1 Relations and Functions

1.3 EXERCISE

SHORT ANSWER TYPE QUESTIONS

Q1. Let A = {a, b, c} and the relation R be defined on A as follows:

$$R = \{(a, a), (b, c), (a, b)\}$$

Then, write minimum number of ordered pairs to be added in R to make R reflexive and transitive.

Sol. Here, $R = \{(a, a), (b, c), (a, b)\}$

for reflexivity; (b, b), (c, c) and for transitivity; (a, c)

Hence, the required ordered pairs are (b, b), (c, c) and (a, c)

Q2. Let D be the domain of the real valued function f defined by $f(x) = \sqrt{25 - x^2}$. Then write D.

Sol. Here, $f(x) = \sqrt{25 - x^2}$

For real value of $f(x)$, $25 - x^2 \geq 0$

$\Rightarrow -x^2 \geq -25 \Rightarrow x^2 \leq 25 \Rightarrow -5 \leq x \leq 5$

Hence, $D \in -5 \leq x \leq 5$ or $[-5, 5]$

Q3. Let $f, g : R \rightarrow R$ be defined by $f(x) = 2x + 1$ and $g(x) = x^2 - 2$ $\forall$ $x \in R$, respectively. Then find gof.

Sol. Here, $f(x) = 2x + 1$ and $g(x) = x^2 - 2$

$\therefore \quad gof = g[f(x)]$

$= [2x + 1]^2 - 2 = 4x^2 + 4x + 1 - 2 = 4x^2 + 4x - 1$

Hence, $gof = 4x^2 + 4x - 1$

Q4. Let $f : R \rightarrow R$ be the function defined by $f(x) = 2x - 3$ $\forall$ $x \in R$. Write f^{-1}.

Sol. Here, $f(x) = 2x - 3$

Let $f(x) = y = 2x - 3$

$\Rightarrow \quad y + 3 = 2x \Rightarrow x = \dfrac{y+3}{2}$

$\therefore \quad f^{-1}(y) = \dfrac{y+3}{2}$ or $f^{-1}(x) = \dfrac{x+3}{2}$

Q5. If A = {a, b, c, d} and the function $f = \{(a, b), (b, d), (c, a), (d, c)\}$, write f^{-1}.

Sol. Let $y = f(x) \quad \therefore x = f^{-1}(y)$

$\therefore$ If $f = \{(a, b), (b, d), (c, a), (d, c)\}$

then $f^{-1} = \{(b, a), (d, b), (a, c), (c, d)\}$

Q6. If $f: R \to R$ is defined by $f(x) = x^2 - 3x + 2$, write $f[f(x)]$.

Sol. Here, $f(x) = x^2 - 3x + 2$

$$\begin{aligned} \therefore \quad f[f(x)] &= [f(x)]^2 - 3f(x) + 2 \\ &= (x^2 - 3x + 2)^2 - 3(x^2 - 3x + 2) + 2 \\ &= x^4 + 9x^2 + 4 - 6x^3 + 4x^2 - 12x - 3x^2 + 9x - 6 + 2 \\ &= x^4 - 6x^3 + 10x^2 - 3x \end{aligned}$$

Hence, $f[f(x)] = x^4 - 6x^3 + 10x^2 - 3x$

Q7. Is $g = \{(1, 1), (2, 3), (3, 5), (4, 7)\}$ a function? If g is described by $g(x) = \alpha x + \beta$, then what value should be assigned to α and β?

Sol. Yes, $g = \{(1, 1), (2, 3), (3, 5), (4, 7)\}$ is a function.

Here, $g(x) = \alpha x + \beta$

For (1, 1), $g(1) = \alpha.1 + \beta$

$1 = \alpha + \beta$ ...(1)

For (2, 3), $g(2) = \alpha.2 + \beta$

$3 = 2\alpha + \beta$ ...(2)

Solving eqs. (1) and (2) we get, $\alpha = 2, \beta = -1$

Q8. Are the following set of ordered pairs functions? If so, examine whether the mapping is injective or surjective.

(*i*) $\{(x, y) : x$ is a person, y is the mother of $x\}$

(*ii*) $\{(a, b) : a$ is a person, b is an ancestor of $a\}$

Sol. (*i*) It represents a function. The image of distinct elements of x under f are not distinct. So, it is not injective but it is surjective.

(*ii*) It does not represent a function as every domain under mapping does not have a unique image.

Q9. If the mapping f and g are given by

$f = \{(1, 2), (3, 5), (4, 1)\}$ and $g = \{(2, 3), (5, 1), (1, 3)\}$ write fog.

Sol.

$$\begin{aligned} fog &= f[g(x)] \\ &= f[g(2)] = f(3) = 5 \\ &= f[g(5)] = f(1) = 2 \\ &= f[g(1)] = f(3) = 5 \end{aligned}$$

Hence, $fog = \{(2, 5), (5, 2), (1, 5)\}$

Q10. Let C be the set of complex numbers. Prove that the mapping $f: C \to R$ given by $f(z) = |z|, \forall z \in C$, is neither one-one nor onto.

Sol. Here, $f(z) = |z| \quad \forall z \in C$

$$\begin{aligned} f(1) &= |1| = 1 \\ f(-1) &= |-1| = 1 \\ f(1) &= f(-1) \end{aligned}$$

But $1 \neq -1$

Therefore, it is not one-one.

Now, let $f(z) = y = |z|$. Here, there is no pre-image of negative numbers. Hence, it is not onto.

Q11. Let the function $f : R \to R$ be defined by $f(x) = \cos x$, $\forall\ x \in R$. Show that f is neither one-one nor onto.

Sol. Here, $f(x) = \cos x\ \forall\ x \in R$

Let $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \in f(x)$

$$f\left(-\frac{\pi}{2}\right) = \cos\left(-\frac{\pi}{2}\right) = \cos\frac{\pi}{2} = 0$$

$$\cos\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} = 0$$

$$f\left(-\frac{\pi}{2}\right) = f\left(\frac{\pi}{2}\right) = 0$$

But $-\frac{\pi}{2} \neq \frac{\pi}{2}$

Therefore, the given function is not one-one. Also it is not onto function as no pre-image of any real number belongs to the range of $\cos x$ i.e., $[-1, 1]$.

Q12. Let X = {1, 2, 3} and Y = {4, 5}. Find whether the following subsets of X × Y are functions from X to Y or not.

(*i*) $f = \{(1, 4), (1, 5), (2, 4), (3, 5)\}$

(*ii*) $g = \{(1, 4), (2, 4), (3, 4)\}$

(*iii*) $h = \{(1, 4), (2, 5), (3, 5)\}$

(*iv*) $k = \{(1, 4), (2, 5)\}$

Sol. Here, given that X = {1, 2, 3}, Y = {4, 5}

$\therefore$ X × Y = {(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5)}

(*i*) $f = \{(1, 4), (1, 5), (2, 4), (3, 5)\}$

f is not a function because there is no unique image of each element of domain under f.

(*ii*) $g = \{(1, 4), (2, 4), (3, 4)\}$

Yes, g is a function because each element of its domain has a unique image.

(*iii*) $h = \{(1, 4), (2, 5), (3, 5)\}$

Yes, it is a function because each element of its domain has a unique image.

(*iv*) $k = \{(1, 4), (2, 5)\}$

Clearly k is also a function.

Q13. If function $f: A \to B$ and $g: B \to A$ satisfy $gof = I_A$, then show that f is one-one and g is onto.

Sol. Let $x_1, x_2 \in gof$

$$gof\{f(x_1)\} = gof\{f(x_2)\}$$

$$\Rightarrow \quad g(x_1) = g(x_2) \qquad [\because gof = I_A]$$

$$\therefore \quad x_1 = x_2$$

Hence, f is one-one. But g is not onto as there is no pre-image of A in B under g.

Q14. Let $f: R \to R$ be the function defined by $f(x) = \dfrac{1}{2 - \cos x}$, $\forall\, x \in R$. Then, find the range of f.

Sol. Given function is $f(x) = \dfrac{1}{2 - \cos x}$, $\forall\, x \in R$.

Range of $\cos x$ is $[-1, 1]$

Let $\quad f(x) = y = \dfrac{1}{2 - \cos x}$

$$\Rightarrow \quad 2y - y\cos x = 1 \quad \Rightarrow \quad y\cos x = 2y - 1$$

$$\Rightarrow \quad \cos x = \frac{2y - 1}{y} = 2 - \frac{1}{y}$$

Now $\quad -1 \le \cos x \le 1$

$$\Rightarrow \quad -1 \le 2 - \frac{1}{y} \le 1 \quad \Rightarrow \quad -1 - 2 \le -\frac{1}{y} \le 1 - 2$$

$$\Rightarrow \quad -3 \le -\frac{1}{y} \le -1 \quad \Rightarrow 3 \ge \frac{1}{y} \ge 1 \quad \Rightarrow \quad \frac{1}{3} \le y \le 1$$

Hence, the range of $f = \left[\dfrac{1}{3}, 1\right]$.

Q15. Let n be a fixed positive integer. Define a relation R in Z as follows $\forall\, a, b \in Z$, $a\,R\,b$ if and only if $a - b$ is divisible by n. Show that R is an equivalence relation.

Sol. Here, $\forall\, a, b \in Z$ and $a\,R\,b$ if and only if $a - b$ is divisible by n.
The given relation is an equivalence relation if it is reflexive, symmetric and transitive.

(*i*) Reflexive:

$a\,R\,a \;\Rightarrow\; (a - a) = 0$ divisible by n

So, R is reflexive.

(*ii*) Symmetric:

$a\,R\,b = b\,R\,a \quad \forall\, a, b \in Z$

$a - b$ is divisible by n (Given)

$\Rightarrow -(b - a)$ is divisible by n

$\Rightarrow b - a$ is divisible by n

$\Rightarrow b \, R \, a$

Hence, R is symmetric.

(*iii*) Transitive:

$a \, R \, b$ and $b \, R \, c \iff a \, R \, c \quad \forall \, a, b, c \in Z$

$a - b$ is divisible by n

$b - c$ is also divisible by n

$\Rightarrow (a - b) + (b - c)$ is divisible by n

$\Rightarrow (a - c)$ is divisible by n

Hence, R is transitive.

So, R is an equivalence relation.

LONG ANSWER TYPE QUESTIONS

Q16. If A = {1, 2, 3, 4}, define relations on A which have properties of being.

(*a*) reflexive, transitive but not symmetric.

(*b*) symmetric but neither reflexive nor transitive

(*c*) reflexive, symmetric and transitive.

Sol. Given that A = {1, 2, 3, 4}

$\therefore \quad ARA = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4), (2, 1), (3, 1), (4, 1), (3, 2), (4, 2), (4, 3)\}$

(*a*) Let $R_1 = \{(1, 1), (2, 2), (1, 2), (2, 3), (1, 3)\}$

So, R_1 is reflexive and transitive but not symmetric.

(*b*) Let $R_2 = \{(2, 3), (3, 2)\}$

So, R_2 is only symmetric.

(*c*) Let $R_3 = \{(1, 1), (1, 2), (2, 1), (2, 4), (1, 4)\}$

So, R_3 is reflexive, symmetric and transitive.

Q17. Let R be relation defined on the set of natural number N as follows:

$R = \{(x, y) : x \in N, y \in N, 2x + y = 41\}$. Find the domain and range of the relation R. Also verify whether R is reflexive, symmetric and transitive.

Sol. Given that $x \in N, y \in N$ and $2x + y = 41$

$\therefore \quad$ Domain of R = {1, 2, 3, 4, 5, ..., 20}

and $\quad$ Range = {39, 37, 35, 33, 31, ..., 1}

Here, $\quad (3, 3) \notin R$

as $\quad 2 \times 3 + 3 \neq 41$

So, R is not reflexive.

R is not symmetric as $(2, 37) \in R$ but $(37, 2) \notin R$

R is not transitive as $(11, 19) \in R$ and $(19, 3) \in R$ but $(11, 3) \notin R$.

Hence, R is neither reflexive, nor symmetric and nor transitive.

Q18. Given A = {2, 3, 4}, B = {2, 5, 6, 7}, construct an example of each of the following:

(*i*) an injective mapping from A to B.
(*ii*) a mapping from A to B which is not injective
(*iii*) a mapping from B to A.

Sol. Here, $A = \{2, 3, 4\}$ and $B = \{2, 5, 6, 7\}$

(*i*) Let $f : A \to B$ be the mapping from A to B
$f = \{(x, y) : y = x + 3\}$
$\therefore f = \{(2, 5), (3, 6), (4, 7)\}$ which is an injective mapping.

(*ii*) Let $g : A \to B$ be the mapping from $A \to B$ such that
$g = \{(2, 5), (3, 5), (4, 2)\}$ which is not an injective mapping.

(*iii*) Let $h : B \to A$ be the mapping from B to A
$h = \{(y, x) : x = y - 2\}$
$h = \{(5, 3), (6, 4), (7, 3)\}$ which is the mapping from B to A.

Q19. Give an example of a map

(*i*) which is one-one but not onto.
(*ii*) which is not one-one but onto.
(*iii*) which is neither one-one nor onto.

Sol. (*i*) Let $f : N \to N$ given by $f(x) = x^2$

Let $x_1, x_2 \in N$ then $f(x_1) = x_1^2$ and $f(x_2) = x_2^2$

Now, $f(x_1) = f(x_2) \Rightarrow x_1^2 = x_2^2 \Rightarrow x_1^2 - x_2^2 = 0$

$\Rightarrow (x_1 + x_2)(x_1 - x_2) = 0$

Since $x_1, x_2 \in N$, so $x_1 + x_2 = 0$ is not possible.

$\therefore \quad x_1 - x_2 = 0 \quad \Rightarrow x_1 = x_2$

$\therefore \quad f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

So, $f(x)$ is one to one function.

Now, Let $f(x) = 5 \in N$

then $\quad x^2 = 5 \quad \Rightarrow \quad x = \pm\sqrt{5} \notin N$

So, f is not onto.

Hence, $f(x) = x^2$ is one-one but not onto.

(*ii*) Let $f : N \times N$, defined by $f(n) = \begin{cases} \dfrac{n+1}{2} & \text{if } n \text{ is odd} \\ \dfrac{n}{2} & \text{if } n \text{ is even} \end{cases}$

Since $f(1) = f(2)$ but $1 \neq 2$,

So, f is not one-one.

Now, let $y \in N$ be any element.

Then $f(n) = y$

$$\Rightarrow \left.\begin{matrix} \dfrac{n+1}{2} & \text{if } n \text{ is odd} \\ \dfrac{n}{2} & \text{if } n \text{ is even} \end{matrix}\right\} = y$$

$$\Rightarrow \qquad n = 2y - 1 \quad \text{if } y \text{ is even}$$
$$n = 2y \quad \text{if } y \text{ is odd or even}$$

$$\Rightarrow \qquad n = \begin{cases} 2y - 1 & \text{if } y \text{ is even} \\ 2y & \text{if } y \text{ is odd or even} \end{cases} \in \text{N} \ \forall \ y \in \text{N}$$

$\therefore$ Every $y \in$ N has pre-image

$$n = \begin{cases} 2y - 1 & \text{if } y \text{ is even} \\ 2y & \text{if } y \text{ is odd or even} \end{cases} \in \text{N}$$

$\therefore f$ is onto.

Hence, f is not one-one but onto.

(*iii*) Let $f : \text{R} \to \text{R}$ be defined as $f(x) = x^2$

Let $x_1 = 2$ and $x_2 = -2$

$$f(x_1) = x_1^2 = (2)^2 = 4$$
$$f(x_2) = x_2^2 = (-2)^2 = 4$$
$$f(2) = f(-2) \quad \text{but } 2 \neq -2$$

So, it is not one-one function.

Let $f(x) = -2 \quad \Rightarrow \quad x^2 = -2 \quad \therefore \quad x = \pm\sqrt{-2} \notin \text{R}$

Which is not possible, so f is not onto.

Hence, f is neither one-one nor onto.

Q20. Let A = R – {3}, B = R – {1}. Let f : A $\to$ B be defined by $f(x) = \dfrac{x-2}{x-3}$, $\forall \ x \in$ A. Then, show that f is bijective.

Sol. Here, A $\in$ R – {3}, B = R – {1}

Given that $f : \text{A} \to \text{B}$ defined by $f(x) = \dfrac{x-2}{x-3} \ \forall \ x \in \text{A}$.

Let $x_1, x_2 \in f(x)$

$$\therefore \qquad f(x_1) = f(x_2)$$

$$\Rightarrow \qquad \frac{x_1 - 2}{x_1 - 3} = \frac{x_2 - 2}{x_2 - 3}$$

$$\Rightarrow \qquad (x_1 - 2)(x_2 - 3) = (x_2 - 2)(x_1 - 3)$$

$$\Rightarrow \quad \cancel{x_1 x_2} - 3x_1 - 2x_2 + \cancel{6} = \cancel{x_1 x_2} - 3x_2 - 2x_1 + \cancel{6}$$

$$\Rightarrow \qquad -x_1 = -x_2 \quad \Rightarrow \quad x_1 = x_2$$

So, it is injective function.

Now, Let $\qquad y = \dfrac{x-2}{x-3}$

$\Rightarrow \ xy - 3y = x - 2 \quad \Rightarrow \ xy - x = 3y - 2$

$\Rightarrow \ x(y - 1) = 3y - 2 \quad \Rightarrow x = \dfrac{3y-2}{y-1}$

$$f(x) = \frac{x-2}{x-3} = \frac{\frac{3y-2}{y-1}-2}{\frac{3y-2}{y-1}-3} \Rightarrow \frac{3y-2-2y+2}{3y-2-3y+3} \Rightarrow y$$

$\Rightarrow \quad f(x) = y \in B.$

So, $f(x)$ is surjective function.

Hence, $f(x)$ is a bijective function.

Q21. Let $A = [-1, 1]$, then discuss whether the following functions defined on A are one-one, onto or bijective.

(*i*) $f(x) = \frac{x}{2}$ (*ii*) $g(x) = |x|$ (*iii*) $h(x) = x\,|x|$ (*iv*) $k(x) = x^2$

Sol. (*i*) Given that $-1 \le x \le 1$

Let $x_1, x_2 \in f(x)$

$$f(x_1) = \frac{1}{x_1} \quad \text{and} \quad f(x_2) = \frac{1}{x_2}$$

$$f(x_1) = f(x_2) \Rightarrow \frac{1}{x_1} = \frac{1}{x_2} \Rightarrow x_1 = x_2$$

So, $f(x)$ is one-one function.

Let $\quad f(x) = y = \frac{x}{2} \Rightarrow x = 2y$

For $y = 1, x = 2 \notin [-1, 1]$

So, $f(x)$ is not onto. Hence, $f(x)$ is not bijective function.

(*ii*) Here, $\quad g(x) = |x|$

$$g(x_1) = g(x_2) \Rightarrow |x_1| = |x_2| \Rightarrow x_1 = \pm x_2$$

So, $g(x)$ is not one-one function.

Let $g(x) = y = |x| \Rightarrow x = \pm y \notin A \;\forall\; y \in A$

So, $g(x)$ is not onto function.

Hence, $g(x)$ is not bijective function.

(*iii*) Here, $\quad h(x) = x|x|$

$$h(x_1) = h\,f(x_2)$$

$$\Rightarrow \quad x_1|x_1| = x_2|x_2| \Rightarrow x_1 = x_2$$

So, $h(x)$ is one-one function.

Now, let $\quad h(x) = y = x|x| = x^2 \text{ or } -x^2$

$$\Rightarrow \quad x = \pm\sqrt{-y} \notin A \;\forall\; y \in A$$

$\therefore$ $h(x)$ is not onto function.

Hence, $h(x)$ is not bijective function.

(*iv*) Here, $\quad k(x) = x^2$

$$k(x_1) = k(x_2)$$

$$\Rightarrow \quad x_1^2 = x_2^2 \Rightarrow x_1 = \pm x_2$$

So, $k(x)$ is not one-one function.

Now, let $k(x) = y = x^2 \Rightarrow x = \pm\sqrt{y}$

If $y = -1 \Rightarrow x = \pm\sqrt{-1} \notin A \; \forall \; y \in A$

$\therefore k(x)$ is not onto function.

Hence, $k(x)$ is not a bijective function.

Q22. Each of the following defines a relation of N

(*i*) x is greater than y, $x, y \in N$

(*ii*) $x + y = 10$, $x, y \in N$

(*iii*) xy is square of an integer $x, y \in N$

(*iv*) $x + 4y = 10$, $x, y \in N$.

Determine which of the above relations are reflexive, symmetric and transitive.

Sol. (*i*) x is greater than y, $x, y \in N$

For reflexivity $x > x \; \forall \; x \in N$ which is not true

So, it is not reflexive relation.

Now, $x > y$ but $y \not> x \; \forall \; x, y \in N$

$\Rightarrow x \, R \, y$ but $y \not{R} \, x$

So, it is not symmetric relation.

For transitivity, $x \, R \, y, y \, R \, z \Rightarrow x \, R \, z \; \forall \; x, y, z \in N$

$$\Rightarrow x > y, y > z \Rightarrow x > z$$

So, it is transitive relation.

(*ii*) Here, $R = \{(x, y) : x + y = 10 \; \forall \; x, y \in N\}$

$R = \{(1, 9), (2, 8), (3, 7), (4, 6), (5, 5), (6, 4), (7, 3), (8, 2), (9, 1)\}$

For reflexive: $5 + 5 = 10$, $5 \, R \, 5 \Rightarrow (x, x) \in R$

So, R is reflexive.

For symmetric: $(1, 9) \in R$ and $(9, 1) \in R$

So, R is symmetric.

For transitive: $(3, 7) \in R$, $(7, 3) \in R$ but $(3, 3) \notin R$

So, R is not transitive.

(*iii*) Here, $R = \{(x, y) : xy$ is a square of an integer, $x, y \in N\}$

For reflexive: $x \, R \, x = x \, . \, x = x^2$ is an integer

[$\because$ Square of an integer is also an integer]

So, R is reflexive.

For symmetric: $x \, R \, y = y \, R \, x \; \forall \; x, y \in N$

$\therefore \quad xy = yx$ (integer)

So, it is symmetric.

For transitive: $x \, R \, y$ and $y \, R \, z \Rightarrow x \, R \, z$

Let $xy = k^2$ and $yz = m^2$

$$x = \frac{k^2}{y} \quad \text{and} \quad z = \frac{m^2}{y}$$

$\therefore \quad xz = \dfrac{k^2 m^2}{y^2}$ which is again a square of an integer.

So, R is transitive.

(*iv*) Here, $R = \{(x, y) : x + 4y = 10, x, y \in N\}$
$R = \{(2, 2), (6, 1)\}$
For reflexivity: $(2, 2) \in R$
So, R is reflexive.
For symmetric: $(x, y) \in R$ but $(y, x) \notin R$
$(6, 1) \in R$ but $(1, 6) \notin R$
So, R is not symmetric.
For transitive: $(x, y) \in R$ but $(y, z) \notin R$ and $(x, z) \in R$
So, R is not transitive.

Q23. Let $A = \{1, 2, 3, ..., 9\}$ and R be the relation in $A \times A$ defined by $(a, b) R (c, d)$ if $a + d = b + c$ for $(a, b), (c, d)$ in $A \times A$. Prove that R is an equivalence relation and also obtain equivalent class $[(2, 5)]$.

Sol. Here, $A = \{1, 2, 3, ..., 9\}$
and $R \to A \times A$ defined by $(a, b) R (c, d) \Rightarrow a + d = b + c$
$\forall (a, b), (c, d) \in A \times A$
For reflexive: $(a, b) R (a, b) = a + b = b + a \quad \forall a, b \in A$ which is true. So, R is reflexive.
For symmetric: $(a, b) R (c, d) = (c, d) R (a, b)$
L.H.S. $a + d = b + c$
R.H.S. $c + b = d + a$
L.H.S. = R.H.S. So, R is symmetric.
For transitive: $(a, b) R (c, d)$ and $(c, d) R (e, f) \Leftrightarrow (a, b) R (e, f)$
$\Rightarrow \quad a + d = b + c$ and $c + f = d + e$
$\Rightarrow \quad a + d = b + c$ and $d + e = c + f$
$\Rightarrow \quad (a + d) - (d + e) = (b + c) - (c + f)$
$\Rightarrow \quad a - e = b - f$
$\Rightarrow \quad a + f = b + e$
$\Rightarrow \quad (a, b) R (e, f)$
So, R is transitive.
Hence, R is an equivalence relation.
Equivalent class of $\{(2, 5)\}$ is $\{(1, 4), (2, 5), (3, 6), (4, 7), (5, 8), (6, 9)\}$

Q24. Using the definition, prove that the function $f : A \to B$ is invertible if and only if f is both one-one and onto.

Sol. A function $f : X \to Y$ is said to be invertible if there exists a function $g : Y \to X$ such that $gof = I_X$ and $fog = I_Y$ and then the inverse of f is denoted by f^{-1}.
A function $f : X \to Y$ is said to be invertible iff f is a bijective function.

Q25. Function $f, g : R \to R$ are defined, respectively, by $f(x) = x^2 + 3x + 1$, $g(x) = 2x - 3$, find
(*i*) fog (*ii*) gof (*iii*) fof (*iv*) gog

Sol. (*i*) $fog \Rightarrow f[g(x)] = [g(x)]^2 + 3[g(x)] + 1$

$= (2x-3)^2 + 3(2x-3) + 1$
$= 4x^2 + 9 - 12x + 6x - 9 + 1 = 4x^2 - 6x + 1$

(ii) $gof \Rightarrow g[f(x)] = 2[x^2 + 3x + 1] - 3$
$= 2x^2 + 6x + 2 - 3 = 2x^2 + 6x - 1$

(iii) $fof \Rightarrow f[f(x)] = [f(x)]^2 + 3[f(x)] + 1$
$= (x^2 + 3x + 1)^2 + 3(x^2 + 3x + 1) + 1$
$= x^4 + 9x^2 + 1 + 6x^3 + 6x + 2x^2 + 3x^2 + 9x + 3 + 1$
$= x^4 + 6x^3 + 14x^2 + 15x + 5$

(iv) $gog \Rightarrow g[g(x)] = 2[g(x)] - 3 = 2(2x-3) - 3 = 4x - 6 - 3 = 4x - 9$

Q26. Let $*$ be the binary operation defined on Q. Find which of the following binary operations are commutative.

(i) $a * b = a - b \ \forall\, a, b \in Q$ (ii) $a * b = a^2 + b^2 \ \forall\, a, b \in Q$
(iii) $a * b = a + ab \ \forall\, a, b \in Q$ (iv) $a * b = (a-b)^2 \ \forall\, a, b \in Q$

Sol. (i) $a * b = a - b \in Q \quad \forall\, a, b \in Q.$
So, $*$ is binary operation.
$a * b = a - b$ and $b * a = b - a \quad \forall\, a, b \in Q$
$a - b \neq b - a$
So, $*$ is not commutative.

(ii) $a * b = a^2 + b^2 \in Q$, so $*$ is a binary operation.
$a * b = b * a$
$\Rightarrow \quad a^2 + b^2 = b^2 + a^2 \quad \forall\, a, b \in Q$
Which is true. So, $*$ is commutative.

