x)]$$

$\Rightarrow \quad \frac{dV}{dx} = \frac{\pi}{3} (r + x) [-r - x + 2r - 2x]$

$\Rightarrow \quad \frac{dV}{dx} = \frac{\pi}{3} (r + x)(r - 3x)$

For maximum or minimum volume,

$$\frac{dV}{dx} = 0$$

$\Rightarrow \quad \frac{\pi}{3}(r + x)(r - 3x) = 0$

$\Rightarrow \quad x = r$ or $x = \frac{r}{3}$

$x = r$ is not possible.

Now Again differentiating w. r. t. x_1, we get

$$\frac{d^2V}{dx^2} = \frac{\pi}{3}[(r+x)(-3) + (r-3x)(1)]$$

$\Rightarrow \quad \frac{d^2V}{dx^2} = \frac{\pi}{3} [-3r - 3x + r - 3x]$

$\Rightarrow \quad \frac{d^2V}{dx^2} = \frac{\pi}{3} (-2r - 6x)$

$\Rightarrow \quad \left.\frac{d^2V}{dx^2}\right|_{x=\frac{r}{3}} = \frac{\pi}{3}\left(-2r - \frac{6r}{3}\right)$

$$= \frac{-4\pi r}{3} < 0$$

So, Volume of cone is maximum when $x = \frac{r}{3}$.

Now, Put $x = \frac{r}{3}$ in (i)

$$AC = r + x$$
$$= r + \frac{r}{3}$$
$$AC = \frac{4r}{3}$$

The altitude of cone is $\frac{4r}{3}$ when its volume is maximum. **Hence Proved.**

From equation (iii),

$$(BC)^2 = r^2 - \left(\frac{r}{3}\right)^2 = r^2 - \frac{r^2}{9}$$
$$(BC)^2 = \frac{8r^2}{9}$$

Maximum volume of cone $= \frac{\pi}{3}\left(\frac{8r^2}{9}\right)\left(\frac{4r}{3}\right)$

[Using (ii)]

$$= \frac{32\pi r^3}{81}$$ **Ans.**

25. If $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, then find A^{-1}. Hence solve the following system of equations: $2x - 3y + 5z = 11$, $3x + 2y - 4z = -5$, $x + y - 2z = -3$. [6]

Solution :

Given, $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$

$$|A| = 2(-4+4) + 3(-6+4) + 5(3-2)$$
$$|A| = 3(-2) + 5(1)$$
$$|A| = -1 \neq 0$$

$\therefore$ A^{-1} exists.

Cofactors of A,

$a_{11} = 0$, $a_{21} = -1$, $a_{31} = 2$,
$a_{12} = 2$, $a_{22} = -9$, $a_{32} = 23$,
$a_{13} = 1$, $a_{23} = -5$, $a_{33} = 13$,

$$\text{adj } A = \begin{bmatrix} 0 & 2 & 1 \\ -1 & -9 & -5 \\ 2 & 23 & 13 \end{bmatrix}^T$$

$$\text{adj } A = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj } A$$

$$A^{-1} = \frac{1}{-1}\begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}$$ ...(i)**Ans.**

Now, given equations are

$$2x - 3y + 5z = 11$$
$$3x + 2y - 4z = -5$$
$$x + y - 2z = -3$$

Here, $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} B = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$

$$AX = B$$

$\therefore$ $X = A^{-1}B$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}\begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$ [From (i)]

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0-5+6 \\ -22-45+69 \\ -11-25+39 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Hence, $x = 1$, $y = 2$ and $z = 3$ **Ans.**

OR

Obtain the inverse of the following matrix using elementary operations :

$$A = \begin{bmatrix} -1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

Solution :

Given, $A = \begin{bmatrix} -1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

$$A = IA$$

$$\begin{bmatrix} -1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}A$$

Applying $R_1 \leftrightarrow R_2$, we get

$$\begin{bmatrix}1&2&3\\-1&1&2\\3&1&1\end{bmatrix}=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}A$$

Applying $R_2 \to R_2 + R_1$, we get

$$\begin{bmatrix}1&2&3\\0&3&5\\3&1&1\end{bmatrix}=\begin{bmatrix}0&1&0\\1&1&0\\0&0&1\end{bmatrix}A$$

Applying $R_3 \to R_3 - 3R_1$, we get

$$\begin{bmatrix}1&2&3\\0&3&5\\0&-5&-8\end{bmatrix}=\begin{bmatrix}0&1&0\\1&1&0\\0&-3&1\end{bmatrix}A$$

Applying $R_2 \to \frac{1}{3}R_2$, we get

$$\begin{bmatrix}1&2&3\\0&1&\frac{5}{3}\\0&-5&-8\end{bmatrix}=\begin{bmatrix}0&1&0\\\frac{1}{3}&\frac{1}{3}&0\\0&-3&1\end{bmatrix}A$$

Applying $R_1 \to R_1 - 2R_2$, we get

$$\begin{bmatrix}1&0&\frac{-1}{3}\\0&1&\frac{5}{3}\\0&-5&-8\end{bmatrix}=\begin{bmatrix}\frac{-2}{3}&\frac{1}{3}&0\\\frac{1}{3}&\frac{1}{3}&0\\0&-3&1\end{bmatrix}A$$

Applying $R_3 \to R_3 + 5R_2$, we get

$$\begin{bmatrix}1&0&\frac{-1}{3}\\0&1&\frac{5}{3}\\0&0&\frac{1}{3}\end{bmatrix}=\begin{bmatrix}\frac{-2}{3}&\frac{1}{3}&0\\\frac{1}{3}&\frac{1}{3}&0\\\frac{5}{3}&\frac{-4}{3}&1\end{bmatrix}A$$

Applying $R_1 \to R_1 - R_3$, we get

$$\begin{bmatrix}1&0&0\\0&1&\frac{5}{3}\\0&0&\frac{1}{3}\end{bmatrix}=\begin{bmatrix}1&-1&0\\\frac{1}{3}&\frac{1}{3}&0\\\frac{5}{3}&\frac{-4}{3}&1\end{bmatrix}A$$

Applying $R_2 \to R_2 - 5R_3$, we get

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&\frac{1}{3}\end{bmatrix}=\begin{bmatrix}1&-1&0\\-8&7&-5\\\frac{5}{3}&\frac{-4}{3}&1\end{bmatrix}A$$

Now, Applying $R_3 \to 3R_3$, we get

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&-1&0\\-8&7&-5\\5&-4&3\end{bmatrix}A$$

Hence, $A^{-1}=\begin{bmatrix}1&-1&1\\-8&7&-5\\5&-4&3\end{bmatrix}$ **Ans.**

Math 2019 (Delhi) SET III

Time allowed : 3 hours **Maximum marks : 100**

Note : Except for the following questions, all the remaining questions have been asked in previous sets.

SECTION-A

1. If $3A - B = \begin{bmatrix}5&0\\1&1\end{bmatrix}$ and $B = \begin{bmatrix}4&3\\2&5\end{bmatrix}$, then find the matrix A. **[1]**

Solution :

Given, $B = \begin{bmatrix}4&3\\2&5\end{bmatrix}$

Let $C = \begin{bmatrix}5&0\\1&1\end{bmatrix}$

$\therefore$ $3A - B = C$

$\Rightarrow$ $A = \frac{1}{3}(B + C)$

$\Rightarrow$ $A = \frac{1}{3}\left(\begin{bmatrix}4&3\\2&5\end{bmatrix}+\begin{bmatrix}5&0\\1&1\end{bmatrix}\right)$

$\Rightarrow$ $A = \frac{1}{3}\begin{bmatrix}9&3\\3&6\end{bmatrix}$

$\therefore$ $A = \begin{bmatrix}3&1\\1&2\end{bmatrix}$ **Ans.**

2. Write the order and the degree of the following differential equation :

$$x^3\left(\frac{d^2y}{dx^2}\right)^2 + x\left(\frac{dy}{dx}\right)^4 = 0$$ **[1]**

Solution :

Given,

$$x^3\left(\frac{d^2y}{dx^2}\right)^2 + x\left(\frac{dy}{dx}\right)^4 = 0$$

$\therefore$ Order of the given differential euqation is 2 and degree is 2. **Ans.**