(iii) $a * b = a + ab \in Q$, so $*$ is a binary operation.
$a * b = a + ab$ and $b * a = b + ba$
$a + ab \neq b + ba \Rightarrow a * b \neq b * a \quad \forall\, a, b \in Q.$
So, $*$ is not commutative.

(iv) $a * b = (a-b)^2 \in Q$, so $*$ is binary operation.
$a * b = (a-b)^2$ and $b * a = (b-a)^2$
$a * b = b * a \Rightarrow (a-b)^2 = (b-a)^2 \quad \forall\, a, b \in Q.$
So, $*$ is commutative.

Q27. If $*$ be binary operation defined on R by $a * b = 1 + ab \ \forall\, a, b \in R$. Then, the operation $*$ is

(i) commutative but not associative
(ii) associative but not commutative
(iii) neither commutative nor associative
(iv) both commutative and associative

Sol. (i): Given that
$a * b = 1 + ab \quad \forall\, a, b \in R$
and $\quad b * a = 1 + ba \quad \forall\, a, b \in R$
$a * b = b * a = 1 + ab$
So, $*$ is commutative.
Now $\quad a * (b * c) = (a * b) * c \qquad \forall\, a, b, c \in R$

L.H.S. $a * (b * c) = a * (1 + bc) = 1 + a(1 + bc) = 1 + a + abc$
R.H.S. $(a * b) * c = (1 + ab) * c = 1 + (1 + ab) . c = 1 + c + abc$
L.H.S. $\neq$ R.H.S.
So, $*$ is not associative.
Hence, $*$ is commutative but not associative.

OBJECTIVE TYPE QUESTIONS

Choose the correct answer out of the given four options in each of the Exercises from 28 to 47 (M.C.Q.)

Q28. Let T be the set of all triangles in the Euclidean plane and let a relation R on T be defined as a R b, if a is congruent to b, $\forall$ a, $b \in$ T. Then R is

(*a*) Reflexive but not transitive
(*b*) Transitive but not symmetric
(*c*) Equivalence
(*d*) None of these

Sol. If $a \cong b$ $\forall$ $a, b \in$ T
then a R $a \Rightarrow a \cong a$ which is true for all $a \in$ T
So, R is reflexive.
Now, a R b and b R a.
i.e., $a \cong b$ and $b \cong a$ which is true for all $a, b \in$ T
So, R is symmetric.
Let a R b and b R c.
$\Rightarrow a \cong b$ and $b \cong a \Rightarrow a \cong c$ $\forall$ $a, b, c \in$ T
So, R is transitive.
Hence, R is equivalence relation.
So, the correct answer is (*c*).

Q29. Consider the non-empty set consisting of children in a family and a relation R defined as a R b, if a is brother of b. Then R is

(*a*) symmetric but not transitive
(*b*) transitive but not symmetric
(*c*) neither symmetric nor transitive
(*d*) both symmetric and transitive

Sol. Here, a R $b \Rightarrow a$ is a brother of b.
a R $a \Rightarrow a$ is a brother of a which is not true.
So, R is not reflexive.
a R $b \Rightarrow a$ is a brother of b.
b R $a \Rightarrow$ which is not true because b may be sister of a.
$\Rightarrow a \text{ R } b \neq b \text{ R } a$
So, R is not symmetric.
Now, a R b, b R $c \Rightarrow a$ R c
$\Rightarrow a$ is the brother of b and b is the brother of c.
$\therefore$ a is also the brother of c.

So, R is transitive.

Hence, correct answer is (*b*).

Q30. The maximum number of equivalence relations on the set A = {1, 2, 3} are

(*a*) 1 (*b*) 2 (*c*) 3 (*d*) 5

Sol. Here, A = {1, 2, 3}

The number of equivalence relations are as follows:

$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 3), (1, 3)\}$

$R_2 = \{(2, 2), (1, 3), (3, 1), (3, 2), (1, 2)\}$

$R_3 = \{(3, 3), (1, 2), (2, 3), (1, 3), (3, 2)\}$

Hence, correct answer is (*d*)

Q31. If a relation R on the set {1, 2, 3} be defined by R = {(1, 2)}, then R is

(*a*) reflexive (*b*) transitive

(*c*) symmetric (*d*) None of these

Sol. Given that: R = {(1, 2)}

$a \not R a$, so it is not reflexive.

$a R b$ but $b \not R a$, so it is not symmetric.

$a R b$ and $b R c \Rightarrow a R c$ which is true.

So, R is transitive.

Hence, correct answer is (*b*).

Q32. Let us define a relation R in R as $a R b$ if $a \geq b$. Then R is

(*a*) an equivalence relation

(*b*) reflexive, transitive but not symmetric

(*c*) symmetric, transitive but not reflexive

(*d*) neither transitive nor reflexive but symmetric.

Sol. Here, $a R b$ if $a \geq b$

$\Rightarrow a R a \Rightarrow a \geq a$ which is true, so it is reflexive.

Let $a R b \Rightarrow a \geq b$, but $b \not\geq a$, so $b \not R a$

R is not symmetric.

Now, $a \geq b, b \geq c \Rightarrow a \geq c$ which is true.

So, R is transitive.

Hence, correct answer is (*b*).

Q33. Let A = {1, 2, 3} and consider the relation R = {(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)}, then R is

(*a*) reflexive but not symmetric

(*b*) reflexive but not transitive

(*c*) symmetric and transitive

(*d*) neither symmetric nor transitive.

Sol. Given that: R = {(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)}

Here, 1 R 1, 2 R 2 and 3 R 3, so R is reflexive.
1 R 2 but $2 \not R 1$ or 2 R 3 but $3 \not R 2$, so, R is not symmetric.
1 R 1 and 1 R 2 $\Rightarrow$ 1 R 3, so, R is transitive.
Hence, the correct answer is (*a*).

Q34. The identity element for the binary operation $*$ defined on $Q \sim \{0\}$ as $a * b = \frac{ab}{2}$ $\forall\, a, b \in Q \sim \{0\}$ is

(*a*) 1 (*b*) 0 (*c*) 2 (*d*) None of these

Sol. Given that: $a * b = \frac{ab}{2}$ $\forall\, a, b \in Q - \{0\}$

Let e be the identity element

$\therefore$ $$a * e = \frac{ae}{2} = a \Rightarrow e = 2$$

Hence, the correct answer is (*c*).

Q35. If the set A contains 5 elements and set B contains 6 elements, then the number of one-one and onto mapping from A to B is

(*a*) 720 (*b*) 120 (*c*) 0 (*d*) None of these

Sol. If A and B sets have m and n elements respectively, then the number of one-one and onto mapping from A to B is

$n!$ if $m = n$

and 0 if $m \neq n$

Here, $m = 5$ and $n = 6$

$5 \neq 6$

So, number of mapping = 0

Hence, the correct answer is (*c*).

Q36. Let A = $\{1, 2, 3, ..., n\}$ and B = $\{a, b\}$. Then the number of surjections from A to B is

(*a*) ${}^nP_2$ (*b*) $2^n - 2$ (*c*) $2^n - 1$ (*d*) None of these

Sol. Here, A = $\{1, 2, 3, ..., n\}$ and B = $\{a, b\}$

Let m be the number of elements of set A

and n be the number of elements of set B

$\therefore$ Number of surjections from A to B is

${}^nC_m \times m!$ as $n \geq m$

Here, $m = 2$ (given)

$\therefore$ Number of surjections from A to B = ${}^nC_2 \times 2!$

$$= \frac{n!}{2!(n-2)!} \times 2! = \frac{n(n-1)(n-2)!}{2!(n-2)!} \times 2 = n(n-1) = n^2 - n$$

Hence, the correct answer is (*d*).

Q37. Let $f : R \to R$ be defined by $f(x) = \frac{1}{x}$, $\forall\, x \in R$ then f is

(*a*) one-one (*b*) onto

(c) bijective (d) f is not defined

Sol. Given that $f(x) = \frac{1}{x}$

Put $x = 0$ $\therefore$ $f(x) = \frac{1}{0} = \infty$

So, $f(x)$ is not defined.

Hence, the correct answer is (d).

Q38. Let $f : R \to R$ be defined by $f(x) = 3x^2 - 5$ and $g : R \to R$ by $g(x) = \frac{x}{x^2+1}$, then gof is

(a) $\frac{3x^2-5}{9x^4-30x^2+26}$ (b) $\frac{3x^2-5}{9x^4-6x^2+26}$

(c) $\frac{3x^2}{x^4+2x^2-4}$ (d) $\frac{3x^2}{9x^4+30x^2-2}$

Sol. Here, $f(x) = 3x^2 - 5$ and $g(x) = \frac{x}{x^2+1}$

$\therefore$ $gof = gof(x) = g[3x^2 - 5]$

$$= \frac{3x^2-5}{(3x^2-5)^2+1} = \frac{3x^2-5}{9x^4+25-30x^2+1}$$

$\therefore$ $gof = \frac{3x^2-5}{9x^4-30x^2+26}$

Hence, the correct answer is (a).

Q39. Which of the following functions from Z to Z are bijections?

(a) $f(x) = x^3$ (b) $f(x) = x + 2$

(c) $f(x) = 2x + 1$ (d) $f(x) = x^2 + 1$

Sol. Given that $f : Z \to Z$

Let $x_1, x_2 \in f(x) \Rightarrow f(x_1) = x_1 + 2, f(x_2) = x_2 + 2$

$f(x_1) = f(x_2) \Rightarrow x_1 + 2 = x_2 + 2 \Rightarrow x_1 = x_2$

So, $f(x)$ is one-one function.

Now, let $y = x + 2$ $\therefore x = y - 2 \in Z$ $\forall y \in Z$

So, $f(x)$ is onto function.

$\therefore f(x)$ is bijective function.

Hence, the correct answer is (b).

Q40. Let $f : R \to R$ be the functions defined by $f(x) = x^3 + 5$. Then $f^{-1}(x)$ is

(a) $(x+5)^{1/3}$ (b) $(x-5)^{1/3}$ (c) $(5-x)^{1/3}$ (d) $5-x$

Sol. Given that $f(x) = x^3 + 5$

Let $y = x^3 + 5 \Rightarrow x^3 = y - 5$

$\therefore$ $x = (y-5)^{1/3} \Rightarrow f^{-1}(x) = (x-5)^{1/3}$

Hence, the correct answer is (b).

Q41. Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be the bijective functions. Then $(gof)^{-1}$ is

(*a*) $f^{-1}og^{-1}$ (*b*) $f og$ (*c*) $g^{-1}of^{-1}$ (*d*) gof

Sol. Here, $f : A \rightarrow B$ and $g : B \rightarrow C$

$\therefore \quad (gof)^{-1} = f^{-1}og^{-1}$

Hence, the correct answer is (*a*).

Q42. Let $f : R - \left\{\frac{3}{5}\right\} \rightarrow R$ be defined by $f(x) = \frac{3x+2}{5x-3}$, then

(*a*) $f^{-1}(x) = f(x)$ (*b*) $f^{-1}(x) = -f(x)$

(*c*) $(fof)x = -x$ (*d*) $f^{-1}(x) = \frac{1}{19}f(x)$

Sol. Given that $f(x) = \frac{3x+2}{5x-3} \ \forall \ x \neq \frac{3}{5}$

Let $\quad y = \frac{3x+2}{5x-3}$

$\Rightarrow \quad y(5x-3) = 3x+2$

$\Rightarrow \quad 5xy - 3y = 3x+2$

$\Rightarrow \quad 5xy - 3x = 3y+2$

$\Rightarrow \quad x(5y-3) = 3y+2$

$\Rightarrow \quad x = \frac{3y+2}{5y-3}$

$\Rightarrow \quad f^{-1}(x) = \frac{3x+2}{5x-3}$

$\Rightarrow \quad f^{-1}(x) = f(x)$

Hence, the correct answer is (*a*).

Q43. Let $f : [0, 1] \rightarrow [0, 1]$ be defined by $f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 1-x, & \text{if } x \text{ is irrational} \end{cases}$.

Then $(fof)x$ is

(*a*) constant (*b*) $1 + x$

(*c*) x (*d*) None of these

Sol. Given that $f : [0, 1] \rightarrow [0, 1]$

$\therefore \quad f = f^{-1}$

So, $\quad (fof)x = x$ (identity element)

Hence, correct answer is (*c*).

Q44. Let $f : [2, \infty) \rightarrow R$ be the function defined by $f(x) = x^2 - 4x + 5$, then the range of f is

(*a*) R (*b*) $[1, \infty)$ (*c*) $[4, \infty)$ (*d*) $[5, \infty)$

Sol. Given that $\quad f(x) = x^2 - 4x + 5$

Let $\quad y = x^2 - 4x + 5$

$\Rightarrow \quad x^2 - 4x + 5 - y = 0$

$$\Rightarrow \quad x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \times 1 \times (5-y)}}{2 \times 1}$$

$$= \frac{4 \pm \sqrt{16 - 20 + 4y}}{2}$$

$$= \frac{4 \pm \sqrt{4y-4}}{2} = \frac{4 \pm 2\sqrt{y-1}}{2} = 2 \pm \sqrt{y-1}$$

$\therefore$ For real value of x, $y - 1 \geq 0 \Rightarrow y \geq 1$.

So, the range is $[1, \infty)$.

Hence, the correct answer is (*b*).

Q45. Let $f : N \to R$ be the function defined by $f(x) = \frac{2x-1}{2}$ and $g : Q \to R$ be another function defined by $g(x) = x + 2$ then, $gof\left(\frac{3}{2}\right)$ is

(*a*) 1 (*b*) -1 (*c*) $\frac{7}{2}$ (*d*) None of these

Sol. Here, $\quad f(x) = \frac{2x-1}{2}$ and $g(x) = x + 2$

$$\therefore \quad gof(x) = g[(f(x)]$$

$$= f(x) + 2$$

$$= \frac{2x-1}{2} + 2 = \frac{2x+3}{2}$$

$$gof\left(\frac{3}{2}\right) = \frac{2 \times \frac{3}{2} + 3}{2} = 3$$

Hence, the correct answer is (*d*).

Q46. Let $f : R \to R$ be defined by $f(x) = \begin{cases} 2x & : x > 3 \\ x^2 & : 1 < x \leq 3 \\ 3x & : x \leq 1 \end{cases}$

then $f(-1) + f(2) + f(4)$ is

(*a*) 9 (*b*) 14 (*c*) 5 (*d*) None of these

Sol. Given that:

$$f(x) = \begin{cases} 2x & : x > 3 \\ x^2 & : 1 < x \leq 3 \\ 3x & : x \leq 1 \end{cases}$$

$\therefore f(-1) + f(2) + f(4) = 3(-1) + (2)^2 + 2(4) = -3 + 4 + 8 = 9$

Hence, the correct answer is (*a*).

Q47. If $f : R \to R$ be given by $f(x) = \tan x$, then $f^{-1}(1)$ is

(a) $\frac{\pi}{4}$ (b) $\left\{n\pi + \frac{\pi}{4} : n \in Z\right\}$

(c) does not exist (d) None of these

Sol. Given that $f(x) = \tan x$

Let $f(x) = y = \tan x \Rightarrow x = \tan^{-1} y$

$\Rightarrow \quad f^{-1}(x) = \tan^{-1}(x)$

$\Rightarrow \quad f^{-1}(1) = \tan^{-1}(1)$

$\Rightarrow \quad f^{-1}(1) = \tan^{-1}\left[\tan\left(\frac{\pi}{4}\right)\right] = \frac{\pi}{4}$

Hence, the correct answer is (a).

Fill in the Blanks in Each of the Exercises 48 to 52.

Q48. Let the relation R be defined in N by $a \, R \, b$ if $2a + 3b = 30$. Then R = ...

Sol. Given that $a \, R \, b : 2a + 3b = 30$

$\Rightarrow \quad 3b = 30 - 2a$

$\Rightarrow \quad b = \frac{30 - 2a}{3}$

for $a = 3, b = 8$

$a = 6, b = 6$

$a = 9, b = 4$

$a = 12, b = 2$

Hence, $R = \{(3, 8), (6, 6), (9, 4), (12, 2)\}$

Q49. Let the relation R be defined on the set $A = \{1, 2, 3, 4, 5\}$ by $R = \{(a, b) : |a^2 - b^2| < 8\}$. Then R is given by ....

Sol. Given that $A = \{1, 2, 3, 4, 5\}$ and $R = \{(a, b) : |a^2 - b^2| < 8\}$

So, clearly, $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 2), (4, 3)$
$(3, 4), (4, 4), (5, 5)\}$

Q50. Let $f = \{(1, 2), (3, 5), (4, 1)\}$ and $g = \{(2, 3), (5, 1), (1, 3)\}$. Then gof = ... and fog = ...

Sol. Here, $f = \{(1, 2), (3, 5), (4, 1)\}$ and $g = \{(2, 3), (5, 1), (1, 3)\}$

$gof(1) = g[f(1)] = g(2) = 3$

$gof(3) = g[f(3)] = g(5) = 1$

$gof(4) = g[f(4)] = g(1) = 3$

$\therefore \quad gof = \{(1, 3), (3, 1), (4, 3)\}$

$fog(2) = f[g(2)] = f(3) = 5$

$$fog(5) = f[g(5)] = f(1) = 2$$
$$fog(1) = f[g(1)] = f(3) = 5$$
$$\therefore \quad fog = \{(2, 5), (5, 2), (1, 5)\}$$

Q51. Let f : R → R be defined by $f(x) = \dfrac{x}{\sqrt{1+x^2}}$, then $(fofof)(x) = \ldots\ldots\ldots\ldots\ldots\ldots$

Sol. Here, $f(x) = \dfrac{x}{\sqrt{1+x^2}} \quad \forall\, x \in R$

$$fofof(x) = fof[f(x)] = f[f\{f(x)\}]$$

$$= f\left[f\left(\frac{x}{\sqrt{1+x^2}}\right)\right] = f\left[\frac{\frac{x}{\sqrt{1+x^2}}}{\sqrt{1+\frac{x^2}{1+x^2}}}\right]$$

$$= f\left[\frac{\frac{x}{\sqrt{1+x^2}}}{\frac{\sqrt{1+x^2+x^2}}{\sqrt{1+x^2}}}\right] = f\left[\frac{x}{\sqrt{1+2x^2}}\right]$$

$$= \left[\frac{\frac{x}{\sqrt{1+2x^2}}}{\sqrt{1+\frac{x^2}{1+2x^2}}}\right] = \left[\frac{\frac{x}{\sqrt{1+2x^2}}}{\frac{\sqrt{1+2x^2+x^2}}{\sqrt{1+2x^2}}}\right] = \frac{x}{\sqrt{1+3x^2}}$$

Hence, $fofof(x) = \dfrac{x}{\sqrt{3x^2+1}}$

Q52. If $f(x) = [4-(x-7)^3]$, then $f^{-1}(x) = \ldots\ldots\ldots\ldots$

Sol. Given that, $\quad f(x) = [4-(x-7)^3]$

Let $\quad y = [4-(x-7)^3]$

$\Rightarrow \quad (x-7)^3 = 4-y$

$\Rightarrow \quad x-7 = (4-y)^{1/3} \Rightarrow x = 7+(4-y)^{1/3}$

Hence, $\quad f^{-1}(x) = 7+(4-x)^{1/3}$

State True or False for the Statements in each of the Exercises 53 to 62.

Q53. Let R = {(3, 1), (1, 3), (3, 3)} be a relation defined on the set A = {1, 2, 3}. Then R is symmetric, transitive but not reflexive.

Sol. Here, R = {(3, 1), (1, 3), (3, 3)}
$(3, 3) \in R$, so R is reflexive.
$(3, 1) \in R$ and $(1, 3) \in R$, so R is symmetric.
Now, $(3, 1) \in R$ and $(1, 3) \in R$ but $(1, 1) \notin R$
So, R is not transitive.
Hence, the statement is 'False'.

Q54. Let f : R → R be the function defined by
$f(x) = \sin(3x + 2)\ \forall\ x \in R$, then f is invertible.

Sol. Given that: $f(x) = \sin(3x + 2)\ \forall\ x \in R$,
$f(x)$ is not one-one.
Hence, the statement is 'False'.

Q55. Every relation which is symmetric and transitive is also reflexive.

Sol. Let R be any relation defined on A = {1, 2, 3}
R = {(1, 2), (2, 1), (2, 3), (1, 3)}
Here, $(1, 2) \in R$ and $(2, 1) \in R$, so R is symmetric.
$(1, 2) \in R$, $(2, 3) \in R \Rightarrow (1, 3) \in R$, so R is transitive.
But $(1, 1) \notin R$, $(2, 2) \notin R$ and $(3, 3) \notin R$.
Hence, the statement is 'False'.

Q56. An integer m is said to be related to another integer n if m is an integral multiple of n. This relation in Z is reflexive, symmetric and transitive.

Sol. Here, $m = kn$ (where k is an integer)
If $k = 1$ $m = n$, so z is reflexive.
Clearly z is not symmetric but z is transitive.
Hence, the statement is 'False'.

Q57. Let A = {0, 1} and N be the set of natural numbers then the mapping $f : N \to A$ defined by $f(2n - 1) = 0$, $f(2n) = 1$, $\forall\ n \in N$ is onto.

Sol. Given that A = [0, 1]
$f(2n - 1) = 0$ and $f(2n) = 1\ \forall\ n \in N$
So, $f : N \to A$ is a onto function.
Hence, the statement is 'True'.

Q58. The relation R on the set A = {1, 2, 3} defined as
R = {(1, 1), (1, 2), (2, 1), (3, 3)} is reflexive, symmetric and transitive.

Sol. Here, R = {(1, 1), (1, 2), (2, 1), (3, 3)}
Here, $(1, 1) \in R$, so R is Reflexive.
$(1, 2) \in R$ and $(2, 1) \in R$, so R is Symmetric.

$(1, 2) \in R$ but $(2, 3) \notin R$

So, R is not transitive.

Hence, the statement is 'False'.

Q59. The composition of functions is commutative.

Sol. Let $f(x) = x^2$ and $g(x) = 2x + 3$

$$fog(x) = f[g(x)] = (2x + 3)^2 = 4x^2 + 9 + 12x$$

$$gof(x) = g[f(x)] = 2x^2 + 3$$

So, $$fog(x) \neq gof(x)$$

Hence, the statement is 'False'.

Q60. The composition of functions is associative.

Sol. Let $f(x) = 2x$, $g(x) = x - 1$ and $h(x) = 2x + 3$

$$fo\{goh(x)\} = fo\{g(2x + 3)\}$$

$$= f(2x + 3 - 1) = f(2x + 2) = 2(2x + 2) = 4x + 4.$$

and $$(fog)oh(x) = (fog)\{h(x)\}$$

$$= fog(2x + 3)$$

$$= f(2x + 3 - 1) = f(2x + 2) = 2(2x + 2) = 4x + 4$$

So, $$fo\{goh(x)\} = \{(fog)oh(x)\} = 4x + 4$$

Hence, the statement is 'True'.

Q61. Every function is invertible.

Sol. Only bijective functions are invertible.

Hence, the statement is 'False'.

Q62. A binary operation on a set has always the identity element.

Sol. '+' is a binary operation on the set N but it has no identity element.

Hence, the statement is 'False'.

❑❑❑

2 Inverse Trigonometric Functions

2.3 EXERCISE

SHORT ANSWER TYPE QUESTIONS

Q1. Find the value of $\tan^{-1}\left(\tan\frac{5\pi}{6}\right)+\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$.

Sol. We know that $\frac{5\pi}{6}\notin\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ and $\frac{13\pi}{6}\notin[0,\pi]$

$$\therefore\ \tan^{-1}\left(\tan\frac{5\pi}{6}\right)+\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$$

$$=\tan^{-1}\left[\tan\left(\pi-\frac{\pi}{6}\right)\right]+\cos^{-1}\left[\cos\left(2\pi+\frac{\pi}{6}\right)\right]$$

$$=\tan^{-1}\left[\tan\left(-\frac{\pi}{6}\right)\right]+\cos^{-1}\left(\cos\frac{\pi}{6}\right)$$

$$=\tan^{-1}\left(-\tan\frac{\pi}{6}\right)+\cos^{-1}\left(\cos\frac{\pi}{6}\right)$$

$$=-\tan^{-1}\left(\tan\frac{\pi}{6}\right)+\cos^{-1}\left(\cos\frac{\pi}{6}\right)\qquad[\because \tan^{-1}(-x)=-\tan^{-1}x]$$

$$=-\frac{\pi}{6}+\frac{\pi}{6}=0$$

Hence, $\tan^{-1}\left(\tan\frac{5\pi}{6}\right)+\cos^{-1}\left(\cos\frac{13\pi}{6}\right)=0$

Q2. Evaluate: $\cos\left[\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)+\frac{\pi}{6}\right]$

Sol. $\cos\left[\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)+\frac{\pi}{6}\right]$

$$=\cos\left[\pi-\cos^{-1}\frac{\sqrt{3}}{2}+\frac{\pi}{6}\right]\qquad[\because \cos^{-1}(-x)=\pi-\cos^{-1}x]$$

$$=\cos\left[\pi-\cancel{\frac{\pi}{6}}+\cancel{\frac{\pi}{6}}\right]=\cos\pi=-1$$

Hence, $\cos\left[\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)+\frac{\pi}{6}\right]=-1.$

Q3. Prove that: $\cot\left(\frac{\pi}{4}-2\cot^{-1}3\right)=7$.

Sol. L.H.S. $\cot\left(\frac{\pi}{4}-2\cot^{-1}3\right)$

$$= \cot\left[\tan^{-1}(1)-2\tan^{-1}\frac{1}{3}\right] \qquad \left[\because \cot^{-1}x=\tan^{-1}\frac{1}{x}\right]$$

$$= \cot\left[\tan^{-1}(1)-\tan^{-1}\frac{2\times\frac{1}{3}}{1-\left(\frac{1}{3}\right)^2}\right] \qquad \left[\because 2\tan^{-1}x=\tan^{-1}\frac{2x}{1-x^2}\right]$$

$$= \cot\left[\tan^{-1}(1)-\tan^{-1}\frac{\frac{2}{3}}{\frac{8}{9}}\right]$$

$$= \cot\left[\tan^{-1}(1)-\tan^{-1}\frac{3}{4}\right]$$

$$= \cot\left[\tan^{-1}\left(\frac{1-\frac{3}{4}}{1+1\times\frac{3}{4}}\right)\right] = \cot\left[\tan^{-1}\left(\frac{\frac{1}{4}}{\frac{7}{4}}\right)\right]$$

$$= \cot\left[\tan^{-1}\frac{1}{7}\right] \qquad \left[\because \tan^{-1}\frac{1}{x}=\cot^{-1}x\right]$$

$= \cot[\cot^{-1}(7)]=7$ R.H.S.

Hence proved.

Q4. Find the value of $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)+\cot^{-1}\left(\frac{1}{\sqrt{3}}\right)+\tan^{-1}\left[\sin\left(\frac{-\pi}{2}\right)\right]$

Sol. $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)+\cot^{-1}\left(\frac{1}{\sqrt{3}}\right)+\tan^{-1}\left[\sin\left(\frac{-\pi}{2}\right)\right]$

$$= -\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) + \tan^{-1}(\sqrt{3}) + \tan^{-1}(-1) \qquad \left[\begin{array}{l} \because \tan^{-1}(-x) = -\tan^{-1} x \\ \tan^{-1} x = \cot^{-1}\left(\frac{1}{x}\right) \\ \sin\left(\frac{-\pi}{2}\right) = -1 \end{array}\right]$$

$$= -\frac{\pi}{6} + \frac{\pi}{3} - \frac{\pi}{4} = \frac{-\pi}{12} \qquad \left[\because \tan^{-1}(-1) = \frac{-\pi}{4}\right]$$

Hence, $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) + \cot^{-1}\left(\frac{1}{\sqrt{3}}\right) + \tan^{-1}\left[\sin\left(\frac{-\pi}{2}\right)\right] = \frac{-\pi}{12}$

Q5. Find the value of $\tan^{-1}\left(\tan\frac{2\pi}{3}\right)$

Sol. We know that $\frac{2\pi}{3} \notin \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

$$\therefore \quad \tan^{-1}\left(\tan\frac{2\pi}{3}\right) = \tan^{-1}\left[\tan\left(\pi - \frac{\pi}{3}\right)\right] = \tan^{-1}\left(-\tan\frac{\pi}{3}\right)$$

$$= -\tan^{-1}\left(\tan\frac{\pi}{3}\right) \quad [\because \tan^{-1}(-x) = -\tan^{-1} x]$$

$$= -\frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

Hence, $\tan^{-1}\left(\tan\frac{2\pi}{3}\right) = \frac{-\pi}{3}$.

Q6. Show that: $2\tan^{-1}(-3) = \frac{-\pi}{2} + \tan^{-1}\left(\frac{-4}{3}\right)$

Sol. L.H.S. $\quad 2\tan^{-1}(-3) = -2\tan^{-1}(3)$

$$= -\cos^{-1}\left[\frac{1-(3)^2}{1+(3)^2}\right] \qquad \left[\because 2\tan^{-1} x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right]$$

$$= -\cos^{-1}\left(\frac{1-9}{1+9}\right) = -\cos^{-1}\left(\frac{-8}{10}\right)$$

$$= -\cos^{-1}\left(\frac{-4}{5}\right) = -\left[\pi - \cos^{-1}\left(\frac{4}{5}\right)\right] = -\pi + \cos^{-1}\frac{4}{5}$$

$$= -\pi + \tan^{-1}\left(\frac{3}{4}\right) \qquad \left[\because \cos^{-1}\frac{4}{5} = \tan^{-1}\frac{3}{4}\right]$$

$$= -\pi + \frac{\pi}{2} - \cot^{-1}\left(\frac{3}{4}\right) \qquad \left[\tan^{-1}x = \frac{\pi}{2} - \cot^{-1}x\right]$$

$$= \frac{-\pi}{2} - \cot^{-1}\left(\frac{3}{4}\right)$$

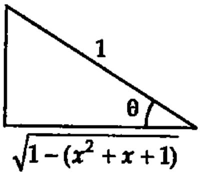
$$= \frac{-\pi}{2} - \tan^{-1}\left(\frac{4}{3}\right) \qquad \left[\because \tan^{-1}x = \cot^{-1}\frac{1}{x}\right]$$

$$= \frac{-\pi}{2} + \tan^{-1}\left(-\frac{4}{3}\right) \text{ R.H.S.}$$

Hence proved.

Q7. Find the real solutions of the equation

$$\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2+x+1} = \frac{\pi}{2}$$



Sol. $\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2+x+1} = \frac{\pi}{2}$

Let $\theta = \sin^{-1}\sqrt{x^2+x+1}$

$\therefore \quad \sin\theta = \sqrt{x^2+x+1}$

$$\Rightarrow \tan\theta = \frac{\sqrt{x^2+x+1}}{\sqrt{-x^2-x}} \quad \Rightarrow \theta = \tan^{-1}\left(\frac{\sqrt{x^2+x+1}}{\sqrt{-x^2-x}}\right)$$

$$\Rightarrow \sin^{-1}\sqrt{x^2+x+1} = \tan^{-1}\left(\sqrt{\frac{x^2+x+1}{-x^2-x}}\right)$$

$$\text{So, } \tan^{-1}\sqrt{x(x+1)} + \tan^{-1}\left(\sqrt{\frac{x^2+x+1}{-x^2-x}}\right) = \frac{\pi}{2}$$

$$\Rightarrow \tan^{-1}\left[\frac{\sqrt{x(x+1)} + \sqrt{\frac{x^2+x+1}{-x(x+1)}}}{1 - \sqrt{x(x+1)} \times \sqrt{\frac{x^2+x+1}{-x(x+1)}}}\right] = \frac{\pi}{2}$$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{x(x+1) + \sqrt{-(x^2+x+1)}}{\sqrt{x(x+1)}}}{1 - \sqrt{-(x^2+x+1)}}\right] = \frac{\pi}{2}$$

$$\Rightarrow \frac{x^2+x-\sqrt{-(x^2+x+1)}}{\left[1-\sqrt{-(x^2+x+1)}\right]\sqrt{x^2+x}} = \tan\frac{\pi}{2} = \frac{1}{0}$$

$$\Rightarrow \left[1-\sqrt{-(x^2+x+1)}\right]\sqrt{x^2+x} = 0$$

$$\Rightarrow \left[1-\sqrt{-(x^2+x+1)}\right] = 0 \quad \text{or} \quad \sqrt{x^2+x} = 0$$

Here, $\sqrt{-(x^2+x+1)} \notin R$

$\therefore \quad \sqrt{x^2+x} = 0$

$\Rightarrow \quad x^2+x=0 \quad \Rightarrow \quad x(x+1)=0$

$\Rightarrow \quad x=0 \quad \text{or} \quad x+1=0 \quad \Rightarrow \quad x=0 \quad \text{or} \quad x=-1$

Hence the real solutions are $x=0$ and $x=-1$.

Alternate Method:

$$\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2+x+1} = \frac{\pi}{2}$$

$$\Rightarrow \quad \tan^{-1}\sqrt{x^2+x} = \frac{\pi}{2} - \sin^{-1}\sqrt{x^2+x+1}$$

$$\Rightarrow \quad \tan^{-1}\sqrt{x^2+x} = \cos^{-1}\sqrt{x^2+x+1} \quad \left[\because \sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}\right]$$

$$\Rightarrow \cos^{-1}\left[\frac{1}{\sqrt{1+x^2+x}}\right] = \cos^{-1}\sqrt{x^2+x+1}$$

$$\left[\because \tan^{-1}x = \cos^{-1}\frac{1}{\sqrt{1+x^2}}\right]$$

$$\Rightarrow \quad \frac{1}{\sqrt{x^2+x+1}} = \sqrt{x^2+x+1}$$

$\Rightarrow \quad x^2+x+1=1 \quad \Rightarrow x^2+x=0$

$\Rightarrow \quad x(x+1)=0 \quad \Rightarrow x=0 \quad \text{or} \quad x+1=0$

$\therefore \quad x=0, x=-1$

Q8. Find the value of the expression

$$\sin\left(2\tan^{-1}\frac{1}{3}\right)+\cos\left(\tan^{-1}2\sqrt{2}\right)$$

Sol. $\sin\left(2\tan^{-1}\frac{1}{3}\right)+\cos\left(\tan^{-1}2\sqrt{2}\right)$

$$\Rightarrow \sin\left[\tan^{-1}\left(\frac{2\times\frac{1}{3}}{1-\left(\frac{1}{3}\right)^2}\right)\right]+\cos\left[\cos^{-1}\frac{1}{\sqrt{1+(2\sqrt{2})^2}}\right]$$

$$\left[\because \tan^{-1}x=\cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)\right]$$

$$\Rightarrow \sin\left[\tan^{-1}\left(\frac{\frac{2}{3}}{1-\frac{1}{9}}\right)\right]+\cos\left[\cos^{-1}\left(\frac{1}{3}\right)\right]$$

$$\Rightarrow \sin\left[\tan^{-1}\left(\frac{3}{4}\right)\right]+\frac{1}{3} \quad \Rightarrow \quad \sin\left[\sin^{-1}\left(\frac{3}{5}\right)\right]+\frac{1}{3}$$

$$\Rightarrow \frac{3}{5}+\frac{1}{3} \quad \Rightarrow \frac{14}{15} \qquad \left[\because \tan^{-1}x=\sin^{-1}\frac{x}{\sqrt{1+x^2}}\right]$$

Hence, $\sin\left(2\tan^{-1}\frac{1}{3}\right)+\cos\left(\tan^{-1}2\sqrt{2}\right)=\frac{14}{15}$.

Q9. If $2\tan^{-1}(\cos\theta)=\tan^{-1}(2\operatorname{cosec}\theta)$, then show that $\theta=\frac{\pi}{4}$

Sol. $2\tan^{-1}(\cos\theta)=\tan^{-1}(2\operatorname{cosec}\theta)$

$$\Rightarrow \tan^{-1}\left(\frac{2\cos\theta}{1-\cos^2\theta}\right)=\tan^{-1}(2\operatorname{cosec}\theta)$$

$$\left[\because 2\tan^{-1}x=\tan^{-1}\frac{2x}{1-x^2}\right]$$

$$\Rightarrow \frac{2\cos\theta}{1-\cos^2\theta}=2\operatorname{cosec}\theta \Rightarrow \frac{2\cos\theta}{\sin^2\theta}=\frac{2}{\sin\theta}$$

$\Rightarrow \cos\theta\sin\theta=\sin^2\theta$

$\Rightarrow \cos\theta\sin\theta-\sin^2\theta=0 \Rightarrow \sin\theta(\cos\theta-\sin\theta)=0$

$\Rightarrow \sin\theta=0$ or $\cos\theta-\sin\theta=0$

$\Rightarrow \sin\theta=0$ or $1-\tan\theta=0$

$\Rightarrow \theta=0$ or $\tan\theta=1$

$\Rightarrow \theta=0°$ or $\theta=\frac{\pi}{4}$ **Hence proved.**

Q10. Show that: $\cos\left(2\tan^{-1}\frac{1}{7}\right)=\sin\left(4\tan^{-1}\frac{1}{3}\right)$

Sol. L.H.S. $\cos\left(2\tan^{-1}\frac{1}{7}\right)$

$$= \cos\left[\cos^{-1}\frac{1-\frac{1}{49}}{1+\frac{1}{49}}\right] \qquad \left[\because 2\tan^{-1}x = \cos^{-1}\frac{1-x^2}{1+x^2}\right]$$

$$= \cos\left[\cos^{-1}\frac{48}{50}\right] = \cos\left[\cos^{-1}\frac{24}{25}\right] = \frac{24}{25}$$

R.H.S. $\sin\left[4\tan^{-1}\frac{1}{3}\right]$

$$= \sin\left[2\tan^{-1}\left(\frac{2\times\frac{1}{3}}{1-\frac{1}{9}}\right)\right] \qquad \left[\because 2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}\right]$$

$$= \sin\left[2\tan^{-1}\left(\frac{\frac{2}{3}}{\frac{8}{9}}\right)\right] = \sin\left[2\tan^{-1}\frac{3}{4}\right]$$

$$= \sin\left[\sin^{-1}\frac{2\times\frac{3}{4}}{1+\frac{9}{16}}\right] \qquad \left[\because 2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2}\right]$$

$$= \sin\left[\sin^{-1}\frac{24}{25}\right] \Rightarrow \frac{24}{25}$$

L.H.S. = R.H.S. **Hence proved.**

Q11. Solve the following equation: $\cos(\tan^{-1}x) = \sin\left(\cot^{-1}\frac{3}{4}\right)$

Sol. Given that $\cos(\tan^{-1}x) = \sin\left(\cot^{-1}\frac{3}{4}\right)$

$$\Rightarrow \cos\left[\cos^{-1}\frac{1}{\sqrt{1+x^2}}\right] = \sin\left[\sin^{-1}\frac{4}{5}\right]$$

$$\left[\begin{aligned} \because \tan^{-1}x &= \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) \\ \cot^{-1}x &= \sin^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) \end{aligned}\right]$$

$$\Rightarrow \qquad \frac{1}{\sqrt{1+x^2}} = \frac{4}{5}$$

Squaring both sides we get,

$$\frac{1}{1+x^2} = \frac{16}{25} \Rightarrow 1+x^2 = \frac{25}{16}$$

$$\Rightarrow \qquad x^2 = \frac{25}{16} - 1 = \frac{9}{16} \Rightarrow x = \pm\frac{3}{4}$$

Hence, $\qquad x = \frac{-3}{4}, \frac{3}{4}.$

LONG ANSWER TYPE QUESTIONS

Q12. Prove that: $\tan^{-1}\left[\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right] = \frac{\pi}{4} + \frac{1}{2}\cos^{-1}x^2$

Sol. L.H.S. $\tan^{-1}\left[\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right]$

Put $x^2 = \cos\theta \quad \therefore \theta = \cos^{-1}x^2$

$$\Rightarrow \tan^{-1}\left[\frac{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}\right]$$

$$\Rightarrow \tan^{-1}\left[\frac{\sqrt{2\cos^2\theta/2}+\sqrt{2\sin^2\theta/2}}{\sqrt{2\cos^2\theta/2}-\sqrt{2\sin^2\theta/2}}\right] \quad \left[\begin{array}{l}\because 1+\cos\theta = 2\cos^2\theta/2 \\ \quad 1-\cos\theta = 2\sin^2\theta/2\end{array}\right]$$

$$\Rightarrow \tan^{-1}\left[\frac{\cos\theta/2+\sin\theta/2}{\cos\theta/2-\sin\theta/2}\right]$$

$$\Rightarrow \tan^{-1}\left[\frac{1+\tan\theta/2}{1-\tan\theta/2}\right] \quad \text{[Dividing the Nr. and Den. by } \cos\theta/2]$$

$$\Rightarrow \tan\left[\tan\left(\frac{\pi}{4} \quad \frac{\theta}{2}\right)\right] \qquad \left[\because \frac{1+\tan\theta}{1-\tan\theta} = \tan\left(\frac{\pi}{4}+\theta\right)\right]$$

$$\Rightarrow \frac{\pi}{4}+\frac{\theta}{2} \Rightarrow \frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2 \text{ R.H.S.} \qquad [\text{Putting } \theta = \cos^{-1}x^2]$$

Hence proved.

Q13. Find the simplified form of $\cos^{-1}\left(\frac{3}{5}\cos x + \frac{4}{5}\sin x\right)$,

where $x \in \left[\frac{-3\pi}{4}, \frac{\pi}{4}\right]$

Sol. Given that $\cos^{-1}\left(\frac{3}{5}\cos x + \frac{4}{5}\sin x\right)$

Put $\quad \frac{3}{5} = \cos y$

$\therefore \quad \sqrt{1-\cos^2 y} = \sin y \Rightarrow \sqrt{1-\frac{9}{25}} = \sin y \Rightarrow \frac{4}{5} = \sin y$

$\therefore \cos^{-1}\left[\frac{3}{5}\cos x + \frac{4}{5}\sin x\right] = \cos^{-1}[\cos y \cos x + \sin y \sin x]$

$= \cos^{-1}[\cos(y-x)] = y - x$

$= \tan^{-1}\frac{4}{3} - x \qquad \left[\tan y = \frac{\sin y}{\cos y} = \frac{4}{3}\right]$

Q14. Prove that: $\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \sin^{-1}\frac{77}{85}$

Sol. L.H.S. $\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5}$

Using $\sin^{-1}x + \sin^{-1}y = \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$

$\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \sin^{-1}\left[\frac{8}{17}\cdot\sqrt{1-\left(\frac{3}{5}\right)^2} + \frac{3}{5}\cdot\sqrt{1-\left(\frac{8}{17}\right)^2}\right]$

$= \sin^{-1}\left[\frac{8}{17}\cdot\sqrt{1-\frac{9}{25}} + \frac{3}{5}\cdot\sqrt{1-\frac{64}{289}}\right]$

$= \sin^{-1}\left[\frac{8}{17}\cdot\sqrt{\frac{16}{25}} + \frac{3}{5}\cdot\sqrt{\frac{225}{289}}\right]$

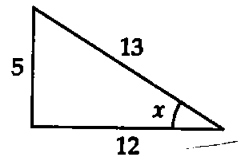
$= \sin^{-1}\left[\frac{8}{17}\cdot\frac{4}{5} + \frac{3}{5}\cdot\frac{15}{17}\right] = \sin^{-1}\left[\frac{32}{85} + \frac{45}{85}\right]$

$= \sin^{-1}\frac{77}{85}$ R.H.S. **Hence proved.**

Q15. Show that: $\sin^{-1}\dfrac{5}{13}+\cos^{-1}\dfrac{3}{5}=\tan^{-1}\dfrac{63}{16}$

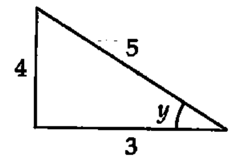
Sol. Let $\sin^{-1}\dfrac{5}{13}=x \quad\Rightarrow\quad \sin x=\dfrac{5}{13}$

$\Rightarrow\quad \tan x=\dfrac{5}{12}$



Let $\cos^{-1}\dfrac{3}{5}=y \quad\Rightarrow\quad \cos y=\dfrac{3}{5}$

$\Rightarrow\quad \tan y=\dfrac{4}{3}$



Now $\quad \tan(x+y)=\dfrac{\tan x+\tan y}{1-\tan x\tan y}$

$$\Rightarrow\quad \tan(x+y)=\frac{\frac{5}{12}+\frac{4}{3}}{1-\frac{5}{12}\times\frac{4}{3}}=\frac{\frac{15+48}{36}}{\frac{36-20}{36}}=\frac{63}{16}$$

$$\Rightarrow\quad x+y=\tan^{-1}\frac{63}{16}$$

$\therefore \sin^{-1}\dfrac{5}{13}+\cos^{-1}\dfrac{3}{5}=\tan^{-1}\dfrac{63}{16}$ **Hence proved.**

Q16. Prove that: $\tan^{-1}\dfrac{1}{4}+\tan^{-1}\dfrac{2}{9}=\sin^{-1}\dfrac{1}{\sqrt{5}}$

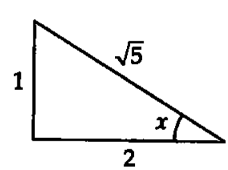
Sol. $\quad \tan^{-1}\dfrac{1}{4}+\tan^{-1}\dfrac{2}{9}=\tan^{-1}\left[\dfrac{\frac{1}{4}+\frac{2}{9}}{1-\frac{1}{4}\times\frac{2}{9}}\right]$

$$\left[\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$$

$$\Rightarrow\quad \tan^{-1}\left[\frac{\frac{9+8}{36}}{\frac{36-2}{36}}\right]=\tan^{-1}\left[\frac{17}{34}\right]$$

Let $\quad \tan^{-1}\left[\dfrac{17}{34}\right]=x$

$\therefore \quad \tan x=\dfrac{17}{34}=\dfrac{1}{2}$



$$\sin x = \frac{1}{\sqrt{5}}$$

$$\therefore \qquad \tan^{-1}\frac{1}{2} = \sin^{-1}\frac{1}{\sqrt{5}} \text{ R.H.S.}$$

Hence, $\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9} = \sin^{-1}\frac{1}{\sqrt{5}}$

Q17. Find the value of $4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$

Sol. $4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$

$$\Rightarrow 2.\left(2\tan^{-1}\frac{1}{5}\right) - \tan^{-1}\frac{1}{239}$$

$$\Rightarrow 2\left[\tan^{-1}\frac{2\times\frac{1}{5}}{1-\frac{1}{25}}\right] - \tan^{-1}\frac{1}{239} \qquad \left[2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}\right]$$

$$\Rightarrow 2\tan^{-1}\frac{5}{12} - \tan^{-1}\frac{1}{239} \quad \Rightarrow \tan^{-1}\left(\frac{2\times\frac{5}{12}}{1-\frac{25}{144}}\right) - \tan^{-1}\left(\frac{1}{239}\right)$$

$$\Rightarrow \tan^{-1}\left(\frac{120}{119}\right) - \tan^{-1}\left(\frac{1}{239}\right)$$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{120}{119}-\frac{1}{239}}{1+\frac{120}{119}\times\frac{1}{239}}\right] \qquad \left[\because \tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}\right]$$

$$\Rightarrow \tan^{-1}\left[\frac{120\times 239 - 119}{119\times 239 + 120}\right] \Rightarrow \tan^{-1}\left[\frac{28680-119}{28441+120}\right]$$

$$\Rightarrow \tan^{-1}\left[\frac{28561}{28561}\right] = \tan^{-1}(1) = \frac{\pi}{4}$$

Q18. Show that $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) = \frac{4-\sqrt{7}}{3}$ and justify why the other value $\frac{4+\sqrt{7}}{3}$ is ignored?

Sol. To prove that $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) = \frac{4-\sqrt{7}}{3}$

L.H.S. Let $\frac{1}{2}\sin^{-1}\frac{3}{4} = \tan^{-1}\theta \qquad [\therefore \tan(\tan^{-1}\theta) = \theta]$

$$\Rightarrow \quad \sin^{-1}\frac{3}{4} = 2\tan^{-1}\theta \Rightarrow \sin^{-1}\frac{3}{4} = \sin^{-1}\left(\frac{2\theta}{1+\theta^2}\right)$$

$$\left[\because 2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2}\right]$$

$$\Rightarrow \quad \frac{2\theta}{1+\theta^2} = \frac{3}{4} \Rightarrow 3 + 3\theta^2 = 8\theta$$

$$\Rightarrow \quad 3\theta^2 - 8\theta + 3 = 0$$

$$\Rightarrow \quad \theta = \frac{-(-8) \pm \sqrt{(-8)^2 - 4\times 3\times 3}}{2\times 3}$$

$$= \frac{8 \pm \sqrt{64-36}}{6} = \frac{8\pm\sqrt{28}}{6} = \frac{8\pm 2\sqrt{7}}{6} = \frac{2(4\pm\sqrt{7})}{6}$$

$$\Rightarrow \quad \theta = \frac{4\pm\sqrt{7}}{3}$$

$$\therefore \quad \theta = \frac{4+\sqrt{7}}{3} \text{ or } \frac{4-\sqrt{7}}{3}$$

$\theta = \dfrac{4+\sqrt{7}}{3}$ is ignored.

Because $\dfrac{-\pi}{2} \le \sin^{-1}\dfrac{3}{4} \le \dfrac{\pi}{2}$

$$\Rightarrow \frac{-\pi}{4} \le \frac{1}{2}\sin^{-1}\frac{3}{4} \le \frac{\pi}{4}$$

$$\Rightarrow \tan\left(\frac{-\pi}{4}\right) \le \tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) \le \tan\left(\frac{\pi}{4}\right)$$

$$\Rightarrow -1 \le \tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) \le 1$$

Hence, $\tan\left(\dfrac{1}{2}\sin^{-1}\dfrac{3}{4}\right) = \dfrac{4-\sqrt{7}}{3}$

Q19. If $a_1, a_2, a_3, \ldots, a_n$ is an arithmetic progression with common difference d, then evaluate the following expression

$$\tan\left[\tan^{-1}\left(\frac{d}{1+a_1a_2}\right) + \tan^{-1}\left(\frac{d}{1+a_2a_3}\right) + \tan^{-1}\left(\frac{d}{1+a_3a_4}\right) + \ldots \right.$$
$$\left. \ldots + \tan^{-1}\left(\frac{d}{1+a_{n-1}.a_n}\right)\right]$$

Sol. If $a_1, a_2, a_3, \ldots, a_n$ are the terms of an arithmetic progression

$\therefore d = a_2 - a_1 = a_3 - a_2 = a_4 - a_3 \ldots$

$$\therefore \tan\left[\tan^{-1}\left(\frac{a_2-a_1}{1+a_1a_2}\right)+\tan^{-1}\left(\frac{a_3-a_2}{1+a_2a_3}\right)+\tan^{-1}\left(\frac{a_4-a_3}{1+a_3a_4}\right)+\dots\right.$$
$$\left.\dots+\tan^{-1}\left(\frac{a_n-a_{n-1}}{1+a_{n-1}.a_n}\right)\right]$$

$$\Rightarrow \tan[(\tan^{-1}a_2-\tan^{-1}a_1)+(\tan^{-1}a_3-\tan^{-1}a_2)+(\tan^{-1}a_4-\tan^{-1}a_3)$$
$$+\dots+(\tan^{-1}a_n-\tan^{-1}a_{n-1})]$$

$$\left[\because \tan^{-1}\frac{x-y}{1+xy}=\tan^{-1}x-\tan^{-1}y\right]$$

$$\Rightarrow \tan[\tan^{-1}a_2-\tan^{-1}a_1+\tan^{-1}a_3-\tan^{-1}a_2+\tan^{-1}a_4-\tan^{-1}a_3$$
$$+\dots+\tan^{-1}a_n-\tan^{-1}a_{n-1}]$$

$$\Rightarrow \tan[\tan^{-1}a_n-\tan^{-1}a_1]$$

$$\Rightarrow \tan\left[\tan^{-1}\left(\frac{a_n-a_1}{1+a_1a_n}\right)\right] \Rightarrow \frac{a_n-a_1}{1+a_1a_n} \qquad [\because \tan(\tan^{-1}x)=x]$$

OBJECTIVE TYPE QUESTIONS

Choose the correct answers from the given four options in each of the Exercises from 20 to 37 (M.C.Q.).

Q20. Which of the following is the principal value branch of $\cos^{-1}x$?

(*a*) $\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$ (*b*) $(0,\pi)$

(*c*) $[0,\pi]$ (*d*) $(0,\pi)-\left\{\frac{\pi}{2}\right\}$

Sol. Principal value branch of $\cos^{-1}x$ is $[0,\pi]$. Hence the correct answer is (*c*).

Q21. Which of the following is the principal value branch of $\operatorname{cosec}^{-1}x$?

(*a*) $\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$ (*b*) $[0,\pi]-\left\{\frac{\pi}{2}\right\}$

(*c*) $\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$ (*d*) $\left[\frac{-\pi}{2},\frac{\pi}{2}\right]-\{0\}$

Sol. Principal value branch of $\operatorname{cosec}^{-1}x$ is

$\left[\frac{-\pi}{2},\frac{\pi}{2}\right]-\{0\}$ as $\operatorname{cosec}^{-1}(0)=\infty$ (not defined).

Hence, the correct answer is (*d*).