SECTION-B

5. Find : $\int \sin x.\log \cos x\, dx.$ **[2]**

Solution :

Let $\quad I = \int \sin x \log \cos x\, dx$

Put $\cos x = t$

$$\sin x\, dx = -dt$$

$$I = \int \underset{\text{I}}{\log t}\,.\,\underset{\text{II}}{\text{I}}\, dt$$

$\Rightarrow \quad I = \left\{\log .t - \int \frac{1}{t}.tdt\right\}$

[Using By Parts]

$\Rightarrow \quad I = -t \log t + t + c$

$\Rightarrow \quad I = -\cos x\,(\log \cos x - 1) + C$

Ans.

6. Evaluate : $\int_{-\pi}^{\pi}(1-x^2)\sin x \cos^2 x\, dx.$ **[2]**

Solution :

Let $\quad I = \int_{-\pi}^{\pi}(1-x^2)\sin x \cos^2 x\, dx$

Let $\quad f(x) = (1-x^2)\sin x \cos^2 x$

Then $\quad f(-x) = -(1-x^2)\sin x \cos^2 x$

$$\begin{bmatrix}\because \cos(-\theta) = \cos\theta \\ \sin(-\theta) = -\sin\theta\end{bmatrix}$$

$\therefore \quad f(x) = -f(x)$

So, $f(x)$ is odd.

$\therefore \int_{-a}^{a} f(x)dx = 0$

when $f(x)$ is odd

$\therefore \quad I = 0$ **Ans.**

OR

Evaluate : $\int_{-1}^{2}\frac{|x|}{x}dx.$

Solution :

Let $\quad I = \int_{-1}^{2}\frac{|x|}{x}dx$

$$I = \int_{-1}^{0}\frac{-x}{x}dx + \int_{0}^{2}\frac{x}{x}dx$$

$\Rightarrow \quad I = \int_{-1}^{0} -1\, dx + \int_{0}^{2} 1\, dx$

$\Rightarrow \quad I = -[x]_{-1}^{0} + 1\,[x]_0^2$

$\Rightarrow \quad I = -[0+1] + [2-0]$

$\Rightarrow \quad I = -1 + 2$

$\Rightarrow \quad I = 1$ **Ans.**

8. Find a matrix A such that 2A – 3B + 5C = 0, where $B = \begin{bmatrix}-2 & 2 & 0 \\ 3 & 1 & 4\end{bmatrix}$ and $C = \begin{bmatrix}2 & 0 & -2 \\ 7 & 1 & 6\end{bmatrix}$. **[2]**

Solution :

Given, $B = \begin{bmatrix}-2 & 2 & 0 \\ 3 & 1 & 1\end{bmatrix}$ and $C = \begin{bmatrix}2 & 0 & -2 \\ 7 & 1 & 6\end{bmatrix}$

and $2A - 3B + 5C = 0$

$\Rightarrow \quad 2A = -5C + 3B$

$\Rightarrow \quad A = \frac{1}{2}[3B - 5C]$

$\Rightarrow \quad A = \frac{1}{2}\left(3\begin{bmatrix}-2 & 2 & 0 \\ 3 & 1 & 4\end{bmatrix} - 5\begin{bmatrix}2 & 0 & -2 \\ 7 & 1 & 6\end{bmatrix}\right)$

$\Rightarrow \quad A = \frac{1}{2}\left(\begin{bmatrix}-6 & 6 & 0 \\ 9 & 3 & 12\end{bmatrix} - \begin{bmatrix}10 & 0 & -10 \\ 35 & 5 & 30\end{bmatrix}\right)$

$\Rightarrow \quad A = \frac{1}{2}\begin{bmatrix}-16 & 6 & 10 \\ -26 & -2 & -18\end{bmatrix}$

$\Rightarrow \quad A = \begin{bmatrix}-8 & 3 & 5 \\ -13 & -1 & -9\end{bmatrix}$ **Ans.**

SECTION–C

13. Using propeties of determinants, prove the following :

$$\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix} = a^3 + b^3 + c^3 - 3abc.$$

[4]

Solution :

$$\text{L.H.S.} = \begin{vmatrix} a & b & c \\ a+b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix}$$