Q22. If $3\tan^{-1}x + \cot^{-1}x = \pi$, then x equals

(*a*) 0 (*b*) 1 (*c*) -1 (*d*) $\frac{1}{2}$

Sol. Given that $3\tan^{-1}x + \cot^{-1}x = \pi$

$\Rightarrow 2\tan^{-1}x + \tan^{-1}x + \cot^{-1}x = \pi$

$\Rightarrow \quad 2\tan^{-1}x + \frac{\pi}{2} = \pi \qquad \left[\because \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}\right]$

$\Rightarrow \quad 2\tan^{-1}x = \pi - \frac{\pi}{2} \quad \Rightarrow 2\tan^{-1}x = \frac{\pi}{2}$

$\Rightarrow \quad \tan^{-1}x = \frac{\pi}{4} \quad \Rightarrow \tan^{-1}x = \tan^{-1}(1)$

$\therefore \quad x = 1$

Hence, the correct answer is (*b*).

Q23. The value of $\sin^{-1}\left[\cos\left(\frac{33\pi}{5}\right)\right]$ is

(*a*) $\frac{3\pi}{5}$ (*b*) $\frac{-7\pi}{5}$ (*c*) $\frac{\pi}{10}$ (*d*) $\frac{-\pi}{10}$

Sol. $\sin^{-1}\left[\cos\left(\frac{33\pi}{5}\right)\right] = \sin^{-1}\left[\cos\left(6\pi + \frac{3\pi}{5}\right)\right]$

$= \sin^{-1}\left[\cos\frac{3\pi}{5}\right] \qquad [\because \cos(2n\pi + x) = \cos x]$

$= \sin^{-1}\left[\cos\left(\frac{\pi}{2} + \frac{\pi}{10}\right)\right]$

$= \sin^{-1}\left[-\sin\left(\frac{\pi}{10}\right)\right] \quad \left[\because \cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta\right]$

$= \sin^{-1}\left[\sin\left(\frac{-\pi}{10}\right)\right] = \frac{-\pi}{10}$

Hence, the correct answer is (*d*).

Q24. The domain of the function $\cos^{-1}(2x-1)$ is

(*a*) [0, 1] (*b*) [–1, 1] (*c*) (–1, 1) (*d*) [0, π]

Sol. The given function is $\cos^{-1}(2x-1)$

Let $\quad f(x) = \cos^{-1}(2x-1)$

$-1 \le 2x - 1 \le 1 \Rightarrow -1 + 1 \le 2x \le 1 + 1$

$0 \le 2x \le 2 \quad \Rightarrow 0 \le x \le 1$

$\therefore$ domain of the given function is [0, 1].

Hence, the correct answer is (*a*)

Q25. The domain of the function defined by $f(x) = \sin^{-1}\sqrt{x-1}$ is

(*a*) [1, 2] (*b*) [– 1, 1] (*c*) [0, 1] (*d*) None of these

Sol. Let $f(x) = \sin^{-1}\sqrt{x-1}$

$\because \sqrt{x-1} \geq 0$ and $-1 \leq \sqrt{x-1} \leq 1$

$\Rightarrow 0 \leq x-1 \leq 1 \Rightarrow 1 \leq x \leq 2 \Rightarrow x \in [1, 2]$

Hence, the correct answer is (*a*).

Q26. If $\cos\left[\sin^{-1}\frac{2}{5} + \cos^{-1}x\right] = 0$, then x is equal to

(*a*) $\frac{1}{5}$ (*b*) $\frac{2}{5}$ (*c*) 0 (*d*) 1

Sol. Given that $\cos\left[\sin^{-1}\frac{2}{5} + \cos^{-1}x\right] = 0$

$$\Rightarrow \sin^{-1}\frac{2}{5} + \cos^{-1}x = \cos^{-1}(0)$$

$$\Rightarrow \sin^{-1}\frac{2}{5} + \cos^{-1}x = \frac{\pi}{2} \Rightarrow \sin^{-1}\frac{2}{5} = \frac{\pi}{2} - \cos^{-1}x$$

$$\Rightarrow \sin^{-1}\frac{2}{5} = \sin^{-1}x \quad \left[\because \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}\right]$$

$$\Rightarrow x = \frac{2}{5}$$

Hence, the correct answer is (*b*).

Q27. The value of sin [2 $\tan^{-1}$ (0.75)] is equal to

(*a*) 0.75 (*b*) 1.5 (*c*) 0.96 (*d*) sin 1.5

Sol. Given that sin [2 $\tan^{-1}$ (0.75)]

$$= \sin\left[2\tan^{-1}\frac{3}{4}\right]$$

$$= \sin\left[\sin^{-1}\frac{2 \times \frac{3}{4}}{1 + \frac{9}{16}}\right] \quad \left[\because 2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2}\right]$$

$$= \sin\left[\sin^{-1}\frac{\frac{3}{2}}{\frac{25}{16}}\right] = \sin\left[\sin^{-1}\frac{24}{25}\right]$$

$$= \sin[\sin^{-1}(0.96)]$$

$= 0.96$

Hence, the correct answer is (*c*).

Q28. The value of $\cos^{-1}\left(\cos\frac{3\pi}{2}\right)$ is equal to

(*a*) $\frac{\pi}{2}$ (*b*) $\frac{3\pi}{2}$ (*c*) $\frac{5\pi}{2}$ (*d*) $\frac{7\pi}{2}$

Sol. $\cos^{-1}\left(\cos\frac{\pi}{2}\right) \neq \frac{3\pi}{2} \quad \because \frac{3\pi}{2} \notin [0, \pi]$

$\Rightarrow \cos^{-1}\left[\cos\left(\pi+\frac{\pi}{2}\right)\right] \Rightarrow \cos^{-1}\left[-\cos\frac{\pi}{2}\right] \Rightarrow \cos^{-1}[0] = \frac{\pi}{2}$

Hence, the correct answer is (*a*).

Q29. The value of expression $2\sec^{-2} 2 + \sin^{-1}\left(\frac{1}{2}\right)$ is

(*a*) $\frac{\pi}{6}$ (*b*) $\frac{5\pi}{6}$ (*c*) $\frac{7\pi}{6}$ (*d*) 1

Sol. $2\sec^{-1}2 + \sin^{-1}\frac{1}{2} = 2\sec^{-1}\left(\sec\frac{\pi}{3}\right) + \sin^{-1}\left(\sin\frac{\pi}{6}\right)$

$= 2.\frac{\pi}{3} + \frac{\pi}{6} = \frac{2\pi}{3} + \frac{\pi}{6} = \frac{5\pi}{6}$

Hence, the correct answer is (*b*).

Q30. If $\tan^{-1}x + \tan^{-1}y = \frac{4\pi}{5}$, then $\cot^{-1}x + \cot^{-1}y$ equals

(*a*) $\frac{\pi}{5}$ (*b*) $\frac{2\pi}{5}$ (*c*) $\frac{3\pi}{5}$ (*d*) π

Sol. Given that $\tan^{-1}x + \tan^{-1}y = \frac{4\pi}{5}$

$\Rightarrow \frac{\pi}{2} - \cot^{-1}x + \frac{\pi}{2} - \cot^{-1}y = \frac{4\pi}{5} \quad \left[\because \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}\right]$

$\Rightarrow \quad \pi - (\cot^{-1}x + \cot^{-1}y) = \frac{4\pi}{5}$

$\Rightarrow \quad \cot^{-1}x + \cot^{-1}y = \pi - \frac{4\pi}{5}$

$\Rightarrow \quad \cot^{-1}x + \cot^{-1}y = \frac{\pi}{5}$

Hence, the correct answer is (*a*).

Q31. If $\sin^{-1}\left(\frac{2a}{1+a^2}\right) + \cos^{-1}\left(\frac{1-a^2}{1+a^2}\right) = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$,

where $a, x \in]0, 1$, then the value of x is

(*a*) 0 (*b*) $\frac{a}{2}$ (*c*) a (*d*) $\frac{2a}{1-a^2}$

Sol. $\sin^{-1}\left(\dfrac{2a}{1+a^2}\right)+\cos^{-1}\left(\dfrac{1-a^2}{1+a^2}\right) = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$

$\Rightarrow\ 2\tan^{-1}a + 2\tan^{-1}a = 2\tan^{-1}x$

$\left[\because 2\tan^{-1}x = \sin^{-1}\dfrac{2x}{1+x^2} = \cos^{-1}\dfrac{1-x^2}{1+x^2} = \tan^{-1}\dfrac{2x}{1-x^2}\right]$

$\Rightarrow\quad 4\tan^{-1}a = 2\tan^{-1}x \quad\Rightarrow 2\tan^{-1}a = \tan^{-1}x$

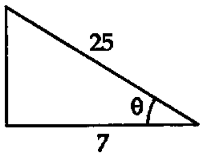
$\Rightarrow\quad \tan^{-1}\dfrac{2a}{1-a^2} = \tan^{-1}x \quad\Rightarrow x = \dfrac{2a}{1-a^2}$

Hence, the correct answer is (*d*).

Q32. The value of $\cot\left[\cos^{-1}\left(\dfrac{7}{25}\right)\right]$ is

(*a*) $\dfrac{25}{24}$ (*b*) $\dfrac{25}{7}$ (*c*) $\dfrac{24}{25}$ (*d*) $\dfrac{7}{24}$

Sol. We have, $\cot\left[\cos^{-1}\left(\dfrac{7}{25}\right)\right]$



Let $\cos^{-1}\dfrac{7}{25} = \theta$

$\therefore\quad \cos\theta = \dfrac{7}{25} \Rightarrow \cot\theta = \dfrac{7}{24}$

$\therefore \cot\left[\cos^{-1}\left(\dfrac{7}{25}\right)\right] = \cot\left[\cot^{-1}\left(\dfrac{7}{24}\right)\right] = \dfrac{7}{24}$

Hence, the correct answer is (*d*).

Q33. The value of expression $\tan\left[\dfrac{1}{2}\cos^{-1}\dfrac{2}{\sqrt{5}}\right]$ is

(*a*) $2+\sqrt{5}$ (*b*) $\sqrt{5}-2$ (*c*) $\dfrac{\sqrt{5}+2}{2}$ (*d*) $5+\sqrt{2}$

Sol. We have, $\tan\left[\dfrac{1}{2}\cos^{-1}\dfrac{2}{\sqrt{5}}\right]$

Let $\theta = \dfrac{1}{2}\cos^{-1}\dfrac{2}{\sqrt{5}}$

$\Rightarrow\quad 2\theta = \cos^{-1}\dfrac{2}{\sqrt{5}} \Rightarrow \cos 2\theta = \dfrac{2}{\sqrt{5}}$

$\Rightarrow\quad \dfrac{1-\tan^2\theta}{1+\tan^2\theta} = \dfrac{2}{\sqrt{5}} \quad \left[\because \cos 2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}\right]$

$\Rightarrow\quad 2+2\tan^2\theta = \sqrt{5}-\sqrt{5}\tan^2\theta$

$\Rightarrow \sqrt{5}\tan^2\theta + 2\tan^2\theta = \sqrt{5}-2 \Rightarrow (\sqrt{5}+2)\tan^2\theta = \sqrt{5}-2$

$$\Rightarrow \qquad \tan^2\theta = \frac{\sqrt{5}-2}{\sqrt{5}+2}$$

$$\Rightarrow \qquad \tan^2\theta = \frac{(\sqrt{5}-2)(\sqrt{5}-2)}{(\sqrt{5}+2)(\sqrt{5}-2)} \Rightarrow \tan^2\theta = \frac{(\sqrt{5}-2)^2}{5-4}$$

$$\Rightarrow \qquad \tan\theta = \pm(\sqrt{5}-2)$$

$$\Rightarrow \qquad \tan\theta = \sqrt{5}-2, \quad \left[-(\sqrt{5}-2) \text{ is not required}\right]$$

Hence, the correct answer is (*b*).

Q34. If $|x| \le 1$, then $2\tan^{-1}x + \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ is equal to

(*a*) $4\tan^{-1}x$ (*b*) 0 (*c*) $\frac{\pi}{2}$ (*d*) π

Sol. Here, we have $2\tan^{-1}x + \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$$= 2\tan^{-1}x + 2\tan^{-1}x \qquad \left[\because 2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2}\right]$$

$$= 4\tan^{-1}x$$

Hence, the correct answer is (*a*).

Q35. If $\cos^{-1}\alpha + \cos^{-1}\beta + \cos^{-1}\gamma = 3\pi$, then $\alpha(\beta+\gamma) + \beta(\gamma+\alpha) + \gamma(\alpha+\beta)$ equals

(*a*) 0 (*b*) 1 (*c*) 6 (*d*) 12

Sol. We have $\cos^{-1}\alpha + \cos^{-1}\beta + \cos^{-1}\gamma = 3\pi$

$\Rightarrow \quad \cos^{-1}\alpha + \cos^{-1}\beta + \cos^{-1}\gamma = \pi + \pi + \pi$

$\Rightarrow \quad \cos^{-1}\alpha = \pi, \cos^{-1}\beta = \pi$ and $\cos^{-1}\gamma = \pi$

$\Rightarrow \quad \alpha = \cos\pi, \beta = \cos\pi$ and $\gamma = \cos\pi$

$\therefore \quad \alpha = -1, \beta = -1$ and $\gamma = -1$

Which gives $\alpha = \beta = \gamma = -1$

So $\quad \alpha(\beta+\gamma) + \beta(\gamma+\alpha) + \gamma(\alpha+\beta)$

$\Rightarrow \quad (-1)(-1-1) + (-1)(-1-1) + (-1)(-1-1)$

$\Rightarrow \quad (-1)(-2) + (-1)(-2) + (-1)(-2) \Rightarrow 2+2+2 \Rightarrow 6$

Hence, the correct answer is (*c*).

Q36. The number of real solutions of the equation

$\sqrt{1+\cos 2x} = \sqrt{2}\cos^{-1}(\cos x)$ in $\left[\frac{\pi}{2}, \pi\right]$ is (*a*) 0 (*b*) 1 (*c*) 2 (*d*) infinite

Sol. We have $\sqrt{1+\cos 2x} = \sqrt{2}\cos^{-1}(\cos x)$

$\Rightarrow \sqrt{2\cos^2 x} = \sqrt{2}x \quad [\because \cos^{-1}(\cos x) = x]$

$\Rightarrow \sqrt{2}\cos x = \sqrt{2}x \Rightarrow \cos x = x$

Which does not satisfy for any value of x.

Hence, the correct answer is (*d*).

Q37. If $\cos^{-1} x > \sin^{-1} x$, then

(*a*) $\frac{1}{\sqrt{2}} < x \le 1$ (*b*) $0 \le x < \frac{1}{\sqrt{2}}$

(*c*) $-1 \le x < \frac{1}{\sqrt{2}}$ (*d*) $x > 0$

Sol. Here, given that $\cos^{-1} x > \sin^{-1} x$

$\Rightarrow \sin[\cos^{-1} x] > x$

$\Rightarrow \sin\left[\sin^{-1}\sqrt{1-x^2}\right] > x \Rightarrow \sqrt{1-x^2} > x$

$\Rightarrow x < \sqrt{1-x^2} \Rightarrow x^2 < 1 - x^2 \Rightarrow 2x^2 < 1$

$\Rightarrow x^2 < \frac{1}{2} \Rightarrow x < \pm\frac{1}{\sqrt{2}}$

We know that $-1 \le x \le 1$

So $-1 \le x < \frac{1}{\sqrt{2}}$.

Hence, the correct answer is (c).

Fill in the Blanks in each of the Exercises 38 to 48.

Q38. The principal value of $\cos^{-1}\left(-\frac{1}{2}\right)$ is ....

Sol. Let $\cos^{-1}\left(-\frac{1}{2}\right) = x \Rightarrow \cos x = -\frac{1}{2}$

$\Rightarrow \cos x = \cos\left(-\frac{\pi}{3}\right) \Rightarrow \cos x = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\frac{2\pi}{3}$

$\therefore x = \frac{2\pi}{3} \in [0, \pi]$

Hence, Principal value of $\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$.

Q39. The value of $\sin^{-1}\left(\sin\frac{3\pi}{5}\right)$ is ....

Sol. $\sin^{-1}\left(\sin\frac{3\pi}{5}\right) \ne \frac{3\pi}{5}$ as $\frac{3\pi}{5} \notin \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

So $\sin^{-1}\left(\sin\frac{3\pi}{5}\right) = \sin^{-1}\sin\left(\pi - \frac{2\pi}{5}\right)$

$$= \sin^{-1}\sin\left(\frac{2\pi}{5}\right) = \frac{2\pi}{5} \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$$

Hence, the value of $\sin^{-1}\left(\sin\frac{3\pi}{5}\right) = \frac{2\pi}{5}$

Q40. If $\cos\left(\tan^{-1}x + \cot^{-1}\sqrt{3}\right) = 0$, then value of x is ....

Sol. Given that

$$\cos\left[\tan^{-1}x + \cot^{-1}\sqrt{3}\right] = 0$$

$$\Rightarrow \quad \tan^{-1}x + \cot^{-1}\sqrt{3} = \cos^{-1}(0)$$

$$\Rightarrow \quad \tan^{-1}x + \cot^{-1}\sqrt{3} = \frac{\pi}{2}$$

$$\Rightarrow \quad \tan^{-1}x = \frac{\pi}{2} - \cot^{-1}\sqrt{3} \qquad \left[\because \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}\right]$$

$$\Rightarrow \quad \tan^{-1}x = \tan^{-1}\sqrt{3} \Rightarrow x = \sqrt{3}$$

Hence, the value of x is $\sqrt{3}$.

Q41. The set of values of $\sec^{-1}\left(\frac{1}{2}\right)$ is ....

Sol. Let $\quad \sec^{-1}\left(\frac{1}{2}\right) = x \Rightarrow \sec x = \frac{1}{2}$

Since, the domain of $\sec^{-1}x$ is $R - \{-1, 1\}$ and $\frac{1}{2} \notin R - \{-1, 1\}$.

Hence, $\sec^{-1}\left(\frac{1}{2}\right)$ has no set of values.

Q42. The principal value of $\tan^{-1}\sqrt{3}$ is ....

Sol. $\tan^{-1}\sqrt{3} = \tan^{-1}\left(\tan\frac{\pi}{3}\right) = \frac{\pi}{3} \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

Hence the principal value of $\tan^{-1}\sqrt{3}$ is $\frac{\pi}{3}$.

Q43. The value of $\cos^{-1}\left(\cos\frac{14\pi}{3}\right)$ is ....

Sol.
$$\cos^{-1}\left(\cos\frac{14\pi}{3}\right) = \cos^{-1}\left[\cos\left(5\pi - \frac{\pi}{3}\right)\right]$$

$$= \cos^{-1}\left[\cos\left(\frac{-\pi}{3}\right)\right] = \cos^{-1}\left[\cos\left(\pi - \frac{\pi}{3}\right)\right]$$

$$= \cos^{-1}\left[\cos\frac{2\pi}{3}\right] = \frac{2\pi}{3} \in [0, \pi]$$

Hence, the value of $\cos^{-1}\left[\cos\frac{14\pi}{3}\right] = \frac{2\pi}{3}$.

Q44. The value of $\cos(\sin^{-1}x + \cos^{-1}x)$, $|x| \le 1$ is ....

Sol. $\cos[\sin^{-1}x + \cos^{-1}x] = \cos\frac{\pi}{2} = 0 \qquad \left[\because \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}\right]$

Hence, the value of $\cos(\sin^{-1}x + \cos^{-1}x) = 0$.

Q45. The value of expression $\tan\left(\frac{\sin^{-1}x + \cos^{-1}x}{2}\right)$, when $x = \frac{\sqrt{3}}{2}$ is ....

Sol. $\tan\left(\frac{\sin^{-1}x + \cos^{-1}x}{2}\right) = \tan\left(\frac{\pi}{4}\right) = 1 \quad \left[\because \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}\right]$

Hence, the value of the given expression is 1.

Q46. If $y = 2\tan^{-1}x + \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ for all x, then ..... $< y <$ ......

Sol.

$$y = 2\tan^{-1}x + \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow \quad y = 2\tan^{-1}x + 2\tan^{-1}x$$

$$\Rightarrow \quad y = 4\tan^{-1}x \quad \left[\because \sin^{-1}\left(\frac{2x}{1+x^2}\right) = 2\tan^{-1}x\right]$$

Now $\frac{-\pi}{2} < \tan^{-1}x < \frac{\pi}{2}$

$\Rightarrow \quad -4 \times \frac{\pi}{2} < 4\tan^{-1}x < 4 \times \frac{\pi}{2} \quad \Rightarrow -2\pi < y < 2\pi$

Hence, the value of y is $(-2\pi, 2\pi)$.

Q47. The result $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$ is true when value of xy is ....

Sol. The given result is true when $xy > -1$.

Q48. The value of $\cot^{-1}(-x)$ for all $x \in R$ in terms of $\cot^{-1}x$ is ....

Sol. $\cot^{-1}(-x) = \pi - \cot^{-1}x, x \in R \qquad [\because \text{as } \cot^{-1}(-x) = \pi - \cot^{-1}x]$

State True or False for the Statement in Each of the Exercises 49 to 55.

Q49. All trigonometric functions have inverse over their respective domains.

Sol. False.

We know that all inverse trigonometric functions are restricted over their domains.

Q50. The value of expression $(\cos^{-1}x)^2$ is equal to $\sec^2 x$.

Sol. False.

We know that $\cos^{-1} x = \sec^{-1}\left(\frac{1}{x}\right) \neq \sec x$

So $(\cos^{-1} x)^2 \neq \sec^2 x$

Q51. The domain of trigonometric functions can be restricted to any one of their branch (not necessarily principal value) in order to obtain their inverse functions.

Sol. True.

We know that all trigonometric functions are restricted over their domains to obtain their inverse functions.

Q52. The least numerical value, either positive or negative of angle θ is called principal value of the inverse trigonometric function.

Sol. True.

Q53. The graph of inverse trigonometric function can be obtained from the graph of their corresponding trigonometric function by interchanging x and y *axes*.

Sol. True.

We know that the domain and range are interchanged in the graph of inverse trigonometric functions to that of their corresponding trigonometric functions.

Q54. The minimum value of n for which $\tan^{-1}\frac{n}{\pi} > \frac{\pi}{4}$, $n \in N$ is valid is 5.

Sol. False.

Given that $\tan^{-1}\frac{n}{\pi} > \frac{\pi}{4}$

$\Rightarrow \quad \frac{n}{\pi} > \tan\frac{\pi}{4} \quad \Rightarrow \quad \frac{n}{\pi} > 1$

$\Rightarrow \quad n > \pi \quad \Rightarrow n > 3.14$

Hence, the value of n is 4.

Q55. The principal value of $\sin^{-1}\left[\cos\left(\sin^{-1}\frac{1}{2}\right)\right]$ is $\frac{\pi}{3}$.

Sol. True.

$$\sin^{-1}\left[\cos\left(\sin^{-1}\frac{1}{2}\right)\right] = \sin^{-1}\left[\cos\left(\sin^{-1}\sin\frac{\pi}{6}\right)\right]$$

$$\sin^{-1}\left[\cos\frac{\pi}{6}\right] = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \sin^{-1}\left(\sin\frac{\pi}{3}\right) = \frac{\pi}{3}$$

3 Matrices

3.3 EXERCISE

SHORT ANSWER TYPE QUESTIONS

Q1. If a matrix has 28 elements, what are the possible orders it can have? What if it has 13 elements?

Sol. The possible orders that a matrix having 28 elements are $\{28 \times 1, 1 \times 28, 2 \times 14, 14 \times 2, 4 \times 7, 7 \times 4\}$. The possible orders of a matrix having 13 elements are $\{1 \times 13, 13 \times 1\}$.

Q2. In the matrix $A = \begin{bmatrix} a & 1 & x \\ 2 & \sqrt{3} & x^2 - y \\ 0 & 5 & \frac{-2}{5} \end{bmatrix}$, write:

(*i*) The order of the matrix A (*ii*) The number of elements

(*iii*) Write elements a_{23}, a_{31}, a_{12}

Sol. (*i*) The order of the given matrix A is 3×3

(*ii*) The number of elements in matrix $A = 3 \times 3 = 9$

(*iii*) a_{ij} = the elements of i^{th} row and j^{th} column.

So, $a_{23} = x^2 - y$, $a_{31} = 0$, $a_{12} = 1$.

Q3. Construct $a_{2 \times 2}$ matrix where

(*i*) $a_{ij} = \dfrac{(i-2j)^2}{2}$ (*ii*) $a_{ij} = |-2i + 3j|$

Sol. Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2 \times 2}$

(*i*) Given that $a_{ij} = \dfrac{(i-2j)^2}{2}$

$$a_{11} = \frac{(1-2\times1)^2}{2} = \frac{1}{2};\ a_{12} = \frac{(1-2\times2)^2}{2} = \frac{9}{2}$$

$$a_{21} = \frac{(2-2\times1)^2}{2} = 0;\ a_{22} = \frac{(2-2\times2)^2}{2} = 2$$

Hence, the matrix $A = \begin{bmatrix} \frac{1}{2} & \frac{9}{2} \\ 0 & 2 \end{bmatrix}$

(ii) Given that $a_{ij} = |-2i + 3j|$

$$a_{11} = |-2\times1+3\times1| = 1;\ a_{12} = |-2\times1+3\times2| = 4$$

$$a_{21} = |-2\times2+3\times1| = -1;\ a_{22} = |-2\times2+3\times2| = 2$$

Hence, the matrix $A = \begin{bmatrix} 1 & 4 \\ -1 & 2 \end{bmatrix}$

Q4. Construct a 3×2 matrix whose elements are given by $a_{ij} = e^{ix} \sin jx$.

Sol. Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}_{3\times2}$

Given that $a_{ij} = e^{ix} \sin jx$

$$a_{11} = e^x \sin x \qquad a_{12} = e^x \sin 2x$$

$$a_{21} = e^{2x} \sin x \qquad a_{22} = e^{2x} \sin 2x$$

$$a_{31} = e^{3x} \sin x \qquad a_{32} = e^{3x} \sin 2x$$

Hence, the matrix $A = \begin{bmatrix} e^x \sin x & e^x \sin 2x \\ e^{2x} \sin x & e^{2x} \sin 2x \\ e^{3x} \sin x & e^{3x} \sin 2x \end{bmatrix}$

Q5. Find the values of a and b if $A = B$, where

$$A = \begin{bmatrix} a+4 & 3b \\ 8 & -6 \end{bmatrix},\quad B = \begin{bmatrix} 2a+2 & b^2+2 \\ 8 & b^2-5b \end{bmatrix}$$

Sol. Given that $A = B$

$$\Rightarrow \begin{bmatrix} a+4 & 3b \\ 8 & -6 \end{bmatrix} = \begin{bmatrix} 2a+2 & b^2+2 \\ 8 & b^2-5b \end{bmatrix}$$

Equating the corresponding elements, we get

$a + 4 = 2a + 2$, $3b = b^2 + 2$ and $b^2 - 5b = -6$

$\Rightarrow$ $2a - a = 2$, $b^2 - 3b + 2 = 0$, $b^2 - 5b + 6 = 0$

$\therefore$ $a = 2$

$\therefore$ $b^2 - 3b + 2 = 0$	$\therefore$ $b^2 - 5b + 6 = 0$
$\Rightarrow$ $b^2 - 2b - b + 2 = 0,$	$b^2 - 3b - 2b + 6 = 0$
$\Rightarrow$ $b(b-2) - 1(b-2) = 0,$	$\Rightarrow$ $b(b-3) - 2(b-3) = 0$
$\Rightarrow$ $(b-1)(b-2) = 0,$	$\Rightarrow$ $(b-2)(b-3) = 0$
$\therefore$ $b = 1, 2$	$\Rightarrow$ $b = 2, 3$

but here 2 is common.

Hence, the value of $a = 2$ and $b = 2$.

Q6. If possible, find the sum of the matrices A and B, where

$$A = \begin{bmatrix} \sqrt{3} & 1 \\ 2 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} x & y & z \\ a & b & 6 \end{bmatrix}.$$

Sol. The order of matrix $A = 2 \times 2$ and the order of matrix $B = 2 \times 3$. Addition of matrices is only possible when they have same order. So, $A + B$ is not possible.

Q7. If $X = \begin{bmatrix} 3 & 1 & -1 \\ 5 & -2 & -3 \end{bmatrix}$ and $Y = \begin{bmatrix} 2 & 1 & -1 \\ 7 & 2 & 4 \end{bmatrix}$, find

(*i*) $X + Y$ (*ii*) $2X - 3Y$

(*iii*) A matrix Z such that $X + Y + Z$ is a zero matrix.

Sol. Given that $X = \begin{bmatrix} 3 & 1 & -1 \\ 5 & -2 & -3 \end{bmatrix}$ and $Y = \begin{bmatrix} 2 & 1 & -1 \\ 7 & 2 & 4 \end{bmatrix}$

(*i*) $$X + Y = \begin{bmatrix} 3 & 1 & -1 \\ 5 & -2 & -3 \end{bmatrix} + \begin{bmatrix} 2 & 1 & -1 \\ 7 & 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3+2 & 1+1 & -1-1 \\ 5+7 & -2+2 & -3+4 \end{bmatrix} = \begin{bmatrix} 5 & 2 & -2 \\ 12 & 0 & 1 \end{bmatrix}$$

(*ii*) $$2X - 3Y = 2\begin{bmatrix} 3 & 1 & -1 \\ 5 & -2 & -3 \end{bmatrix} - 3\begin{bmatrix} 2 & 1 & -1 \\ 7 & 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times3 & 2\times1 & -2\times1 \\ 2\times5 & -2\times2 & -2\times3 \end{bmatrix} - \begin{bmatrix} 3\times2 & 1\times3 & -1\times3 \\ 3\times7 & 3\times2 & 3\times4 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 2 & -2 \\ 10 & -4 & -6 \end{bmatrix} - \begin{bmatrix} 6 & 3 & -3 \\ 21 & 6 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 6-6 & 2-3 & -2+3 \\ 10-21 & -4-6 & -6-12 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ -11 & -10 & -18 \end{bmatrix}$$

(*iii*) $X + Y + Z = 0$

$$\Rightarrow \begin{bmatrix} 3 & 1 & -1 \\ 5 & -2 & -3 \end{bmatrix} + \begin{bmatrix} 2 & 1 & -1 \\ 7 & 2 & 4 \end{bmatrix} + \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where $Z = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 3+2+a & 1+1+b & -1-1+c \\ 5+7+d & -2+2+e & -3+4+f \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 5+a & 2+b & -2+c \\ 12+d & e & 1+f \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Equating the corresponding elements, we get

$5+a=0 \Rightarrow a=-5, \quad 2+b=0 \Rightarrow b=-2, \quad -2+c=0 \Rightarrow c=2$

$12+d=0 \Rightarrow d=-12, \quad e=0, \quad 1+f=0 \Rightarrow f=-1$

Hence, the matrix $Z=\begin{bmatrix}-5 & -2 & 2\\ -12 & 0 & -1\end{bmatrix}$

Q8. Find non-zero values of x satisfying the matrix equation:

$$x\begin{bmatrix}2x & 2\\ 3 & x\end{bmatrix}+2\begin{bmatrix}8 & 5x\\ 4 & 4x\end{bmatrix}=2\begin{bmatrix}(x^2+8) & 24\\ (10) & 6x\end{bmatrix}$$

Sol. The given equation can be written as

$$\begin{bmatrix}2x^2 & 2x\\ 3x & x^2\end{bmatrix}+\begin{bmatrix}16 & 10x\\ 8 & 8x\end{bmatrix}=\begin{bmatrix}(2x^2+16) & 48\\ 20 & 12x\end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix}2x^2+16 & 12x\\ 3x+8 & x^2+8x\end{bmatrix}=\begin{bmatrix}2x^2+16 & 48\\ 20 & 12x\end{bmatrix}$$

Equating the corresponding elements we get

$$12x=48, \qquad 3x+8=20, \qquad x^2+8x=12x$$

$$\therefore\ x=\frac{48}{12}=4, \qquad 3x=20-8=12, \qquad \Rightarrow\ x^2=12x-8x=4x$$

$$\therefore \qquad x=4, \qquad \Rightarrow \qquad x^2-4x=0$$

$$x=0, x=4$$

Hence, the non-zero values of x is 4.

Q9. If $A=\begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix}$ and $B=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$, show that

$(A+B)(A-B)\neq A^2-B^2$

Sol. Given that $A=\begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix}$ and $B=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$

$$A+B=\begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix}+\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$$

$$\Rightarrow \qquad A+B=\begin{bmatrix}0+0 & 1-1\\ 1+1 & 1+0\end{bmatrix} \Rightarrow A+B=\begin{bmatrix}0 & 0\\ 2 & 1\end{bmatrix}$$

$$A-B=\begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix}-\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$$

$$\Rightarrow \qquad A-B=\begin{bmatrix}0-0 & 1+1\\ 1-1 & 1-0\end{bmatrix} \Rightarrow A-B=\begin{bmatrix}0 & 2\\ 0 & 1\end{bmatrix}$$

$$\therefore (A+B)\cdot(A-B) = \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0+0 & 0+0 \\ 0+0 & 4+1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}$$

$$\begin{aligned} \text{Now, R.H.S.} &= A^2 - B^2 \\ &= A\cdot A - B\cdot B \\ &= \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0+1 & 0+1 \\ 0+1 & 1+1 \end{bmatrix} - \begin{bmatrix} 0-1 & 0+0 \\ 0+0 & -1+0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1+1 & 1-0 \\ 1-0 & 2+1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \end{aligned}$$

Hence, $\begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} \neq \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$

Hence, $(A+B).(A-B) \neq A^2 - B^2$

Q10. Find the value of x if

$$\begin{bmatrix} 1 & x & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 2 \\ 2 & 5 & 1 \\ 15 & 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ x \end{bmatrix} = O$$

Sol. Given that $\begin{bmatrix} 1 & x & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 2 \\ 2 & 5 & 1 \\ 15 & 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ x \end{bmatrix} = O$

$$\Rightarrow \begin{bmatrix} 1+2x+15 & 3+5x+3 & 2+x+2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ x \end{bmatrix} = O$$

$$\Rightarrow \begin{bmatrix} 2x+16 & 5x+6 & x+4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ x \end{bmatrix} = O$$

$\Rightarrow$ $[2x+16+10x+12+x^2+4x] = 0;$ $\Rightarrow x^2+16x+28=0$

$\Rightarrow$ $x^2+14x+2x+28=0;$ $\Rightarrow x(x+14)+2(x+14)=0$

$\Rightarrow$ $(x+2)(x+14)=0;$ $x+2=0$ or $x+14=0$

$\therefore$ $x=-2$ or $x=-14$

Hence, the values of x are -2 and -14.

Q11. Show that $A = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$ satisfies the equation $A^2 - 3A - 7I = O$

and hence find A^{-1}.

Sol. Given that $A = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$

$A^2 = A \cdot A$

$$= \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 25-3 & 15-6 \\ -5+2 & -3+4 \end{bmatrix} = \begin{bmatrix} 22 & 9 \\ -3 & 1 \end{bmatrix}$$

$A^2 - 3A - 7I = O$

$$\text{L.H.S.} \begin{bmatrix} 22 & 9 \\ -3 & 1 \end{bmatrix} - 3\begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} - 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 22 & 9 \\ -3 & 1 \end{bmatrix} - \begin{bmatrix} 15 & 9 \\ -3 & -6 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 22-15-7 & 9-9-0 \\ -3+3-0 & 1+6-7 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{R.H.S.}$$

We are given $A^2 - 3A - 7I = O$

$\Rightarrow \quad A^{-1}[A^2 - 3A - 7I] = A^{-1}O$

[Pre-multiplying both sides by A^{-1}]

$\Rightarrow \quad A^{-1}A \cdot A - 3A^{-1} \cdot A - 7A^{-1} I = O \qquad [A^{-1}O = O]$

$\Rightarrow \quad I \cdot A - 3I - 7A^{-1} I = O$

$\Rightarrow \quad A - 3I - 7A^{-1} = O$

$\Rightarrow \quad -7A^{-1} = 3I - A$

$$\Rightarrow \quad A^{-1} = \frac{1}{-7}[3I - A]$$

$$\Rightarrow \quad A^{-1} = \frac{1}{-7}\left[3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 5 & 3 \\ -1 & -2 \end{pmatrix}\right]$$

$$= \frac{1}{-7}\left[\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} - \begin{pmatrix} 5 & 3 \\ -1 & -2 \end{pmatrix}\right]$$

$$= \frac{1}{-7}\begin{bmatrix} 3-5 & 0-3 \\ 0+1 & 3+2 \end{bmatrix} = \frac{1}{-7}\begin{bmatrix} -2 & -3 \\ 1 & 5 \end{bmatrix}$$

Hence, $A^{-1} = -\frac{1}{7}\begin{bmatrix} -2 & -3 \\ 1 & 5 \end{bmatrix}$

Q12. Find the matrix A satisfying the matrix equation:

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} A \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Sol. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2}$

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}_{2\times 2} \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2} \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}_{2\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$$

$$\Rightarrow \quad \begin{bmatrix} 2a+c & 2b+d \\ 3a+2c & 3b+2d \end{bmatrix}_{2\times 2} \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}_{2\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$$

$$\Rightarrow \quad \begin{bmatrix} -6a-3c+10b+5d & 4a+2c-6b-3d \\ -9a-6c+15b+10d & 6a+4c-9b-6d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Equating the corresponding elements, we get,

$$-6a-3c+10b+5d = 1 \qquad \text{...(1)}$$

$$-9a-6c+15b+10d = 0 \qquad \text{...(2)}$$

$$4a+2c-6b-3d = 0 \qquad \text{...(3)}$$

$$6a+4c-9b-6d = 1 \qquad \text{...(4)}$$

Multiplying eq. (1) by 2 and subtracting eq. (2), we get,

$$\begin{array}{cccccccccc} - & 12a & - & \cancel{6c} & + & 20b & + & \cancel{10d} & = & 2 \\ & 9a & - & \cancel{6c} & + & 15b & + & \cancel{10d} & = & 0 \\ (+) & & (+) & & (-) & & (-) & & (-) & \\ \hline - & 3a & & & + & 5b & & & = & 2 \end{array}$$

$$-3a+5b = 2 \qquad \text{...(5)}$$

Now, multiplying eq. (3) by 2 and subtracting eq. (4), we get

$$\begin{array}{cccccccccc} & 8a & + & \cancel{4c} & - & 12b & - & \cancel{6d} & = & 0 \\ & 6a & + & \cancel{4c} & - & 9b & - & \cancel{6d} & = & 1 \\ (-) & & (-) & & (+) & & (+) & & (-) & \\ \hline & 2a & & & - & 3b & & & = & -1 \end{array}$$

$$2a-3b = -1 \qquad \text{...(6)}$$

Solving eq. (5) and (6) *i.e.,*

$$-3a+5b = 2$$

$$2a-3b = -1$$

$$\begin{array}{lcl} 2\times(-3a+5b=2) & \Rightarrow & -6a+10b = 4 \\ 3\times(2a-3b=-1) & \Rightarrow & 6a-9b = -3 \\ & \text{Adding} & b = 1 \end{array}$$

Putting the value of b in eq. (6), we get,

$$2a - 3\times 1 = -1$$

$$\Rightarrow \quad 2a-3=-1 \quad \Rightarrow \quad 2a = 3-1 \quad \Rightarrow \quad 2a = 2$$

$$\therefore \qquad a = 1$$

Now, putting the values of a and b in equations (1) and (3)

$$-6\times 1 - 3c + 10\times 1 + 5d = 1$$

$$\Rightarrow \qquad -6-3c+10+5d = 1$$

$\Rightarrow \qquad -3c+5d+4=1 \quad \Rightarrow -3c+5d=-3 \qquad ...(7)$

and from eq. (3)

$4\times1+2c-6\times1-3d=0; \Rightarrow 4+2c-6-3d=0$

$\Rightarrow \qquad -2+2c-3d=0 \quad \Rightarrow \quad 2c-3d=2 \qquad ...(8)$

Solving eq. (7) and (8) we get,

$2\times(-3c+5d=-3) \quad \Rightarrow \quad -6c+10d=-6$

$3\times(2c-3d=2) \quad \Rightarrow \quad 6c-9d=6$

Adding $\quad d=0$

Putting the value of d in eq. (8) we get,

$2c-3\times0=2 \quad \Rightarrow \quad 2c=2 \Rightarrow c=1$

Hence, $A=\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}$.

Q13. Find A, if $\begin{bmatrix}4\\1\\3\end{bmatrix}A=\begin{bmatrix}-4 & 8 & 4\\-1 & 2 & 1\\-3 & 6 & 3\end{bmatrix}$

Sol. Order of $\begin{bmatrix}4\\1\\3\end{bmatrix}$ is 3×1 and order of $\begin{bmatrix}-4 & 8 & 4\\-1 & 2 & 1\\-3 & 6 & 3\end{bmatrix}$ is 3×3. So, the order of matrix A must be 1×3.

Let $A=\begin{bmatrix}a & b & c\end{bmatrix}_{1\times3}$

$$\begin{bmatrix}4\\1\\3\end{bmatrix}_{3\times1}\begin{bmatrix}a & b & c\end{bmatrix}_{1\times3}=\begin{bmatrix}-4 & 8 & 4\\-1 & 2 & 1\\-3 & 6 & 3\end{bmatrix}_{3\times3}$$

$$\Rightarrow \qquad \begin{bmatrix}4a & 4b & 4c\\a & b & c\\3a & 3b & 3c\end{bmatrix}=\begin{bmatrix}-4 & 8 & 4\\-1 & 2 & 1\\-3 & 6 & 3\end{bmatrix}$$

Equating the corresponding elements to get the values of a, b and c

$4a=-4, \qquad 4b=8, \qquad 4c=4$

$\therefore \quad a=-1 \qquad \therefore \quad b=2 \qquad \therefore \ c=1$

Hence, matrix $A=\begin{bmatrix}-1 & 2 & 1\end{bmatrix}$

Q14. If $A=\begin{bmatrix}3 & -4\\1 & 1\\2 & 0\end{bmatrix}$ and $B=\begin{bmatrix}2 & 1 & 2\\1 & 2 & 4\end{bmatrix}$, then verify $(BA)^2\neq B^2A^2$.

Sol. Here, $B = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}_{2\times3}$ and $A = \begin{bmatrix} 3 & -4 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}_{3\times2}$

$$\therefore \quad BA = \begin{bmatrix} 6+1+4 & -8+1+0 \\ 3+2+8 & -4+2+0 \end{bmatrix}_{2\times2} \Rightarrow BA = \begin{bmatrix} 11 & -7 \\ 13 & -2 \end{bmatrix}$$

L.H.S. $(BA)^2 = (BA)\cdot(BA) = \begin{bmatrix} 11 & -7 \\ 13 & -2 \end{bmatrix}\begin{bmatrix} 11 & -7 \\ 13 & -2 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 121-91 & -77+14 \\ 143-26 & -91+4 \end{bmatrix} \Rightarrow \begin{bmatrix} 30 & -63 \\ 117 & -87 \end{bmatrix}$$

R.H.S. $B^2 = B\cdot B = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}_{2\times3} \cdot \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}_{2\times3}$

Here, number of columns of first *i.e.*, 3 is not equal to the number of rows of second matrix *i.e.*, 2.

So, B^2 is not possible. Similarly, A^2 is also not possible.

Hence, $(BA)^2 \neq B^2A^2$

Q15. If possible, find BA and AB, where

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}, B = \begin{bmatrix} 4 & 1 \\ 2 & 3 \\ 1 & 2 \end{bmatrix}$$

Sol. $BA = \begin{bmatrix} 4 & 1 \\ 2 & 3 \\ 1 & 2 \end{bmatrix}_{3\times2} \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}_{2\times3}$

$$BA = \begin{bmatrix} 8+1 & 4+2 & 8+4 \\ 4+3 & 2+6 & 4+12 \\ 2+2 & 1+4 & 2+8 \end{bmatrix}_{3\times3} = \begin{bmatrix} 9 & 6 & 12 \\ 7 & 8 & 16 \\ 4 & 5 & 10 \end{bmatrix}_{3\times3}$$

Now $AB = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}_{2\times3} \begin{bmatrix} 4 & 1 \\ 2 & 3 \\ 1 & 2 \end{bmatrix}_{3\times2}$

$$= \begin{bmatrix} 8+2+2 & 2+3+4 \\ 4+4+4 & 1+6+8 \end{bmatrix}_{2\times2} = \begin{bmatrix} 12 & 9 \\ 12 & 15 \end{bmatrix}_{2\times2}$$

Hence, $\quad BA = \begin{bmatrix} 9 & 6 & 12 \\ 7 & 8 & 16 \\ 4 & 5 & 10 \end{bmatrix}$ and $AB = \begin{bmatrix} 12 & 9 \\ 12 & 15 \end{bmatrix}$.

Q16. Show by an example that for $A \neq O$ and $B \neq O$, $AB = O$.

Sol. Let $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\Rightarrow \qquad AB = \begin{bmatrix} 1-1 & 1-1 \\ -1+1 & -1+1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

Hence, $\quad A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

Q17. Given $A = \begin{bmatrix} 2 & 4 & 0 \\ 3 & 9 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ 2 & 8 \\ 1 & 3 \end{bmatrix}$. Is $(AB)' = B'A'$?

Sol. Here, $A = \begin{bmatrix} 2 & 4 & 0 \\ 3 & 9 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 4 \\ 2 & 8 \\ 1 & 3 \end{bmatrix}$

$$AB = \begin{bmatrix} 2 & 4 & 0 \\ 3 & 9 & 6 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 8 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2+8+0 & 8+32+0 \\ 3+18+6 & 12+72+18 \end{bmatrix} = \begin{bmatrix} 10 & 40 \\ 27 & 102 \end{bmatrix}$$

L.H.S. $(AB)' = \begin{bmatrix} 10 & 27 \\ 40 & 102 \end{bmatrix}$

Now $\quad B = \begin{bmatrix} 1 & 4 \\ 2 & 8 \\ 1 & 3 \end{bmatrix} \quad \Rightarrow \quad B' = \begin{bmatrix} 1 & 2 & 1 \\ 4 & 8 & 3 \end{bmatrix}$

$A = \begin{bmatrix} 2 & 4 & 0 \\ 3 & 9 & 6 \end{bmatrix} \quad \Rightarrow \quad A' = \begin{bmatrix} 2 & 3 \\ 4 & 9 \\ 0 & 6 \end{bmatrix}$

R.H.S. $\quad B'A' = \begin{bmatrix} 1 & 2 & 1 \\ 4 & 8 & 3 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 4 & 9 \\ 0 & 6 \end{bmatrix}$

$$= \begin{bmatrix} 2+8+0 & 3+18+6 \\ 8+32+0 & 12+72+18 \end{bmatrix} = \begin{bmatrix} 10 & 27 \\ 40 & 102 \end{bmatrix} = \text{L.H.S.}$$

Hence, L.H.S. = R.H.S.

Q18. Solve for x and y: $x\begin{bmatrix} 2 \\ 1 \end{bmatrix} + y\begin{bmatrix} 3 \\ 5 \end{bmatrix} + \begin{bmatrix} -8 \\ -11 \end{bmatrix} = O$

Sol. Given that: $x\begin{bmatrix} 2 \\ 1 \end{bmatrix} + y\begin{bmatrix} 3 \\ 5 \end{bmatrix} + \begin{bmatrix} -8 \\ 11 \end{bmatrix} =$

L.H.S. $x\begin{bmatrix} 2 \\ 1 \end{bmatrix} + y\begin{bmatrix} 3 \\ 5 \end{bmatrix} + \begin{bmatrix} -8 \\ -11 \end{bmatrix} = O$

$$\Rightarrow \quad \begin{bmatrix} 2x \\ x \end{bmatrix} + \begin{bmatrix} 3y \\ 5y \end{bmatrix} + \begin{bmatrix} -8 \\ -11 \end{bmatrix} = O \Rightarrow \begin{bmatrix} 2x+3y-8 \\ x+5y-11 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Comparing the corresponding elements of both sides, we get,

$$2x+3y-8=0 \quad \Rightarrow \quad 2x+3y=8 \qquad \text{...(1)}$$

$$x+5y-11=0 \quad \Rightarrow \quad x+5y=11 \qquad \text{...(2)}$$

Multiplying eq. (1) by 1 and eq. (2) by 2, and then on subtracting, we get,

$$\begin{array}{rcrcr} 2x & + & 3y & = & 8 \\ 2x & + & 10y & = & 22 \\ (-) & & (-) & & (-) \\ \hline & & -7y & = & -14 \end{array}$$

$\therefore \quad y = 2$

Putting y = 2 in eq. (2) we get,

$$x + 5 \times 2 = 11 \Rightarrow x + 10 = 11$$

$$\therefore \qquad x = 11 - 10 = 1$$

Hence, the values of x and y are 1 and 2 respectively.

Q19. If X and Y are 2×2 matrices, then solve the following matrix equations for X and Y.

$$2X + 3Y = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}, 3X + 2Y = \begin{bmatrix} -2 & 2 \\ 1 & -5 \end{bmatrix}.$$

Sol. Given that:

$$2X + 3Y = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} \qquad \text{...(1)}$$

$$3X + 2Y = \begin{bmatrix} -2 & 2 \\ 1 & -5 \end{bmatrix} \qquad \text{...(2)}$$

Multiplying eq. (1) by 3 and eq. (2) by 2, we get,

$$3[2X+3Y] = 3\begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} \Rightarrow 6X+9Y = \begin{bmatrix} 6 & 9 \\ 12 & 0 \end{bmatrix} \quad ...(3)$$

$$2[3X+2Y] = 2\begin{bmatrix} -2 & 2 \\ 1 & -5 \end{bmatrix} \Rightarrow 6X+4Y = \begin{bmatrix} -4 & 4 \\ 2 & -10 \end{bmatrix} \quad ...(4)$$

On subtracting eq. (4) from eq. (3) we get

$$5Y = \begin{bmatrix} 6+4 & 9-4 \\ 12-2 & 0+10 \end{bmatrix}$$

$$5Y = \begin{bmatrix} 10 & 5 \\ 10 & 10 \end{bmatrix} \Rightarrow Y = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}$$

Now, putting the value of Y in equation (1) we get,

$$2X + 3\begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}$$

$$\Rightarrow 2X + \begin{bmatrix} 6 & 3 \\ 6 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} \Rightarrow 2X = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} - \begin{bmatrix} 6 & 3 \\ 6 & 6 \end{bmatrix}$$

$$\Rightarrow 2X = \begin{bmatrix} 2-6 & 3-3 \\ 4-6 & 0-6 \end{bmatrix} \Rightarrow 2X = \begin{bmatrix} -4 & 0 \\ -2 & -6 \end{bmatrix}$$

$$\Rightarrow X = \frac{1}{2}\begin{bmatrix} -4 & 0 \\ -2 & -6 \end{bmatrix} \Rightarrow X = \begin{bmatrix} -2 & 0 \\ -1 & -3 \end{bmatrix}$$

Hence, $X = \begin{bmatrix} -2 & 0 \\ -1 & -3 \end{bmatrix}$ and $Y = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}$.

Q20. If $A = [3 \quad 5]$, $B = [7 \quad 3]$, then find a non-zero matrix C such that $AC = BC$.

Sol. Given that: $A = [3 \quad 5]_{1\times 2}$, $B = [7 \quad 3]_{1\times 2}$

Let $C = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{2\times 1}$

$$AC = [3 \quad 5]_{1\times 2}\begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{2\times 1} = [3\alpha + 5\beta]$$

$$BC = [7 \quad 3]_{1\times 2}\begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{2\times 1} = [7\alpha + 3\beta]$$

$AC = BC$ (Given)

$\Rightarrow [3\alpha + 5\beta] = [7\alpha + 3\beta]$

$\Rightarrow 3\alpha + 5\beta = 7\alpha + 3\beta$

$$\Rightarrow \quad 3\alpha - 7\alpha = 3\beta - 5\beta$$

$$\Rightarrow \quad -4\alpha = -2\beta$$

$$\therefore \quad \frac{\alpha}{\beta} = \frac{1}{2}$$

So, let $\alpha = K$ and $\beta = 2K$, K is some real number.

Hence, matrix $C = \begin{bmatrix} K \\ 2K \end{bmatrix}_{2\times 1}$ or $\begin{bmatrix} K & K \\ 2K & 2K \end{bmatrix}_{2\times 2}$ etc.

Q21. Give an example of matrices A, B and C such that AB = AC, where A is non-zero matrix, but $B \neq C$.

Sol. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} \Rightarrow AB = \begin{bmatrix} 1+0 & 2+0 \\ 0+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1+0 & 2+0 \\ 0+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

Hence, AB = AC for matrix A is non-zero and $B \neq C$.

Q22. If $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 3 \\ 3 & -4 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$

verify: (*i*) (AB)C = A(BC) (*ii*) A(B + C) = AB + AC

Sol. Given that $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 3 \\ 3 & -4 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$

(*i*) To verify: (AB)C = A(BC)

$$AB = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 2+6 & 3-8 \\ -4+3 & -6-4 \end{bmatrix} = \begin{bmatrix} 8 & -5 \\ -1 & -10 \end{bmatrix}$$

L.H.S.

$$(AB)C = \begin{bmatrix} 8 & -5 \\ -1 & -10 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 8+5 & 0+0 \\ -1+10 & 0+0 \end{bmatrix} = \begin{bmatrix} 13 & 0 \\ 9 & 0 \end{bmatrix}$$

$$BC = \begin{bmatrix} 2 & 3 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 2-3 & 0+0 \\ 3+4 & 0+0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 7 & 0 \end{bmatrix}$$

R.H.S.

$$A(BC) = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 7 & 0 \end{bmatrix} = \begin{bmatrix} -1+14 & 0+0 \\ 2+7 & 0+0 \end{bmatrix} = \begin{bmatrix} 13 & 0 \\ 9 & 0 \end{bmatrix}$$

L.H.S. = R.H.S.