Applying $R_1 \to R_1 + R_3$, we get

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix}$$

Taking $(a+b+c)$ from R_1, we get

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we get

$$= (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ a-b & 2b-c-a & c-2a+b \\ b+c & a-b & a-c \end{vmatrix}$$

Now, Expanding along R_1, we get

$$= (a+b+c)\,[(1)\{(2b-c-a)(a-c)-(a-b)(c-2a+b)\}]$$
$$= (a+b+c)\,[(2ba-2bc-ac+c^2-a^2+ac) -(ac-2a^2+ab-bc+2ab-b^2)]$$
$$= (a+b+c)\,[2ba-2bc+c^2-a^2-ac+2a^2 -ab+bc-2ab+b^2]$$
$$= (a+b+c)\,[a^2+b^2+c^2-ab-bc-ca]$$
$$= a^3+b^3+c^3-3abc = \text{R.H.S.}$$

Hence Proved.

20. Find : $\int \frac{\cos x}{(1+\sin x)(2+\sin x)}dx$ **[4]**

Solution :

Let $\quad I = \int \frac{\cos x}{(1+\sin x)(2+\sin x)}dx$

Put $\quad \sin x = t$

$\cos x\, dx = dt$

$\therefore \quad I = \int \frac{dt}{(1+t)(2+t)}$

Let $\frac{1}{(1+t)(2+t)} = \frac{A}{1+t} + \frac{B}{2+t}$

$$1 = A(2+t) + B(1+t)$$

Comparing coefficient of t

$$0 = A + B$$
$$A = -B \quad \text{...(i)}$$

Comparing constant terms

$$1 = 2A + B$$
$$1 = -2B + B$$
$$-1 = B$$
$$\therefore \quad A = 1$$

$$\frac{1}{(1+t)(2+t)} = \frac{1}{(1+t)} - \frac{1}{(2+t)}$$

Now, $\quad I = \int \frac{dt}{1+t} - \int \frac{dt}{2+t}$

$\Rightarrow \quad I = \log|1+t| - \log|2+t| + c$

$\Rightarrow \quad I = \log\left|\frac{1+t}{2+t}\right| + C$

$$\left[\because \log m - \log n = \log\left(\frac{m}{n}\right)\right]$$

$\Rightarrow \quad I = \log\left|\frac{1+\sin x}{2+t}\right| + C$ **Ans.**

21. Solve the differential equation :

$\frac{dy}{dx} - \frac{2x}{1+x^2}y = x^2 + 2.$ **[4]**

Solution :

Given, $\frac{dy}{dx} - \frac{2x}{1+x^2}y = x^2+2$

Here, $\quad P = -\frac{2x}{1+x^2}$ and $Q = x^2+2$

$$\text{I.F.} = e^{\int P dx} = e^{\int -\frac{2x}{1+x^2}dx}$$

$$\text{I.F.} = \frac{1}{1+x^2}$$

$\therefore$ The solution of given differential equation is

$$y(\text{I.F}) = \int(\text{I.F.} \times Q)\,dx$$

$\Rightarrow \quad y\left(\frac{1}{1+x^2}\right) = \int \frac{x^2+2}{1+x^2}dx$

$\Rightarrow \quad y\frac{1}{1+x^2} = \int\left(1+\frac{1}{1+x^2}\right)dx$

$\Rightarrow \quad y\frac{1}{1+x^2} = x + \tan^{-1}x + C$

or $\quad y = (1+x^2)(x + \tan + x + C)$ **Ans.**

OR

Solve the differential equations :

$(x+1)\frac{dy}{dx} = 2e^{-1} - 1;\ y(0) = 0.$ **[4]**

Solution :

Given,

$$(x+1)\frac{dy}{dx} = 2e^{-y} - 1$$

$\Rightarrow \quad \frac{dy}{2e^{-y}-1} = \frac{dx}{x+1}$

$\Rightarrow \quad \int \frac{e^y dy}{2-e^y} = \int \frac{dx}{x+1}$

Put $2-e^y = t$

$-e^y dy = dt$

$e^y dy = -dt$

$\therefore \quad \int -\frac{dt}{t} = \int \frac{dx}{x+1}$

$\Rightarrow \quad -\log t = \log |x+1| + \log C$

$\Rightarrow \quad -\log t^{-1} = \log [C(x+1)]$

$\Rightarrow \quad \frac{1}{t} = C(x+1)$

$\Rightarrow \quad \frac{1}{2-e^y} = C(x+1)$ ...(i)