So, (AB)C = A(BC)

(*ii*) To verify: A(B + C) = AB + AC

$$\text{L.H.S.} \quad B+C = \begin{bmatrix} 2 & 3 \\ 3 & -4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 2+1 & 3+0 \\ 3-1 & -4+0 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 2 & -4 \end{bmatrix}$$

$$\text{L.H.S.} A(B+C) = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 3 \\ 2 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 3+4 & 3-8 \\ -6+2 & -6-4 \end{bmatrix} = \begin{bmatrix} 7 & -5 \\ -4 & -10 \end{bmatrix}$$

$$\text{R.H.S.} \quad AB = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 3 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 2+6 & 3-8 \\ -4+3 & -6-4 \end{bmatrix} = \begin{bmatrix} 8 & -5 \\ -1 & -10 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1-2 & 0+0 \\ -2-1 & 0+0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ -3 & 0 \end{bmatrix}$$

$$\text{R.H.S.} \quad AB+AC = \begin{bmatrix} 8 & -5 \\ -1 & -10 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ -3 & 0 \end{bmatrix} = \begin{bmatrix} 8-1 & -5+0 \\ -1-3 & -10+0 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -5 \\ -4 & -10 \end{bmatrix}$$

L.H.S. = R.H.S.

Hence, A(B + C) = AB + AC

Q23. If $P = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ and $Q = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$, prove that

$$PQ = \begin{bmatrix} xa & 0 & 0 \\ 0 & yb & 0 \\ 0 & 0 & zc \end{bmatrix} = QP$$

Sol. Given that: $P = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ and $Q = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

$$PQ = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

$$PQ = \begin{bmatrix} xa+0+0 & 0+0+0 & 0+0+0 \\ 0+0+0 & 0+yb+0 & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+0+zc \end{bmatrix}$$

$$PQ = \begin{bmatrix} xa & 0 & 0 \\ 0 & yb & 0 \\ 0 & 0 & zc \end{bmatrix}$$

Now $$QP = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$$

$$QP = \begin{bmatrix} xa+0+0 & 0+0+0 & 0+0+0 \\ 0+0+0 & 0+yb+0 & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+0+zc \end{bmatrix}$$

$$QP = \begin{bmatrix} xa & 0 & 0 \\ 0 & yb & 0 \\ 0 & 0 & zc \end{bmatrix}$$

Hence, $PQ = QP$.

Q24. If $\begin{bmatrix} 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} -1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = A$ find A.

Sol. Given that: $\begin{bmatrix} 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} -1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = A$

L.H.S. $$\begin{bmatrix} 2 & 1 & 3 \end{bmatrix}_{1\times 3}\begin{bmatrix} -1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}_{3\times 3}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}_{3\times 1}$$

$$\Rightarrow \quad \begin{bmatrix} -2-1+0 & 0+1+3 & -2+0+3 \end{bmatrix}_{1\times3} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}_{3\times1}$$

$$\Rightarrow \quad \begin{bmatrix} -3 & 4 & 1 \end{bmatrix}_{1\times3} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}_{3\times1}$$

$$\Rightarrow \quad \begin{bmatrix} -3+0-1 \end{bmatrix}_{1\times1} = \begin{bmatrix} -4 \end{bmatrix}_{1\times1}$$

Hence, matrix $A = [-4]$

Q25. If $A = \begin{bmatrix} 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 5 & 3 & 4 \\ 8 & 7 & 6 \end{bmatrix}$ and $C = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix}$, verify that $A(B+C) = (AB+AC)$.

Sol. Given that: $A = \begin{bmatrix} 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 5 & 3 & 4 \\ 8 & 7 & 6 \end{bmatrix}$ and $C = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix}$.

L.H.S. $\quad (B+C) = \begin{bmatrix} 5 & 3 & 4 \\ 8 & 7 & 6 \end{bmatrix} + \begin{bmatrix} -1 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 5-1 & 3+2 & 4+1 \\ 8+1 & 7+0 & 2+6 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 5 \\ 9 & 7 & 8 \end{bmatrix}$$

$$A(B+C) = \begin{bmatrix} 2 & 1 \end{bmatrix}_{1\times2} \begin{bmatrix} 4 & 5 & 5 \\ 9 & 7 & 8 \end{bmatrix}_{2\times3}$$

$$= \begin{bmatrix} 8+9 & 10+7 & 10+8 \end{bmatrix}_{1\times3}$$

$$A(B+C) = \begin{bmatrix} 17 & 17 & 18 \end{bmatrix}$$

R.H.S. $\quad AB = \begin{bmatrix} 2 & 1 \end{bmatrix}_{1\times2} \begin{bmatrix} 5 & 3 & 4 \\ 8 & 7 & 6 \end{bmatrix}_{2\times3}$

$$= \begin{bmatrix} 10+8 & 6+7 & 8+6 \end{bmatrix}_{1\times3} = \begin{bmatrix} 18 & 13 & 14 \end{bmatrix}_{1\times3}$$

$$AC = \begin{bmatrix} 2 & 1 \end{bmatrix}_{1\times2} \begin{bmatrix} -1 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix}_{2\times3}$$

$$= \begin{bmatrix} -2+1 & 4+0 & 2+2 \end{bmatrix}_{1\times3} = \begin{bmatrix} -1 & 4 & 4 \end{bmatrix}_{1\times3}$$

$$AB + AC = \begin{bmatrix} 18 & 13 & 14 \end{bmatrix}_{1\times3} + \begin{bmatrix} -1 & 4 & 4 \end{bmatrix}_{1\times3}$$

$$= \begin{bmatrix} 18-1 & 13+4 & 14+4 \end{bmatrix}_{1\times3}$$

$$AB + AC = \begin{bmatrix} 17 & 17 & 18 \end{bmatrix}_{1\times3}$$

L.H.S. = R.H.S.

Hence, $A(B+C) = (AB+AC)$ is verified.

Q26. If $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix}$ then verify that $A^2 + A = A(A + I)$, where I is 3×3 unit matrix.

Sol. Given that: $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix}$

$$A^2 = A \cdot A$$

$$= \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+0 & 0+0-1 & -1+0-1 \\ 2+2+0 & 0+1+3 & -2+3+3 \\ 0+2+0 & 0+1+1 & 0+3+1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -2 \\ 4 & 4 & 4 \\ 2 & 2 & 4 \end{bmatrix}$$

L.H.S. $\quad A^2 + A = \begin{bmatrix} 1 & -1 & -2 \\ 4 & 4 & 4 \\ 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1+1 & -1+0 & -2-1 \\ 4+2 & 4+1 & 4+3 \\ 2+0 & 2+1 & 4+1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -3 \\ 6 & 5 & 7 \\ 2 & 3 & 5 \end{bmatrix}$$

R.H.S. $\quad A(A + I) = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix}\left[\begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]$

$$= \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 & -1 \\ 2 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2+0+0 & 0+0-1 & -1+0-2 \\ 4+2+0 & 0+2+3 & -2+3+6 \\ 0+2+0 & 0+2+1 & 0+3+2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -3 \\ 6 & 5 & 7 \\ 2 & 3 & 5 \end{bmatrix}$$

L.H.S. = R.H.S.

$A^2 + A = A(A + I)$. Hence verified.

Q27. If $A=\begin{bmatrix}0 & -1 & 2\\ 4 & 3 & -4\end{bmatrix}$ and $B=\begin{bmatrix}4 & 0\\ 1 & 3\\ 2 & 6\end{bmatrix}$, then verify that:

(*i*) $(A')'=A$ (*ii*) $(AB)'=B'A'$ (*iii*) $(kA)'=(kA')$

Sol. Given that: $A=\begin{bmatrix}0 & -1 & 2\\ 4 & 3 & -4\end{bmatrix}$, $B=\begin{bmatrix}4 & 0\\ 1 & 3\\ 2 & 6\end{bmatrix}$

(*i*) $$A' = \begin{bmatrix}0 & -1 & 2\\ 4 & 3 & -4\end{bmatrix}'_{2\times 3} = \begin{bmatrix}0 & 4\\ -1 & 3\\ 2 & -4\end{bmatrix}_{3\times 2}$$

$$(A')' = \begin{bmatrix}0 & 4\\ -1 & 3\\ 2 & -4\end{bmatrix}'_{3\times 2} = \begin{bmatrix}0 & -1 & 2\\ 4 & 3 & -4\end{bmatrix}_{2\times 3} = A$$

Hence, $(A')'=A$

(*ii*) L.H.S. $$AB = \begin{bmatrix}0 & -1 & 2\\ 4 & 3 & -4\end{bmatrix}_{2\times 3}\begin{bmatrix}4 & 0\\ 1 & 3\\ 2 & 6\end{bmatrix}_{3\times 2}$$

$$= \begin{bmatrix}0-1+4 & 0-3+12\\ 16+3-8 & 0+9-24\end{bmatrix}_{2\times 2} = \begin{bmatrix}3 & 9\\ 11 & -15\end{bmatrix}_{2\times 2}$$

$$(AB)' = \begin{bmatrix}3 & 11\\ 9 & -15\end{bmatrix}_{2\times 2}$$

R.H.S. $$B' = \begin{bmatrix}4 & 0\\ 1 & 3\\ 2 & 6\end{bmatrix}' = \begin{bmatrix}4 & 1 & 2\\ 0 & 3 & 6\end{bmatrix}$$

$$A' = \begin{bmatrix}0 & -1 & 2\\ 4 & 3 & -4\end{bmatrix}' = \begin{bmatrix}0 & 4\\ -1 & 3\\ 2 & -4\end{bmatrix}$$

$$B'A' = \begin{bmatrix}4 & 1 & 2\\ 0 & 3 & 6\end{bmatrix}_{2\times 3}\begin{bmatrix}0 & 4\\ -1 & 3\\ 2 & -4\end{bmatrix}_{3\times 2}$$

$$= \begin{bmatrix}0-1+4 & 16+3-8\\ 0-3+12 & 0+9-24\end{bmatrix}_{2\times 2} = \begin{bmatrix}3 & 11\\ 9 & -15\end{bmatrix}_{2\times 2}$$

L.H.S. = R.H.S.

Hence, $(AB)'=B'A'$ is verified.

(*iii*) L.H.S. $kA = k\begin{bmatrix} 0 & -1 & 2 \\ 4 & 3 & -4 \end{bmatrix} = \begin{bmatrix} 0 & -k & 2k \\ 4k & 3k & -4k \end{bmatrix}$

$$(kA)' = \begin{bmatrix} 0 & 4k \\ -k & 3k \\ 2k & -4k \end{bmatrix}$$

R.H.S. $kA' = k\begin{bmatrix} 0 & -1 & 2 \\ 4 & 3 & -4 \end{bmatrix}' = k\begin{bmatrix} 0 & 4 \\ -1 & 3 \\ 2 & -4 \end{bmatrix} = \begin{bmatrix} 0 & 4k \\ -k & 3k \\ 2k & -4k \end{bmatrix}$

Hence, L.H.S. = R.H.S.

$(kA)' = (kA')$ is verified.

Q28. If $A = \begin{bmatrix} 1 & 2 \\ 4 & 1 \\ 5 & 6 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 6 & 4 \\ 7 & 3 \end{bmatrix}$, then verify that:

(*i*) $(2A + B)' = 2A' + B'$ (*ii*) $(A - B)' = A' - B'$

Sol. Given that: $A = \begin{bmatrix} 1 & 2 \\ 4 & 1 \\ 5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 6 & 4 \\ 7 & 3 \end{bmatrix}$

(*i*) To verify that: $(2A + B)' = 2A' + B'$

L.H.S. $(2A + B)' = \left[2\begin{pmatrix} 1 & 2 \\ 4 & 1 \\ 5 & 6 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 6 & 4 \\ 7 & 3 \end{pmatrix}\right]' = \left[\begin{pmatrix} 2 & 4 \\ 8 & 2 \\ 10 & 12 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 6 & 4 \\ 7 & 3 \end{pmatrix}\right]'$

$$= \begin{bmatrix} 2+1 & 4+2 \\ 8+6 & 2+4 \\ 10+7 & 12+3 \end{bmatrix}' = \begin{bmatrix} 3 & 6 \\ 14 & 6 \\ 17 & 15 \end{bmatrix}' = \begin{bmatrix} 3 & 14 & 17 \\ 6 & 6 & 15 \end{bmatrix}$$

R.H.S. $2A' + B' = 2\begin{bmatrix} 1 & 2 \\ 4 & 1 \\ 5 & 6 \end{bmatrix}' + \begin{bmatrix} 1 & 2 \\ 6 & 4 \\ 7 & 3 \end{bmatrix}'$

$$= 2\begin{bmatrix} 1 & 4 & 5 \\ 2 & 1 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 6 & 7 \\ 2 & 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 8 & 10 \\ 4 & 2 & 12 \end{bmatrix} + \begin{bmatrix} 1 & 6 & 7 \\ 2 & 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2+1 & 8+6 & 10+7 \\ 4+2 & 2+4 & 12+3 \end{bmatrix} = \begin{bmatrix} 3 & 14 & 17 \\ 6 & 6 & 15 \end{bmatrix}$$

Hence, L.H.S. = R.H.S.

$(2A + B)' = 2A' + B'$ is verified.

(*ii*) To verify that: $(A - B)' = A' - B'$

$$\text{L.H.S. } (A-B)' = \left[\begin{pmatrix} 1 & 2 \\ 4 & 1 \\ 5 & 6 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 6 & 4 \\ 7 & 3 \end{pmatrix}\right]'$$

$$= \begin{bmatrix} 1-1 & 2-2 \\ 4-6 & 1-4 \\ 5-7 & 6-3 \end{bmatrix}' = \begin{bmatrix} 0 & 0 \\ -2 & -3 \\ -2 & 3 \end{bmatrix}' = \begin{bmatrix} 0 & -2 & -2 \\ 0 & -3 & 3 \end{bmatrix}$$

$$\text{R.H.S. } A' - B' = \begin{bmatrix} 1 & 2 \\ 4 & 1 \\ 5 & 6 \end{bmatrix}' - \begin{bmatrix} 1 & 2 \\ 6 & 4 \\ 7 & 3 \end{bmatrix}' = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 1 & 6 \end{bmatrix} - \begin{bmatrix} 1 & 6 & 7 \\ 2 & 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1-1 & 4-6 & 5-7 \\ 2-2 & 1-4 & 6-3 \end{bmatrix} = \begin{bmatrix} 0 & -2 & -2 \\ 0 & -3 & 3 \end{bmatrix}$$

Hence, L.H.S. = R.H.S.

$(A - B)' = A' - B'$ is verified.

Q29. Show that $A'A$ and AA' are both symmetric matrices for any matrix A.

Sol. Let $P = A'A$

$\Rightarrow$ $P' = (A'A)'$

$\Rightarrow$ $P' = A'(A')'$ $\qquad [(AB)' = B'A']$

$\Rightarrow$ $P' = A'A$ $\qquad [\because (A')' = A]$

$\Rightarrow$ $P' = P$

Hence, $A'A$ is a symmetric matrix.

Now, Let $Q = AA'$

$\Rightarrow$ $Q' = (AA')'$

$\Rightarrow$ $Q' = (A')'A'$ $\qquad [(AB)' = B'A']$

$\Rightarrow$ $Q' = AA'$ $\qquad [\because (A')' = A]$

$\Rightarrow$ $Q' = Q$

Hence, AA' is also a symmetric matrix.

Q30. Let A and B be square matrices of the order 3×3. Is $(AB)^2 = A^2B^2$? Give reasons.

Sol. Given that A and B are the matrices of the order 3×3.

$$(AB)^2 = AB \cdot AB$$
$$= AA \cdot BB$$
$$= A^2 \cdot B^2$$

Hence, $(AB)^2 = A^2B^2$

Q31. Show that if A and B are square matrices such that AB = BA then $(A+B)^2 = A^2 + 2AB + B^2$.

Sol. To prove that $(A+B)^2 = A^2 + 2AB + B^2$

L.H.S. $(A+B)^2 = (A+B)\cdot(A+B)$ $\quad [\because A^2 = A\cdot A]$

$= A\cdot A + AB + BA + B\cdot B$

$= A^2 + AB + AB + B^2$ $\quad [AB = BA]$

$= A^2 + 2AB + B^2$ R.H.S.

So, L.H.S. = R.H.S.

Q32. Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}, B = \begin{bmatrix} 4 & 0 \\ 1 & 5 \end{bmatrix}, C = \begin{bmatrix} 2 & 0 \\ 1 & -2 \end{bmatrix}$ and $a = 4$, $b = -2$.

Show that:

(*a*) $A + (B + C) = (A + B) + C$
(*b*) $A(BC) = (AB)C$
(*c*) $(a + b)B = aB + bB$
(*d*) $a(C - A) = aC - aA$
(*e*) $(A^T)^T = A$
(*f*) $(bA)^T = bA^T$
(*g*) $(AB)^T = B^T A^T$
(*h*) $(A - B)C = AC - BC$
(*i*) $(A - B)^T = A^T - B^T$

Sol. (*a*) To prove that: $A + (B + C) = (A + B) + C$

L.H.S. $A + (B + C) = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} + \left[\begin{pmatrix} 4 & 0 \\ 1 & 5 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 1 & -2 \end{pmatrix}\right]$

$= \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} + \begin{bmatrix} 4+2 & 0+0 \\ 1+1 & 5-2 \end{bmatrix}$

$= \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} + \begin{bmatrix} 6 & 0 \\ 2 & 3 \end{bmatrix}$

$= \begin{bmatrix} 1+6 & 2+0 \\ -1+2 & 3+3 \end{bmatrix} = \begin{bmatrix} 7 & 2 \\ 1 & 6 \end{bmatrix}$

R.H.S. $(A + B) + C = \left[\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} + \begin{pmatrix} 4 & 0 \\ 1 & 5 \end{pmatrix}\right] + \begin{pmatrix} 2 & 0 \\ 1 & -2 \end{pmatrix}$

$= \begin{bmatrix} 1+4 & 2+0 \\ -1+1 & 3+5 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 1 & -2 \end{bmatrix}$

$= \begin{bmatrix} 5 & 2 \\ 0 & 8 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 1 & -2 \end{bmatrix}$

$= \begin{bmatrix} 5+2 & 2+0 \\ 0+1 & 8-2 \end{bmatrix} = \begin{bmatrix} 7 & 2 \\ 1 & 6 \end{bmatrix}$

Hence, L.H.S. = R.H.S.

$A + (B + C) = (A + B) + C$ Hence proved.

(*b*) To prove that: A(BC) = (AB)C

$$\text{L.H.S. } A(BC) = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\left[\begin{pmatrix} 4 & 0 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & -2 \end{pmatrix}\right]$$

$$= \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 8+0 & 0+0 \\ 2+5 & 0-10 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 8 & 0 \\ 7 & -10 \end{bmatrix}$$

$$= \begin{bmatrix} 8+14 & 0-20 \\ -8+21 & 0-30 \end{bmatrix} = \begin{bmatrix} 22 & -20 \\ 13 & -30 \end{bmatrix}$$

$$\text{R.H.S. } (AB)C = \left[\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 1 & 5 \end{pmatrix}\right]\begin{bmatrix} 2 & 0 \\ 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 4+2 & 0+10 \\ -4+3 & 0+15 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 10 \\ -1 & 15 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 12+10 & 0-20 \\ -2+15 & 0-30 \end{bmatrix} = \begin{bmatrix} 22 & -20 \\ 13 & -30 \end{bmatrix}$$

Hence, L.H.S. = R.H.S.

A(BC) = (AB)C Hence proved.

(*c*) To prove that: $(a+b)B = aB + bB$

Here, $a = 4$ and $b = -2$

$$\text{L.H.S. } (a+b)B = (4-2)\begin{bmatrix} 4 & 0 \\ 1 & 5 \end{bmatrix} = 2\begin{bmatrix} 4 & 0 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 2 & 10 \end{bmatrix}$$

$$\text{R.H.S. } aB + bB = 4\begin{bmatrix} 4 & 0 \\ 1 & 5 \end{bmatrix} - 2\begin{bmatrix} 4 & 0 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 16 & 0 \\ 4 & 20 \end{bmatrix} - \begin{bmatrix} 8 & 0 \\ 2 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} 16-8 & 0-0 \\ 4-2 & 20-10 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 2 & 10 \end{bmatrix}$$

Hence, L.H.S. = R.H.S.

$(a+b)B = aB + bB$ Hence proved.

(*d*) To prove that: $a(C - A) = aC - aA$

$$\text{L.H.S. } a(C-A) = 4\left[\begin{pmatrix} 2 & 0 \\ 1 & -2 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}\right]$$

$$= 4\begin{bmatrix} 2-1 & 0-2 \\ 1+1 & -2-3 \end{bmatrix} = 4\begin{bmatrix} 1 & -2 \\ 2 & -5 \end{bmatrix} = \begin{bmatrix} 4 & -8 \\ 8 & -20 \end{bmatrix}$$

R.H.S. $aC - aA = 4\begin{bmatrix} 2 & 0 \\ 1 & -2 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 8 & 0 \\ 4 & -8 \end{bmatrix} - \begin{bmatrix} 4 & 8 \\ -4 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 8-4 & 0-8 \\ 4+4 & -8-12 \end{bmatrix} = \begin{bmatrix} 4 & -8 \\ 8 & -20 \end{bmatrix}$$

Hence, L.H.S. = R.H.S.

$a(C - A) = aC - aA$ Hence proved.

(*e*) To prove that: $(A^T)^T = A$

L.H.S. $A^T = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}^T = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

$$(A^T)^T = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = A \quad \text{R.H.S.}$$

Hence, $(A^T)^T = A$

(*f*) To prove that: $(bA)^T = bA^T$

L.H.S. $(bA)^T = \left[-2\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}\right]^T = \begin{bmatrix} -2 & -4 \\ 2 & -6 \end{bmatrix}^T = \begin{bmatrix} -2 & 2 \\ -4 & -6 \end{bmatrix}$

R.H.S. $bA^T = -2\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}^T = -2\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ -4 & -6 \end{bmatrix}$

Hence, L.H.S. = R.H.S.

$(bA)^T = bA^T$ Hence proved.

(*g*) To prove that: $(AB)^T = B^TA^T$

L.H.S. $(AB)^T = \left[\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 1 & 5 \end{pmatrix}\right]^T$

$$= \begin{bmatrix} 4+2 & 0+10 \\ -4+3 & 0+15 \end{bmatrix}^T = \begin{bmatrix} 6 & 10 \\ -1 & 15 \end{bmatrix}^T = \begin{bmatrix} 6 & -1 \\ 10 & 15 \end{bmatrix}$$

R.H.S. $B^TA^T = \begin{bmatrix} 4 & 0 \\ 1 & 5 \end{bmatrix}^T \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}^T$

$$= \begin{bmatrix} 4 & 1 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 4+2 & -4+3 \\ 0+10 & 0+15 \end{bmatrix} = \begin{bmatrix} 6 & -1 \\ 10 & 15 \end{bmatrix}$$

Hence, L.H.S. = R.H.S.

$(AB)^T = B^TA^T$ Hence proved.

(*h*) To prove that: $(A-B)C = AC - BC$

L.H.S. $(A-B)C = \left[\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} - \begin{pmatrix} 4 & 0 \\ 1 & 5 \end{pmatrix}\right]\begin{bmatrix} 2 & 0 \\ 1 & -2 \end{bmatrix}$

$$= \begin{bmatrix} 1-4 & 2-0 \\ -1-1 & 3-5 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 2 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -6+2 & 0-4 \\ -4-2 & 0+4 \end{bmatrix} = \begin{bmatrix} -4 & -4 \\ -6 & 4 \end{bmatrix}$$

R.H.S. $AC - BC = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & -2 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & -2 \end{bmatrix}$

$$= \begin{bmatrix} 2+2 & 0-4 \\ -2+3 & 0-6 \end{bmatrix} - \begin{bmatrix} 8+0 & 0+0 \\ 2+5 & 0-10 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -4 \\ 1 & -6 \end{bmatrix} - \begin{bmatrix} 8 & 0 \\ 7 & -10 \end{bmatrix}$$

$$= \begin{bmatrix} 4-8 & -4-0 \\ 1-7 & -6+10 \end{bmatrix} = \begin{bmatrix} -4 & -4 \\ -6 & 4 \end{bmatrix}$$

Hence, L.H.S. = R.H.S.

$(A-B)C = AC - BC$

(*i*) To prove that: $(A-B)^T = A^T - B^T$

L.H.S. $(A-B)^T = \left[\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} - \begin{pmatrix} 4 & 0 \\ 1 & 5 \end{pmatrix}\right]^T$

$$= \begin{bmatrix} 1-4 & 2-0 \\ -1-1 & 3-5 \end{bmatrix}^T = \begin{bmatrix} -3 & 2 \\ -2 & -2 \end{bmatrix}^T = \begin{bmatrix} -3 & -2 \\ 2 & -2 \end{bmatrix}$$

R.H.S. $A^T - B^T = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}^T - \begin{bmatrix} 4 & 0 \\ 1 & 5 \end{bmatrix}^T = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 1 \\ 0 & 5 \end{bmatrix}$

$$= \begin{bmatrix} 1-4 & -1-1 \\ 2-0 & 3-5 \end{bmatrix} = \begin{bmatrix} -3 & -2 \\ 2 & -2 \end{bmatrix}$$

Hence, L.H.S. = R.H.S.

$(A-B)^T = A^T - B^T$ Hence proved.

Q33. If $A = \begin{bmatrix} \cos q & \sin q \\ -\sin q & \cos q \end{bmatrix}$, then show that

$$A^2 = \begin{bmatrix} \cos 2q & \sin 2q \\ -\sin 2q & \cos 2q \end{bmatrix}$$

Sol. Given that

$$A = \begin{bmatrix} \cos q & \sin q \\ -\sin q & \cos q \end{bmatrix}$$

$$A = A \cdot A = \begin{bmatrix} \cos q & \sin q \\ -\sin q & \cos q \end{bmatrix} \begin{bmatrix} \cos q & \sin q \\ -\sin q & \cos q \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2 q - \sin^2 q & \cos q \sin q + \sin q \cos q \\ \sin q \cos q - \cos q \sin q & -\sin^2 q + \cos^2 q \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2q & \sin 2q \\ -\sin 2q & \cos 2q \end{bmatrix} \quad \begin{bmatrix} \because \cos^2 A - \sin^2 A = \cos 2A \\ 2\sin A \cos A = \sin 2A \end{bmatrix}$$

Hence proved.

Q34. If $A = \begin{bmatrix} 0 & -x \\ x & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $x^2 = -1$, then show that $(A + B)^2 = A^2 + B^2$.

Sol. Given that: $A = \begin{bmatrix} 0 & -x \\ x & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

L.H.S. $\quad (A + B)^2 = (A + B) \cdot (A + B)$

$$= \left[\begin{pmatrix} 0 & -x \\ x & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] \cdot \left[\begin{pmatrix} 0 & -x \\ x & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]$$

$$= \begin{bmatrix} 0+0 & -x+1 \\ x+1 & 0+0 \end{bmatrix} \cdot \begin{bmatrix} 0+0 & -x+1 \\ x+1 & 0+0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -x+1 \\ x+1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & -x+1 \\ x+1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 + (-x+1)(x+1) & 0+0 \\ 0+0 & (x+1)(-x+1)+0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - x^2 & 0 \\ 0 & 1 - x^2 \end{bmatrix}$$

Put $x^2 = -1$ (given)

$$= \begin{bmatrix} 1+1 & 0 \\ 0 & 1+1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

R.H.S.

$$A^2 + B^2 = A \cdot A + B \cdot B$$

$$= \begin{bmatrix} 0 & -x \\ x & 0 \end{bmatrix}\begin{bmatrix} 0 & -x \\ x & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0-x^2 & 0+0 \\ 0+0 & -x^2+0 \end{bmatrix} + \begin{bmatrix} 0+1 & 0+0 \\ 0+0 & 1+0 \end{bmatrix}$$

$$= \begin{bmatrix} -x^2 & 0 \\ 0 & -x^2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -x^2+1 & 0+0 \\ 0+0 & -x^2+1 \end{bmatrix}$$

$$= \begin{bmatrix} -x^2+1 & 0 \\ 0 & -x^2+1 \end{bmatrix} = \begin{bmatrix} 1+1 & 0 \\ 0 & 1+1 \end{bmatrix} \quad [\because x^2 = -1]$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Hence, L.H.S. = R.H.S.

$$(A+B)^2 = A^2 + B^2$$

Q35. Verify that $A^2 = I$ when $A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$.

Sol. Given that: $A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$

L.H.S. $A^2 = A \cdot A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$

$$= \begin{bmatrix} 0+4-3 & 0-3+3 & 0+4-4 \\ 0-12+12 & 4+9-12 & -4-12+16 \\ 0-12+12 & 3+9-12 & -3-12+16 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I \quad \text{R.H.S.}$$

Hence, $A^2 = I$ is verified.

Q36. Prove by Mathematical Induction that $(A')^n = (A^n)'$, where $n \in N$ for any square matrix A.

Sol. To prove that $(A')^n = (A^n)'$

Let $P(n)$: $(A')^n = (A^n)'$

Step 1: Put $n = 1$, P(1): $A' = A'$ which is true for $n = 1$

Step 2: Put $n = K$, P(K): $(A')^K = (A^K)'$ Let it be true for $n = K$

Step 3: Put $n = K + 1$, P(K + 1): $(A')^{K+1} = (A^{K+1})'$

L.H.S.
$$\begin{aligned}(A')^{K+1} &= (A')^K \cdot (A') \\ &= (A^K)' \cdot (A') \quad \text{(From step 2)} \\ &= (A^K \cdot A)' \\ &= (A^{K+1})' \quad \text{R.H.S.}\end{aligned}$$

The given statement is true for P(K+1) whenever it is true for P(K), where $K \in N$.

Q37. Find inverse, by elementary row operations (if possible), of the following matrices.

(*i*) $\begin{bmatrix} 1 & 3 \\ -5 & 7 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix}$

Sol. (*i*) Let $A = \begin{bmatrix} 1 & 3 \\ -5 & 7 \end{bmatrix}$

$$|A| = 1 \times 7 - (-5) \times 3 = 7 + 15 = 22 \neq 0$$

So, A is invertible.

Let $A = IA$

$$\begin{bmatrix} 1 & 3 \\ -5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$$

$R_2 \to R_2 + 5R_1$

$$\Rightarrow \begin{bmatrix} 1 & 3 \\ 0 & 22 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix} A$$

$R_2 \to \frac{1}{22} R_2$

$$\Rightarrow \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{5}{22} & \frac{1}{22} \end{bmatrix} A$$

$R_1 \to R_1 - 3R_2$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{7}{22} & \frac{-3}{22} \\ \frac{5}{22} & \frac{1}{22} \end{bmatrix} A$$

So $$A^{-1} = \begin{bmatrix} \frac{7}{22} & \frac{-3}{22} \\ \frac{5}{22} & \frac{1}{22} \end{bmatrix} \Rightarrow \frac{1}{22}\begin{bmatrix} 7 & -3 \\ 5 & 1 \end{bmatrix}$$

Hence, inverse of $\begin{bmatrix} 1 & 3 \\ -5 & 7 \end{bmatrix}$ is $\frac{1}{22}\begin{bmatrix} 7 & -3 \\ 5 & 1 \end{bmatrix}$

(*ii*) Let $$A = \begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix}$$

$|A| = 1 \times 6 - (-3)(-2) = 6 - 6 = 0$

$|A| = 0$ so A is not invertible.

Hence, inverse of $\begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix}$ is not possible.

Q38. If $\begin{bmatrix} xy & 4 \\ z+6 & x+y \end{bmatrix} = \begin{bmatrix} 8 & w \\ 0 & 6 \end{bmatrix}$, then find the values of x, y, z and w.

Sol. Given that: $\begin{bmatrix} xy & 4 \\ z+6 & x+y \end{bmatrix} = \begin{bmatrix} 8 & w \\ 0 & 6 \end{bmatrix}$

Equating the corresponding elements,

$xy = 8, w = 4, z + 6 = 0 \Rightarrow z = -6, x + y = 6$

Now, solving $x + y = 6$ ...(*i*)

and $xy = 8$ ...(*ii*)

From eqn. (*i*), $y = 6 - x$ ...(*iii*)

Putting the value of y in eqn. (*ii*) we get,

$$x(6 - x) = 8 \Rightarrow 6x - x^2 = 8$$
$$\Rightarrow x^2 - 6x + 8 = 0 \Rightarrow x^2 - 4x - 2x + 8 = 0$$
$$\Rightarrow x(x-4) - 2(x-4) = 0 \Rightarrow (x-4)(x-2) = 0$$
$$\therefore x = 4, 2$$

From eqn. (*iii*), $y = 2, 4$.

Hence, $x = 4$ or 2, $y = 2$ or 4, $z = -6$ and $w = 4$.

Q39. If $A = \begin{bmatrix} 1 & 5 \\ 7 & 12 \end{bmatrix}$ and $B = \begin{bmatrix} 9 & 1 \\ 7 & 8 \end{bmatrix}$, find a matrix C such that $3A + 5B + 2C$ is a null matrix.

Sol. Order of matrices A and B is 2×2.

$\therefore$ Order of matrix C must be 2×2.

Let $$C = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$\therefore \quad 3A + 5B + 2C = 0$

$$\Rightarrow 3\begin{bmatrix} 1 & 5 \\ 7 & 12 \end{bmatrix} + 5\begin{bmatrix} 9 & 1 \\ 7 & 8 \end{bmatrix} + 2\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & 15 \\ 21 & 36 \end{bmatrix} + \begin{bmatrix} 45 & 5 \\ 35 & 40 \end{bmatrix} + \begin{bmatrix} 2a & 2b \\ 2c & 2d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3+45+2a & 15+5+2b \\ 21+35+2c & 36+40+2d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 48+2a & 20+2b \\ 56+2c & 76+2d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Equating the corresponding elements, we get,

$48 + 2a = 0 \Rightarrow 2a = -48 \Rightarrow a = -24$

$20 + 2b = 0 \Rightarrow 2b = -20 \Rightarrow b = -10$

$56 + 2c = 0 \Rightarrow 2c = -56 \Rightarrow c = -28$

$76 + 2d = 0 \Rightarrow 2d = -76 \Rightarrow d = -38$

Hence, $$C = \begin{bmatrix} -24 & -10 \\ -28 & -38 \end{bmatrix}$$

Q40. If $A = \begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}$, then find $A^2 - 5A - 14I$. Hence, find A^3.

Sol. Given that: $A = \begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}$

$$A^2 = A.A = \begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 9+20 & -15-10 \\ -12-8 & 20+4 \end{bmatrix} = \begin{bmatrix} 29 & -25 \\ -20 & 24 \end{bmatrix}$$

$$\therefore \quad A^2 - 5A - 14I = \begin{bmatrix} 29 & -25 \\ -20 & 24 \end{bmatrix} - 5\begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix} - 14\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 29 & -25 \\ -20 & 24 \end{bmatrix} - \begin{bmatrix} 15 & -25 \\ -20 & 10 \end{bmatrix} - \begin{bmatrix} 14 & 0 \\ 0 & 14 \end{bmatrix}$$

$$= \begin{bmatrix} 29 & -25 \\ -20 & 24 \end{bmatrix} - \begin{bmatrix} 29 & -25 \\ -20 & 24 \end{bmatrix}$$

$$= \begin{bmatrix} 29-29 & -25+25 \\ -20+20 & 24-24 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence, $A^2 - 5A - 14I = O$

Now, multiplying both sides by A, we get,

$A^2 . A - 5A . A - 14IA = OA$

$\Rightarrow \quad A^3 - 5A^2 - 14A = 0$

$\Rightarrow \quad A^3 = 5A^2 + 14A$

$$\Rightarrow \quad A^3 = 5\begin{bmatrix} 29 & -25 \\ -20 & 24 \end{bmatrix} + 14\begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 145 & -125 \\ -100 & 120 \end{bmatrix} + \begin{bmatrix} 42 & -70 \\ -56 & 28 \end{bmatrix}$$

$$= \begin{bmatrix} 145+42 & -125-70 \\ -100-56 & 120+28 \end{bmatrix} = \begin{bmatrix} 187 & -195 \\ -156 & 148 \end{bmatrix}$$

Hence, $A^3 = \begin{bmatrix} 187 & -195 \\ -156 & 148 \end{bmatrix}$

Q41. Find the values of a, b, c and d if

$$3\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & 6 \\ -1 & 2d \end{bmatrix} + \begin{bmatrix} 4 & a+b \\ c+d & 3 \end{bmatrix}.$$

Sol. Given that: $3\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & 6 \\ -1 & 2d \end{bmatrix} + \begin{bmatrix} 4 & a+b \\ c+d & 3 \end{bmatrix}$

$$\begin{bmatrix} 3a & 3b \\ 3c & 3d \end{bmatrix} = \begin{bmatrix} a+4 & 6+a+b \\ -1+c+d & 2d+3 \end{bmatrix}$$

Equating the corresponding elements, we get,

$3a = a + 4 \quad \Rightarrow 3a - a = 4 \quad \Rightarrow 2a = 4 \quad \Rightarrow a = 2$

$3b = 6 + a + b \quad \Rightarrow 3b - b - a = 6 \quad \Rightarrow 2b - a = 6 \quad \Rightarrow 2b - 2 = 6$

$\Rightarrow 2b = 8$

$\Rightarrow b = 4$

$3c = -1 + c + d \quad \Rightarrow 3c - c - d = -1 \Rightarrow 2c - d = -1$

and $3d = 2d + 3 \quad \Rightarrow 3d - 2d = 3 \quad \Rightarrow d = 3$

Now $\quad 2c - d = -1$

$\Rightarrow \quad 2c-3=-1 \quad \Rightarrow \quad 2c=3-1 \quad \Rightarrow 2c=2$

$\therefore \quad c=1$

$\therefore a=2, b=4, c=1$ and $d=3$.

Q42. Find the matrix A such that

$$\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix} A = \begin{bmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$$

Sol. Order of matrix $\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix}$ is 3×2 and the matrix

$\begin{bmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$ is 3×3

$\therefore$ Order of matrix A must be 2×3

Let $A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}_{2\times 3}$

So, $$\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$$

$$\begin{bmatrix} 2a-d & 2b-e & 2c-f \\ a+0 & b+0 & c+0 \\ -3a+4d & -3b+4e & -3c+4f \end{bmatrix} = \begin{bmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$$

Equating the corresponding elements, we get,

$2a-d=-1$ and $a=1 \Rightarrow 2\times 1-d=-1 \Rightarrow d=2+1 \Rightarrow d=3$

$2b-e=-8$ and $b=-2 \Rightarrow 2(-2)-e=-8 \Rightarrow -4-e=-8$

$\Rightarrow e=4$

$2c-f=-10$ and $c=-5 \Rightarrow 2(-5)-f=-10 \Rightarrow -10-f=-10$

$\Rightarrow f=0$

$a=1, b=-2, c=-5, d=3, e=4$ and $f=0$

Hence, $$A = \begin{bmatrix} 1 & -2 & -5 \\ 3 & 4 & 0 \end{bmatrix}.$$

Q43. If $A = \begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix}$, find $A^2+2A+7I$.

Sol. Given that: $A = \begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix}$

$$A^2 = A \cdot A = \begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 1+8 & 2+2 \\ 4+4 & 8+1 \end{bmatrix} = \begin{bmatrix} 9 & 4 \\ 8 & 9 \end{bmatrix}$$

$$A^2 + 2A + 7I = \begin{bmatrix} 9 & 4 \\ 8 & 9 \end{bmatrix} + 2\begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 4 \\ 8 & 9 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ 8 & 2 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 9+2+7 & 4+4+0 \\ 8+8+0 & 9+2+7 \end{bmatrix} = \begin{bmatrix} 18 & 8 \\ 16 & 18 \end{bmatrix}$$

Hence, $A^2 + 2A + 7I = \begin{bmatrix} 18 & 8 \\ 16 & 18 \end{bmatrix}$.

Q44. If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, and $A^{-1} = A'$, find value of α.

Sol. Here, $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

Given that: $A^{-1} = A'$

Pre-multiplying both sides by A

$AA^{-1} = AA'$

$\Rightarrow \quad I = AA'$ $\quad [\because AA^{-1} = I]$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & -\sin\alpha\cos\alpha + \sin\alpha\cos\alpha \\ -\sin\alpha\cos\alpha + \cos\alpha\sin\alpha & \sin^2\alpha + \cos^2\alpha \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence, it is true for all values of α.

Q45. If the matrix $\begin{bmatrix} 0 & a & 3 \\ 2 & b & -1 \\ c & 1 & 0 \end{bmatrix}$ is a skew symmetric matrix, find the values of a, b and c.

Sol. Let $A = \begin{bmatrix} 0 & a & 3 \\ 2 & b & -1 \\ c & 1 & 0 \end{bmatrix}$ $A' = \begin{bmatrix} 0 & 2 & c \\ a & b & 1 \\ 3 & -1 & 0 \end{bmatrix}$

For skew symmetric matrix, $A' = -A$.

$$\begin{bmatrix} 0 & 2 & c \\ a & b & 1 \\ 3 & -1 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & a & 3 \\ 2 & b & -1 \\ c & 1 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 0 & 2 & c \\ a & b & 1 \\ 3 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -a & -3 \\ -2 & -b & 1 \\ -c & -1 & 0 \end{bmatrix}$$

Equating the corresponding elements, we get

$a = -2$, $b = -b \Rightarrow 2b = 0 \Rightarrow b = 0$ and $c = -3$

Hence, $a = -2$, $b = 0$ and $c = -3$.

Q46. If $P(x) = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$, then show that

$P(x).P(y) = P(x+y) = P(y).P(x)$

Sol. Given that:

$$P(x) = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$$

$$P(y) = \begin{bmatrix} \cos y & \sin y \\ -\sin y & \cos y \end{bmatrix} \qquad \text{[Replacing } x \text{ by } y\text{]}$$

$$P(x).P(y) = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}\begin{bmatrix} \cos y & \sin y \\ -\sin y & \cos y \end{bmatrix}$$

$$= \begin{bmatrix} \cos x \cos y - \sin x \sin y & \cos x \sin y + \sin x \cos y \\ -\sin x \cos y - \cos x \sin y & -\sin x \sin y + \cos x \cos y \end{bmatrix}$$

$$= \begin{bmatrix} \cos(x+y) & \sin(x+y) \\ -\sin(x+y) & \cos(x+y) \end{bmatrix}$$

$$= P(x+y)$$

Now

$$P(y).P(x) = \begin{bmatrix} \cos y & \sin y \\ -\sin y & \cos y \end{bmatrix}\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$$

$$= \begin{bmatrix} \cos x \cos y - \sin x \sin y & \sin x \cos y + \cos x \sin y \\ -\cos x \sin y - \cos y \sin x & -\sin x \sin y + \cos x \cos y \end{bmatrix}$$

$$= \begin{bmatrix} \cos(x+y) & \sin(x+y) \\ -\sin(x+y) & \cos(x+y) \end{bmatrix}$$

$$= P(x+y)$$

Hence, $P(x).P(y) = P(x+y) = P(y).P(x)$.

Q47. If A is a square matrix such that $A^2 = A$, show that $(I + A)^3 = 7A + I$.

Sol. To show that: $(I + A)^3 = 7A + I$

L.H.S. $(I + A)^3 = I^3 + A^3 + 3I^2A + 3IA^2$

$\Rightarrow I + A^2.A + 3IA + 3IA^2$

$\Rightarrow I + A.A + 3IA + 3IA$ $\quad [\because A^2 = A]$

$\Rightarrow I + A^2 + 3IA + 3IA$

$\Rightarrow I + A + 3IA + 3IA$ $\quad [\because A^2 = A]$

$\Rightarrow I + A + 3A + 3A \Rightarrow 7A + I$ R.H.S.

L.H.S. = R.H.S. Hence, Proved.

Q48. If A and B are square matrices of the same order and B is a skew symmetric matrix, show that $A'BA$ is a skew symmetric.

Sol. Given that B is a skew symmetric matrix $\therefore B' = -B$

Let $P = A'BA$

$\Rightarrow$ $P' = (A'BA)'$

$= A'B'(A')'$ $\quad [(AB)' = B'A']$

$= A'(-B)A$

$= -A'BA = -P$

So $P' = -P$

Hence, $A'BA$ is a skew symmetric matrix.

LONG ANSWER TYPE QUESTIONS

Q49. If $AB = BA$ for any two square matrices, prove by mathematical induction that $(AB)^n = A^nB^n$.

Sol. Let $P(n) : (AB)^n = A^nB^n$

Step 1:

Put $n = 1$, $P(1) : AB = AB$ (True)

Step 2:

Put $n = k$, $P(k) : (AB)^k = A^kB^k$ (Let it be true for any $k \in N$)

Step 3:

Put $n = k+1$, $P(k+1) : (AB)^{k+1} = A^{k+1}B^{k+1}$

L.H.S. $(AB)^{k+1} = (AB)^k.AB$

$= A^kB^k.AB$ $\quad$ [from Step 2]

$= A^{k+1}B^{k+1}$ R.H.S.

L.H.S. = R.H.S.

Hence, if $P(n)$ is true for $P(k)$ then it is true for $P(k+1)$.

Q50. Find x, y, z if $A = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}$ satisfies $A' = A^{-1}$.

Sol. Given that: $A = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}$ and $A' = A^{-1}$

Pre-multiplying both sides by A we get,

$$AA' = AA^{-1}$$

$$\Rightarrow \quad AA' = I$$

$$\Rightarrow \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix} \begin{bmatrix} 0 & x & x \\ 2y & y & -y \\ z & -z & z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0+4y^2+z^2 & 0+2y^2-z^2 & 0-2y^2+z^2 \\ 0+2y^2-z^2 & x^2+y^2+z^2 & x^2-y^2-z^2 \\ 0-2y^2+z^2 & x^2-y^2-z^2 & x^2+y^2+z^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equating the corresponding elements, we get,

$$4y^2 + z^2 = 1 \qquad \text{...(i)}$$

$$2y^2 - z^2 = 0 \qquad \text{...(ii)}$$

Adding (*i*) and (*ii*) we get, $6y^2 = 1 \Rightarrow y^2 = \frac{1}{6} \Rightarrow y = \pm\frac{1}{\sqrt{6}}$

From eqn. (*i*), we get,

$$4y^2 + z^2 = 1$$

$$\Rightarrow 4\left(\frac{1}{\sqrt{6}}\right)^2 + z^2 = 1 \Rightarrow \frac{2}{3} + z^2 = 1 \Rightarrow z^2 = 1 - \frac{2}{3} = \frac{1}{3}$$

$$\therefore \qquad z = \pm\frac{1}{\sqrt{3}}$$

$$x^2 + y^2 + z^2 = 1 \qquad \text{...(iii)}$$

$$x^2 - y^2 - z^2 = 0 \qquad \text{...(iv)}$$

Adding (*iii*) and (*iv*) we get,

$$2x^2 = 1 \Rightarrow x^2 = \frac{1}{2} \Rightarrow x = \pm\frac{1}{\sqrt{2}}$$

Hence, $x = \pm\frac{1}{\sqrt{2}}$, $y = \pm\frac{1}{\sqrt{6}}$ and $z = \pm\frac{1}{\sqrt{3}}$.

Q51. If possible, using elementary row transformation, find the inverse of the following matrices.

(i) $\begin{bmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 3 & -3 \\ -1 & -2 & 2 \\ 1 & 1 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

Sol. (i) Here, $A = \begin{bmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{bmatrix}$ for elementary row transformation we put

$$A = IA$$

$$\begin{bmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_2 \to R_2 + R_1$

$$\begin{bmatrix} 2 & -1 & 3 \\ -3 & 2 & 4 \\ -3 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 2 & -1 & 3 \\ -3 & 2 & 4 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} A$$

$R_1 \to R_1 + R_2$

$$\begin{bmatrix} -1 & 1 & 7 \\ -3 & 2 & 4 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} A$$

$R_2 \to R_2 - 3R_1$

$$\begin{bmatrix} -1 & 1 & 7 \\ 0 & -1 & -17 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ -5 & -2 & 0 \\ -1 & -1 & 1 \end{bmatrix} A$$

$R_1 \to R_1 + R_2$ and $R_3 \to -1 \,.\, R_3$

$$\begin{bmatrix} -1 & 0 & -10 \\ 0 & -1 & -17 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -1 & 0 \\ -5 & -2 & 0 \\ 1 & 1 & -1 \end{bmatrix} A$$

$R_1 \to R_1 + 10R_3$ and $R_2 \to R_2 + 17R_3$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 9 & -10 \\ 12 & 15 & -17 \\ 1 & 1 & -1 \end{bmatrix} A$$

$R_1 \to -1.R_1$ and $R_2 \to -1.R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -7 & -9 & 10 \\ -12 & -15 & 17 \\ 1 & 1 & -1 \end{bmatrix} A$$

Hence, $$A^{-1} = \begin{bmatrix} -7 & -9 & 10 \\ -12 & -15 & 17 \\ 1 & 1 & -1 \end{bmatrix}$$

(*ii*) Here, $$A = \begin{bmatrix} 2 & 3 & -3 \\ -1 & -2 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

Put $$A = IA$$

$$\begin{bmatrix} 2 & 3 & -3 \\ -1 & -2 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_1 \to R_1 - 2R_3$ and $R_2 \to R_2 + R_1$

$$\begin{bmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_1 \to R_1 + R_2$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} A$$

First row on L.H.S. contains all zeros, so the inverse of the given matrix A does not exist.
Hence, matrix A has no inverse.

(*iii*) Here, $$A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

Put $$A = IA$$

$$\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_1 \to 3R_1 - R_2$

$$\begin{bmatrix} 1 & -1 & -3 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_2 \to R_2 - 5R_1$

$$\begin{bmatrix} 1 & -1 & -3 \\ 0 & 6 & 15 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 0 \\ -15 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_2 \to R_2 - 5R_3$

$$\begin{bmatrix} 1 & -1 & -3 \\ 0 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 0 \\ -15 & 6 & -5 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & -1 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 0 \\ -15 & 6 & -5 \\ 15 & -6 & 6 \end{bmatrix} A$$

$R_1 \to R_1 + R_2$

$$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} -12 & 5 & -5 \\ -15 & 6 & -5 \\ 15 & -6 & 6 \end{bmatrix} A$$

$R_3 \to \frac{1}{3}R_3$

$$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -12 & 5 & -5 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix} A$$

$R_1 \to R_1 + 3R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix} A$$

Hence, $$A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

Q52. Express the matrix $\begin{bmatrix} 2 & 3 & 1 \\ 1 & -1 & 2 \\ 4 & 1 & 2 \end{bmatrix}$ as the sum of symmetric and a skew symmetric matrix.

Sol. We know that any square matrix can be expressed as the sum of symmetric and skew symmetric matrix *i.e.*

$A = \frac{1}{2}[A + A'] + \frac{1}{2}[A - A']$.

Let $P = \frac{1}{2}[A + A']$ and $Q = \frac{1}{2}[A - A']$

So $$P = \frac{1}{2}\left[\begin{pmatrix} 2 & 3 & 1 \\ 1 & -1 & 2 \\ 4 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 1 & 4 \\ 3 & -1 & 1 \\ 1 & 2 & 2 \end{pmatrix}\right]$$

$$= \frac{1}{2}\begin{bmatrix} 2+2 & 3+1 & 1+4 \\ 1+3 & -1-1 & 2+1 \\ 4+1 & 1+2 & 2+2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 4 & 4 & 5 \\ 4 & -2 & 3 \\ 5 & 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 & 5/2 \\ 2 & -1 & 3/2 \\ 5/2 & 3/2 & 2 \end{bmatrix}$$

$$P' = \begin{bmatrix} 2 & 2 & 5/2 \\ 2 & -1 & 3/2 \\ 5/2 & 3/2 & 2 \end{bmatrix} = P$$

As $P' = P$ $\quad\therefore$ P is a symmetric matrix.

Now $Q = \frac{1}{2}[A - A']$

$$= \frac{1}{2}\left[\begin{pmatrix} 2 & 3 & 1 \\ 1 & -1 & 2 \\ 4 & 1 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 1 & 4 \\ 3 & -1 & 1 \\ 1 & 2 & 2 \end{pmatrix}\right]$$

$$= \frac{1}{2}\left[\begin{pmatrix} 2-2 & 3-1 & 1-4 \\ 1-3 & -1+1 & 2-1 \\ 4-1 & 1-2 & 2-2 \end{pmatrix}\right] = \frac{1}{2}\left[\begin{pmatrix} 0 & 2 & -3 \\ -2 & 0 & 1 \\ 3 & -1 & 0 \end{pmatrix}\right]$$

$$= \begin{bmatrix} 0 & 1 & -3/2 \\ -1 & 0 & 1/2 \\ 3/2 & -1/2 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & -1 & 3/2 \\ 1 & 0 & -1/2 \\ -3/2 & 1/2 & 0 \end{bmatrix} = -Q.$$

As $Q = -Q$ $\therefore$ Q is a skew symmetric matrix.

So $\qquad A = P + Q$

$$A = \begin{bmatrix} 2 & 2 & 5/2 \\ 2 & -1 & 3/2 \\ 5/2 & 3/2 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 & -3/2 \\ -1 & 0 & 1/2 \\ 3/2 & -1/2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2+0 & 2+1 & \frac{5}{2}-\frac{3}{2} \\ 2-1 & -1+0 & \frac{3}{2}+\frac{1}{2} \\ \frac{5}{2}+\frac{3}{2} & \frac{3}{2}-\frac{1}{2} & 2+0 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 1 & -1 & 2 \\ 4 & 1 & 2 \end{bmatrix} = A$$

Hence, the required relation is

$$A = \begin{bmatrix} 2 & 2 & 5/2 \\ 2 & -1 & 3/2 \\ 5/2 & 3/2 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 & -3/2 \\ -1 & 0 & 1/2 \\ 3/2 & -1/2 & 0 \end{bmatrix}$$

OBJECTIVE TYPE QUESTIONS

Choose the correct answer from the given four options in each of the Exercises 53 to 67.