Put $x = 0$ and $y = 0$ in (i), we get

$\Rightarrow \quad \frac{1}{2} = C$

$\therefore \quad \frac{1}{2-e^y} = \frac{1}{2}(x+1)$ **Ans.**

SECTION–D

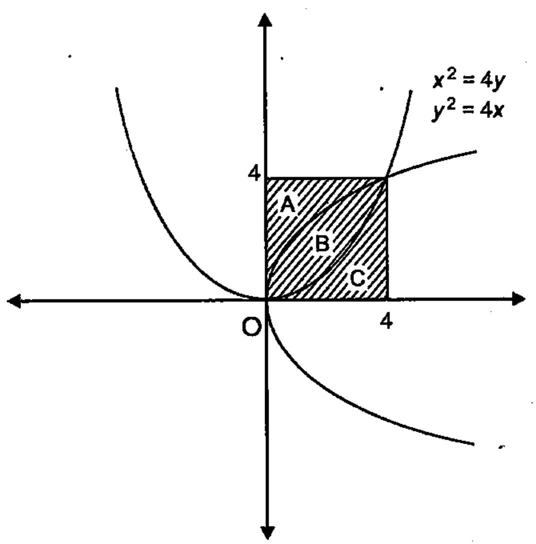
26. Prove that the curves $y^2 = 4x$ and $x^2 = 4y$ divide the by of the squre bounded by $x = 0, x = 4, y = 4$ and $y = 0$ into three equal parts. **[6]**

Solution :

Given, $y^2 = 4x$ ...(i)

$x^2 = 4y$ ...(ii)

$x = 0, x = y, y = 0$ and $y = 4$



For point of interaction put $y = \frac{x^2}{4}$ in (iii)

$\Rightarrow \quad \frac{x^4}{16} = 4x$

$\Rightarrow \quad x^4 = 64x$

$\Rightarrow \quad x^4 - 64x = 0$

$\Rightarrow \quad x(x^3 - 64) = 0$

$\Rightarrow \quad x = 0, x = 4$

Area of part A,

$I = \int_0^4 x \text{ of the curve I } dy$

$I = \int_0^4 \frac{y^2}{4} dy$

$I = \frac{1}{4 \times 3}\left[y^3\right]_0^4$

$I = \frac{1}{12}[4^3]$

$I = \frac{64}{12}$

$I = \frac{16}{3}$ sq. units

Area of Part B,

$I = \int_0^4 y \text{ of the I}^{st} \text{ curve } dx - \int_0^4 y \text{ of the 2}^{nd} \text{ curve } dx$

$I = \int_0^4 2\sqrt{x}dx - \int_0^4 \frac{x^2}{4} dx$

$I = 2 \times \frac{2}{3}\left[x^{3/2}\right]_0^4 - \frac{1}{12}\left[x^3\right]_0^4$

$I = \frac{4}{3}[4^{3/2}] - \frac{1}{12}[4^3]$

$I = \frac{4}{3} \times 8 - \frac{1}{12} \times 64$

$I = \frac{32}{3} - \frac{16}{3}$

$I = \frac{16}{3}$ sq units

Area of part C,

$I = \int_0^4 y \text{ of the 2}^{nd} \text{ curve } dx$

$I = \int_0^4 \frac{x^2}{4} dx$

$I = \frac{1}{4}\int_0^4 x^2 dx$

$I = \frac{1}{12}\left[x^3\right]_0^4$

$I = \frac{1}{12} \times 64$

$I = \frac{16}{3}$ sq. units

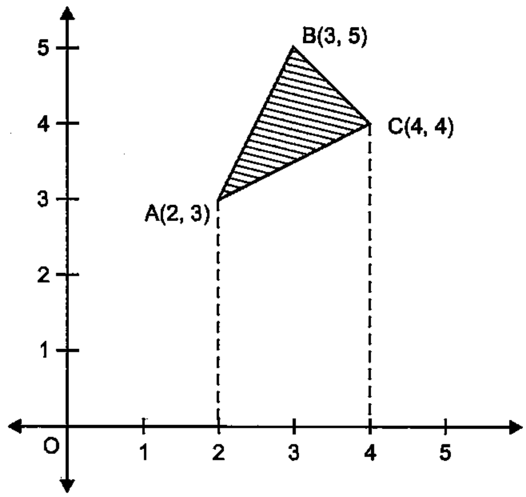
$\therefore$ The curves $y^2 = 4x$ and $x^2 = 4y$ divides the area of the square into three equal parts. **Hence Proved.**

OR

Using integration, find the area of the triangle whose vertices are (2, 3), (3, 5) and (4, 4).

Solution :

Given, the vertices of ΔABC, A(2, 3), B (3, 5) and C (4, 4).



Equation of line AB,

$$(y-3) = \frac{5-3}{3-2}(x-2)$$
$$(y-3) = 2(x-2)$$
$$y = 2x-4+3$$
$$y = 2x-1$$

Equation of line BC,

$$(y-5) = \frac{4-5}{4-3}(x-3)$$
$$(y-5) = -(x-3)$$
$$y-5 = -x-3$$
$$y = -x+8$$

Equation of line AC,

$$(y-3) = \frac{4-3}{4-2}(x-2)$$
$$(y-3) = \frac{1}{2}(x-2)$$
$$y = \frac{x-2}{2}+3$$
$$y = \frac{x+4}{2}$$

$$I = \int_2^3 (2n-1)\,dx + \int_3^4 (-x+8)\,dx - \int_2^4 \frac{x+4}{2}\,dx$$

Area of Δ ABC

$$I = \int_2^3 y_{AB}dx + \int_3^4 y_{BC}\,dx - \int_2^3 y_{AC}dx$$

$$I = [x^2]_2^3 - [x]_2^3 - \frac{1}{2}[x^2]_3^4 + 8[x]_3^4 - \frac{1}{2}\left[\frac{x^2}{2}\right]_2^4 + 2[x]_2^4$$

$$I = [9-4]-[3-2]-\frac{1}{2}[16-9] + 8[4-3]-\frac{1}{4}[16-4]+2[4-2]$$

$$I = 5-1-\frac{7}{2}+8-\frac{12}{4}+4$$

$$I = 4-\frac{7}{2}+8-3+4$$

$$I = 13-\frac{7}{2}$$

$$I = \frac{19}{2} \text{ Sq. units}$$ **Ans.**

29. Two cards are drawn simultaneously (or successively without replacement) from a well shuffled pack of 52 cards. Find the mean and variance of the number of kings. [4]

Solution :

Let S = Number of kings.

and F = number of non kings

$\therefore$ $P(S) = \frac{4}{52}$ and $P(F) = \frac{48}{52}$

Let x_i be the event that king is drawn.

x_i	Px_i	x_iPx_i	x_i^2	$Px_i\,x_i^2$
0	$\frac{48}{52}\times\frac{47}{51}\times 1=\frac{2256}{2652}$	0	0	0
1	$\frac{4}{52}\times\frac{48}{51}\times 2=\frac{384}{2652}$	$\frac{384}{2652}$	1	$\frac{384}{2652}$
2	$\frac{4}{52}\times\frac{3}{51}\times 1=\frac{12}{2652}$	$\frac{12}{2652}$	4	$\frac{48}{2652}$

Px_i	$2256/2652$	$384/2652$	$12/2652$
x_i	0	1	2

$$\text{Mean } (\bar{X}) = \Sigma x_i^2 P(x_i)$$
$$= \frac{384+12}{2652} = \frac{396}{2652}$$

$$\text{Variance } (\sigma^2) = \Sigma x_i^2 P(x_i) - (\bar{X})^2$$
$$= \frac{432}{2652} = \left(\frac{396}{2652}\right)^2$$
$$= \frac{36}{221} - \left(\frac{33}{221}\right)^2$$
$$= \frac{(36\times 221)-(33\times 33)}{(221\times 221)}$$
$$= \frac{7956-1089}{48841} = \frac{6867}{48841}$$ **Ans.**