Q53. The matrix $P = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 4 & 0 \\ 4 & 0 & 0 \end{bmatrix}$ is a

(*a*) square matrix (*b*) diagonal matrix

(*c*) unit matrix (*d*) None

Sol. Given that $A = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 4 & 0 \\ 4 & 0 & 0 \end{bmatrix}$

Here number of columns and the number of rows are equal *i.e.*, 3. So, A is a square matrix.

Hence, the correct option is (*a*).

Q54. Total number of possible matrices of order 3×3 with each entry 2 or 0 is

(*a*) 9 (*b*) 27 (*c*) 81 (*d*) 512

Sol. Total number of possible matrices of order 3×3 with each entry 0 or $2 = 2^{3\times3} = 2^9 = 512$.

Hence, the correct option is (*d*).

Q55. If $\begin{bmatrix} 2x+y & 4x \\ 5x-7 & 4x \end{bmatrix} = \begin{bmatrix} 7 & 7y-13 \\ y & x+6 \end{bmatrix}$, then the value of x and y is

(*a*) $x=3, y=1$ (*b*) $x=2, y=3$

(*c*) $x=2, y=4$ (*d*) $x=3, y=3$

Sol. Given that: $\begin{bmatrix} 2x+y & 4x \\ 5x-7 & 4x \end{bmatrix} = \begin{bmatrix} 7 & 7y-13 \\ y & x+6 \end{bmatrix}$

Equating the corresponding elements, we get,

$$2x + y = 7 \quad \ldots(i)$$

and $$4x = x + 6 \quad \ldots(ii)$$

from eqn. (*ii*) $4x - x = 6$

$3x = 6$

$\therefore$ $x = 2$

from eqn. (*i*) $2 \times 2 + y = 7$

$4 + y = 7 \quad \therefore y = 7 - 4 = 3$

Hence, the correct option is (*b*).

Q56. If $$A = \frac{1}{\pi}\begin{bmatrix} \sin^{-1}(x\pi) & \tan^{-1}\left(\frac{x}{\pi}\right) \\ \sin^{-1}\left(\frac{x}{\pi}\right) & \cot^{-1}(\pi x) \end{bmatrix}$$

$$B = \frac{1}{\pi}\begin{bmatrix} -\cos^{-1}(x\pi) & \tan^{-1}\left(\frac{x}{\pi}\right) \\ \sin^{-1}\left(\frac{x}{\pi}\right) & -\tan^{-1}(\pi x) \end{bmatrix}$$

then A – B is equal to

(*a*) I (*b*) O (*c*) 2I (*d*) $\frac{1}{2}$I

Sol. Given that: $$A = \frac{1}{\pi}\begin{bmatrix} \sin^{-1}(x\pi) & \tan^{-1}\left(\frac{x}{\pi}\right) \\ \sin^{-1}\left(\frac{x}{\pi}\right) & \cot^{-1}(\pi x) \end{bmatrix}$$

and $$B = \frac{1}{\pi}\begin{bmatrix} -\cos^{-1}(x\pi) & \tan^{-1}\left(\frac{x}{\pi}\right) \\ \sin^{-1}\left(\frac{x}{\pi}\right) & -\tan^{-1}(\pi x) \end{bmatrix}$$

$$A - B = \frac{1}{\pi}\begin{bmatrix} \sin^{-1}(x\pi) & \tan^{-1}\left(\frac{x}{\pi}\right) \\ \sin^{-1}\left(\frac{x}{\pi}\right) & \cot^{-1}(\pi x) \end{bmatrix} - \frac{1}{\pi}\begin{bmatrix} -\cos^{-1}(x\pi) & \tan^{-1}\left(\frac{x}{\pi}\right) \\ \sin^{-1}\left(\frac{x}{\pi}\right) & -\tan^{-1}(\pi x) \end{bmatrix}$$

$$= \frac{1}{\pi}\begin{bmatrix} \sin^{-1}(x\pi) + \cos^{-1}(x\pi) & \tan^{-1}\left(\frac{x}{\pi}\right) - \tan^{-1}\left(\frac{x}{\pi}\right) \\ \sin^{-1}\left(\frac{x}{\pi}\right) - \sin^{-1}\left(\frac{x}{\pi}\right) & \cot^{-1}(\pi x) + \tan^{-1}(\pi x) \end{bmatrix}$$

$$= \frac{1}{\pi}\begin{bmatrix} \frac{\pi}{2} & 0 \\ 0 & \frac{\pi}{2} \end{bmatrix} \qquad \begin{bmatrix} \because \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2} \\ \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2} \end{bmatrix}$$

$$= \frac{1}{\pi} \times \frac{\pi}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{2}I$$

Hence, the correct option is (*d*).

Q57. If A and B are two matrices of the order $3 \times m$ and $3 \times n$, respectively and $m = n$, then the order of matrix $(5A - 2B)$ is

(*a*) $m \times 3$ (*b*) 3×3 (*c*) $m \times n$ (*d*) $3 \times n$

Sol. As we know that the addition and subtraction of two matrices is only possible when they have same order. It is also given that $m = n$.

$\therefore$ Order of $(5A - 2B)$ is $3 \times n$

Hence, the correct option is (*d*).

Q58. If $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then A^2 is equal to

(*a*) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (*b*) $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ (*c*) $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ (*d*) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Sol. Given that $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$A^2 = A \,.\, A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0+1 & 0+0 \\ 0+0 & 1+0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence, the correct option is (*d*).

Q59. If matrix $A = [a_{ij}]_{2 \times 2}$, where $a_{ij} = 1$ if $i \neq j$
$= 0$ if $i = j$

then A^2 is equal to

(a) I (b) A (c) 0 (d) None of these

Sol. Given that: $A = [a_{ij}]_{2 \times 2}$

Let $$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times 2}$$

$$a_{11} = 0 \qquad [\because i = j]$$
$$a_{12} = 1 \qquad [\because i \neq j]$$
$$a_{21} = 1 \qquad [\because i \neq j]$$
$$a_{22} = 0 \qquad [\because i = j]$$

$\therefore$ $$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Now, $$A^2 = A \,.\, A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0+1 & 0+0 \\ 0+0 & 1+0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Hence, the correct option is (a).

Q60. The matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ is a

(a) Identity matrix (b) symmetric matrix
(c) skew symmetric matrix (d) none of these

Sol. Let $$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$A' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} = A$$

$A' = A$, so A is a symmetric matrix.
Hence, the correct option is (b).

Q61. The matrix $\begin{bmatrix} 0 & -5 & 8 \\ 5 & 0 & 12 \\ -8 & -12 & 0 \end{bmatrix}$ is a

(a) diagonal matrix (b) symmetric matrix
(c) skew symmetric matrix (d) scalar matrix

Sol. Let
$$A = \begin{bmatrix} 0 & -5 & 8 \\ 5 & 0 & 12 \\ -8 & -12 & 0 \end{bmatrix}$$
$$A' = \begin{bmatrix} 0 & 5 & -8 \\ -5 & 0 & -12 \\ 8 & 12 & 0 \end{bmatrix}$$
$$\Rightarrow \qquad A' = -\begin{bmatrix} 0 & -5 & 8 \\ 5 & 0 & 12 \\ -8 & -12 & 0 \end{bmatrix} = -A$$

$A' = -A$, so A is a skew symmetric matrix.

Hence, the correct option is (*c*).

Q62. If A is a matrix of order $m \times n$ and B is a matrix such that AB′ and B′A are both defined, then order of B is

(*a*) $m \times m$ (*b*) $n \times n$ (*c*) $n \times m$ (*d*) $m \times n$

Sol. Order of matrix $A = m \times n$

Let order of matrix B be $K \times P$

Order of matrix $B' = P \times K$

If AB′ is defined then the order of AB′ is $m \times K$ if $n = P$

If B′A is defined then order of B′A is $P \times n$ when $K = m$

Now, order of $B' = P \times K$

$\therefore$ Order of $B = K \times P$

$= m \times n$ $[\because K = m, P = n]$

Hence, the correct option is (*d*).

Q63. If A and B are matrices of same order, then $(AB' - BA')$ is a

(*a*) skew symmetric matrix (*b*) null matrix

(*c*) symmetric matrix (*d*) unit matrix

Sol. Let $P = (AB' - BA')$

$P' = (AB' - BA')'$

$= (AB')' - (BA')'$

$= (B')'A' - (A')'B'$ $[\because (AB)' = B'A']$

$= BA' - AB'$

$= -(AB' - BA') = -P$

$P' = -P$, so it is a skew symmetric matrix.

Hence, the correct option is (*a*).

Q64. If A is a square matrix such that $A^2 = I$, then $(A - I)^3 + (A + I)^3 - 7A$ is equal to

(*a*) A (*b*) $I - A$ (*c*) $I + A$ (*d*) 3A

Sol. $(A-I)^3+(A+I)^3-7A = A^3-I^3-3A^2I+3AI^2+A^3+I^3+3A^2I$
$$+3AI^2-7A$$
$$= 2A^3+6AI^2-7A$$
$$= 2A.A^2+6AI-7A$$
$$= 2AI+6AI-7A \qquad [A^2=I]$$
$$= 8AI-7A=8A-7A$$
$$= A$$

Hence, the correct option is (*a*).

65. For any two matrices A and B, we have

(*a*) $AB=BA$ (*b*) $AB \neq BA$

(*c*) $AB=O$ (*d*) None of the above

Sol. We know that for any two matrices A and B, we may have $AB=BA$, $AB \neq BA$ and $AB=0$, but it is not always true.
Hence, the correct option is (*d*).

Q66. On using elementary column operation $C_2 \to C_2-2C_1$, in the following matrix equation

$\begin{bmatrix}1 & -3\\ 2 & 4\end{bmatrix} = \begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}\begin{bmatrix}3 & 1\\ 2 & 4\end{bmatrix}$, we have:

(*a*) $\begin{bmatrix}1 & -5\\ 0 & 4\end{bmatrix} = \begin{bmatrix}1 & -1\\ -2 & 2\end{bmatrix}\begin{bmatrix}3 & -5\\ 2 & 0\end{bmatrix}$

(*b*) $\begin{bmatrix}1 & -5\\ 0 & 4\end{bmatrix} = \begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}\begin{bmatrix}3 & -5\\ -0 & 2\end{bmatrix}$

(*c*) $\begin{bmatrix}1 & -5\\ 2 & 0\end{bmatrix} = \begin{bmatrix}1 & -3\\ 0 & 1\end{bmatrix}\begin{bmatrix}3 & 1\\ -2 & 4\end{bmatrix}$

(*d*) $\begin{bmatrix}1 & -5\\ 2 & 0\end{bmatrix} = \begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}\begin{bmatrix}3 & -5\\ 2 & 0\end{bmatrix}$

Sol. Given that: $\begin{bmatrix}1 & -3\\ 2 & 4\end{bmatrix} = \begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}\begin{bmatrix}3 & 1\\ 2 & 4\end{bmatrix}$

Using $C_2 \to C_2-2C_1$, we get
$$\begin{bmatrix}1 & -5\\ 2 & 0\end{bmatrix} = \begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}\begin{bmatrix}3 & -5\\ 2 & 0\end{bmatrix}$$

Hence, the correct option is (*d*).

Q67. On using elementary row operation $R_1 \to R_1-3R_2$ in the following matrix equation

$$\begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}, \text{ we have:}$$

(*a*) $\begin{bmatrix} -5 & -7 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -7 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

(*b*) $\begin{bmatrix} -5 & -7 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} -1 & -3 \\ 1 & 1 \end{bmatrix}$

(*c*) $\begin{bmatrix} -5 & -7 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & -7 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

(*d*) $\begin{bmatrix} 4 & 2 \\ -5 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & -3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

Sol. We have, $\begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

Using elementary row transformation $R_1 \to R_1 - 3R_2$,

$$\begin{bmatrix} -5 & -7 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -7 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

Hence, the correct option is (*a*).

Fill in the Blanks in Each of the Exercises 68-81.

Q68. .... matrix is both symmetric and skew symmetric matrix.

Sol. Null matrix *i.e.* $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is both symmetric and skew symmetric matrix.

Q69. Sum of two skew symmetric matrices is always .... matrix.

Sol. Let A and B be any two matrices

$\therefore$ For skew symmetric matrices

$$A = -A' \qquad ...(i)$$

and $$B = -B' \qquad ...(ii)$$

Adding (*i*) and (*ii*) we get

$$A + B = -A' - B'$$

$\Rightarrow$ $A + B = -(A' + B')$, so $A + B$ is skew symmetric matrix.

Hence, the sum of two skew symmetric matrices is always **skew symmetric matrix.**

Q70. The negative of a matrix is obtained by multiplying it by ....

Sol. Let A be a matrix

$\therefore \qquad -A = -1 \,.\, A$

Hence, negative of a matrix is obtained by multiplying it by -1.

Q71. The protuct of any matrix by the scalar .... is the null matrix.

Sol. Let A be any matrix

$\therefore \qquad 0 \,.\, A = A \,.\, 0 = 0$

Hence, the product of any matrix by the scalar **0** is the null matrix.

Q72. A matrix which is not a square matrix is called a .... matrix.

Sol. A matrix which is not a square matrix is called a **rectangular matrix.**

Q73. Matrix multiplication is .... over addition.

Sol. Matrix multiplication is **distributive** over addition. Let A, B and C be any matrices.

So, (*i*) $A(B + C) = AB + AC$

(*ii*) $(A + B)C = AC + BC$

Q74. If A is a symmetric matrix, then A^3 is a .... matrix.

Sol. Let A be a symmetric matrix

$$\therefore \qquad A' = A$$

$$(A^3)' = (A')^3 = A^3 \qquad [\because (A')^k = (A^k)']$$

Hence, if A is a symmetric matrix, then A^3 is a **symmetric matrix.**

Q75. If A is a skew symmetric matrix, then A^2 is a ....

Sol. If A is a skew symmetric matrix,

$$\therefore \qquad A' = -A$$

$$(A^2)' = (A')^2$$

$$= (-A)^2$$

$$= A^2$$

Hence, A^2 is a **symmetric matrix.**

Q76. If A and B are square matrices of the same order then

(*i*) $(AB)' =$ ....

(*ii*) $(kA)' =$ .... (k is any scalar quantity)

(*iii*) $[k(A - B)]' =$ ....

Sol. (*i*) $(AB)' = B'A'$

(*ii*) $(kA)' = k \,.\, A'$

(*iii*) $[k(A - B)]' = k(A - B)' = k(A' - B')$

Q77. If A is a skew symmetric, then kA is a .... (k is any scalar)

Sol. If A is a skew symmetric matrix

$$\therefore \quad A' = -A$$

$$(kA)' = kA' = k(-A) = -kA$$

Hence, kA is a **skew symmetric** matrix.

Q78. If A and B are symmetric matrices, then

(*i*) AB – BA is a ....

(*ii*) BA – 2AB is a ....

Sol. (*i*) Let

$$\begin{aligned} P &= (AB - BA) \\ P' &= (AB - BA)' \\ &= (AB)' - (BA)' \\ &= B'A' - A'B' && [\because (AB)' = B'A'] \\ &= BA - AB && [\because A' = A \text{ and } B' = B] \\ &= -(AB - BA) \\ &= -P \end{aligned}$$

Hence, (AB – BA) is a **skew symmetric matrix.**

(*ii*) Let

$$\begin{aligned} Q &= (BA - 2AB) \\ Q' &= (BA - 2AB)' \\ &= (BA)' - (2AB)' \\ &= A'B' - 2(AB)' && [\because (kA)' = kA'] \\ &= A'B' - 2B'A' \\ &= AB - 2BA && [\because A' = A \text{ and } B' = B] \\ &= -(2BA - AB) \end{aligned}$$

Hence, (BA – 2AB) is neither a symmetric nor a skew symmetric matrix.

Q79. If A is a symmetric matrix, then B′AB is ....

Sol. If A is a symmetric matrix

$$\therefore \quad A' = A$$

Let

$$\begin{aligned} P &= B'AB \\ P' &= (B'AB)' \\ &= B'A'(B')' && [\because (AB)' = B'A'] \\ &= B'AB && [\because A' = A \text{ and } (B')' = B] \\ \therefore \quad P' &= P \end{aligned}$$

So, P is a symmetric matrix.

Hence, B′AB is a symmetric matrix.

Q80. If A and B are symmetric matrices of same order, then AB is symmetric if and only if ....

Sol. Given that $A' = A$

and $B' = B$

Let $P = AB$

$$P' = (AB)'$$
$$= B'A'$$
$$P' = BA \qquad [\because A' = A \text{ and } B' = B]$$
$$= P$$

Hence, AB is symmetric if and only if **AB = BA**.

Q81. In applying one or more row operations while finding A^{-1} by elementary row operations, we obtain all zeros in one or more, then A^{-1} ....

Sol. A^{-1} does not exist if we apply one or more row operations while finding A^{-1} by elementary row operations, obtain all zeroes in one or more rows.

State (Exercises 82 to 101) which of the following statements are True or False

Q82. A matrix denotes a number.

Sol. False.

A matrix is an array of elements, numbers or functions having rows and columns.

Q83. Matrices of any order can be added.

Sol. False.

The matrices having same order can only be added.

Q84. Two matrices are equal if they have same number of rows and same number of columns.

Sol. False.

The two matrices are said to be equal if their corresponding elements are same.

Q85. Matrices of different orders can not be subtracted.

Sol. True.

For addition and subtraction, the order of the two matrices should be same.

Q86. Matrix addition is associative as well as commutative.

Sol. True.

If A, B and C are the matrices of addition then

$$A + (B + C) = (A + B) + C \qquad \text{(associative)}$$
$$A + B = B + A \qquad \text{(commutative)}$$

Q87. Matrix multiplication is commutative.

Sol. False.

Since $AB \neq BA$ if AB and BA are well defined.

Q88. A square matrix where every element is unity is called an identity matrix.

Sol. False.

Since, in identity matrix all the elements of principal diagonal are unity rest are zero.

e.g., $$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

Q89. If A and B are two square matrices of the same order, then $A + B = B + A$.

Sol. True.

If A and B are square matrices then their addition is commutative *i.e.*, $A + B = B + A$.

Q90. If A and B are two matrices of the same order, then $A - B = B - A$.

Sol. False.

Since subtraction of any two matrices of the same order is not commutative *i.e.*, $A - B \neq B - A$.

Q91. If matrix $AB = O$, then $A = O$ or $B = O$ or both A and B are null matrices.

Sol. False.

Since for any two non-zero matrices A and B, we may get $AB = 0$.

Q92. Transpose of a column matrix is a column matrix.

Sol. False.

Transpose of a column matrix is a row matrix.

e.g., $A = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}_{3 \times 1}$ $\quad \therefore A' = \begin{bmatrix} 2 & 3 & 5 \end{bmatrix}_{1 \times 3}$

Q93. If A and B are two square matrices of the same order, then $AB = BA$.

Sol. False.

For two square matrices A and B, $AB = BA$ is not always true.

Q94. If each of the three matrices of the same order are symmetric, then their sum is a symmetric matrix.

Sol. True.

Let A, B and C be three matrices of the same order.

Given that $A' = A$, $B' = B$ and $C' = C$

Let $\quad P = A + B + C$

$\Rightarrow \quad P' = (A + B + C)'$

$= A' + B' + C'$

$= A + B + C$

$= P$

So, $A + B + C$ is also a symmetric matrix.

Q95. If A and B are any two matrices of the same order, then $(AB)' = A'B'$.

Sol. False.

Since $(AB)' = B'A'$.

Q96. If $(AB)' = B'A'$, where A and B are not square matrices, then number of rows in A is equal to number of columns in B and number of columns in A is equal to number of rows in B.

Sol. True.

Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{p \times q}$

AB is defined when $n = P$

$\therefore$ Order of $AB = m \times q$

$\Rightarrow$ Order of $(AB)' = q \times m$

Order of B' is $q \times p$ and order of A' is $n \times m$

$\therefore$ $B'A'$ is defined when $P = n$

and the order of $B'A'$ is $q \times m$

Hence, order of $(AB)'$ = Order of $B'A'$ *i.e.*, $q \times m$.

Q97. If A, B and C are square matrices of same order, then $AB = AC$ always implies that $B = C$.

Sol. False.

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 0 \\ 3 & 4 \end{bmatrix}$

$$\therefore \quad AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Here $AB = AC = 0$ but $B \neq C$.

Q98. AA' is always a symmetric matrix of any matrix A.

Sol. True.

Let $\quad P = AA'$

$$\begin{aligned} P' &= (AA')' \\ &= (A')'.A' \qquad [(AB)' = B'A'] \\ &= AA' \\ &= P \end{aligned}$$

So, P is symmetric matrix.

Hence, AA' is always a symmetric matrix.

Q99. If $A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$ then AB and BA are defined and equal.

Sol. False.

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 4 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$$

Since AB is defined

$$\therefore \quad AB = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 4 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4+12-2 & 6+15-1 \\ 2+16+4 & 3+20+2 \end{bmatrix} = \begin{bmatrix} 14 & 20 \\ 22 & 25 \end{bmatrix}$$

BA is also defined.

$$\therefore \quad BA = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & -1 \\ 1 & 4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4+3 & 6+12 & -2+6 \\ 8+5 & 12+20 & -4+10 \\ 4+1 & 6+4 & -2+2 \end{bmatrix} = \begin{bmatrix} 7 & 18 & 4 \\ 13 & 32 & 6 \\ 5 & 10 & 0 \end{bmatrix}$$

So $\quad AB \neq BA$

Q100. If A is a skew symmetric matrix, then A^2 is a symmetric matrix.

Sol. True.

$$(A^2)' = (A')^2$$
$$= [-A]^2 \qquad [\because A' = -A]$$
$$= A^2$$

So, A^2 is a symmetric matrix.

Q101. $(AB)^{-1} = A^{-1}B^{-1}$ where A and B are invertible matrices satisfying commutative property with respective to multiplication.

Sol. True.

If A and B are invertible matrices of the same order

$\therefore \qquad (AB)^{-1} = (BA)^{-1} \qquad [\because AB = BA]$

But $\qquad (AB)^{-1} = A^{-1}B^{-1} \qquad$ [Given]

$\therefore \qquad (BA)^{-1} = B^{-1}A^{-1}$

So $\qquad A^{-1}B^{-1} = B^{-1}A^{-1}$

$\therefore$ A and B satisfy commutative property w.r.t. multiplication.

4 Determinants

4.3 EXERCISE

SHORT ANSWER TYPE QUESTIONS

Using the properties of determinants in Exercise 1 to 6, evaluate:

Q1. $\begin{vmatrix} x^2-x+1 & x-1 \\ x+1 & x+1 \end{vmatrix}$

Sol. Let $\Delta = \begin{vmatrix} x^2-x+1 & x-1 \\ x+1 & x+1 \end{vmatrix}$

$C_1 \to C_1 - C_2$

$$= \begin{vmatrix} x^2-2x+2 & x-1 \\ 0 & x+1 \end{vmatrix}$$

$$= (x+1)(x^2-2x+2) - 0$$

$$= x^3 - 2x^2 + 2x + x^2 - 2x + 2 = x^3 - x^2 + 2$$

Q2. $\begin{vmatrix} a+x & y & z \\ x & a+y & z \\ x & y & a+z \end{vmatrix}$

Sol. Let $\Delta = \begin{vmatrix} a+x & y & z \\ x & a+y & z \\ x & y & a+z \end{vmatrix}$

$C_1 \to C_1 + C_2 + C_3$

$$= \begin{vmatrix} a+x+y+z & y & z \\ a+x+y+z & a+y & z \\ a+x+y+z & y & a+z \end{vmatrix} = (a+x+y+z)\begin{vmatrix} 1 & y & z \\ 1 & a+y & z \\ 1 & y & a+z \end{vmatrix}$$

(Taking $a+x+y+z$ common)

$R_1 \to R_1 - R_2,\ R_2 \to R_2 - R_3$

$$= (a+x+y+z)\begin{vmatrix} 0 & -a & 0 \\ 0 & a & -a \\ 1 & y & a+z \end{vmatrix}$$

Expanding along $C_1 = (a+x+y+z)\left|1(a^2-0)\right| = a^2(a+x+y+z)$

Q3. $\begin{vmatrix} 0 & xy^2 & xz^2 \\ x^2y & 0 & yz^2 \\ x^2z & zy^2 & 0 \end{vmatrix}$

Sol. Let $\Delta = \begin{vmatrix} 0 & xy^2 & xz^2 \\ x^2y & 0 & yz^2 \\ x^2z & zy^2 & 0 \end{vmatrix}$

Taking x^2, y^2 and z^2 common from C_1, C_2 and C_3 respectively

$$= x^2y^2z^2 \begin{vmatrix} 0 & x & x \\ y & 0 & y \\ z & z & 0 \end{vmatrix}$$

Expanding along R_1

$$= x^2y^2z^2 \left[0\begin{vmatrix} 0 & y \\ z & 0 \end{vmatrix} - x\begin{vmatrix} y & y \\ z & 0 \end{vmatrix} + x\begin{vmatrix} y & 0 \\ z & z \end{vmatrix} \right]$$

$$= x^2y^2z^2[-x(0-yz) + x(yz-0)]$$

$$= x^2y^2z^2(xyz + xyz) = x^2y^2z^2(2xyz) = 2x^3y^3z^3$$

Q4. $\begin{vmatrix} 3x & -x+y & -x+z \\ x-y & 3y & z-y \\ x-z & y-z & 3z \end{vmatrix}$

Sol. Let $\Delta = \begin{vmatrix} 3x & -x+y & -x+z \\ x-y & 3y & z-y \\ x-z & y-z & 3z \end{vmatrix}$

$C_1 \to C_1 + C_2 + C_3$

$$= \begin{vmatrix} x+y+z & -x+y & -x+z \\ x+y+z & 3y & z-y \\ x+y+z & y-z & 3z \end{vmatrix}$$

Taking $(x+y+z)$ common from C_1

$$= (x+y+z)\begin{vmatrix} 1 & -x+y & -x+z \\ 1 & 3y & z-y \\ 1 & y-z & 3z \end{vmatrix}$$

$R_1 \to R_1 - R_2$, $R_2 \to R_2 - R_3$

$$= (x+y+z)\begin{vmatrix} 0 & -x-2y & -x+y \\ 0 & 2y+z & -y-2z \\ 1 & y-z & 3z \end{vmatrix}$$

Expanding along C_1

$$= (x+y+z)\left[1\begin{vmatrix}-x-2y & -x+y \\ 2y+z & -y-2z\end{vmatrix}\right]$$

$$= (x+y+z)\left[(-x-2y)(-y-2z)-(2y+z)(-x+y)\right]$$

$$= (x+y+z)\,(xy+2zx+2y^2+4yz+2xy-2y^2+zx-zy)$$

$$= (x+y+z)\,(3xy+3zx+3yz) = 3(x+y+z)\,(xy+yz+zx)$$

Q5. $\begin{vmatrix}x+4 & x & x \\ x & x+4 & x \\ x & x & x+4\end{vmatrix}$

Sol. Let $\Delta = \begin{vmatrix}x+4 & x & x \\ x & x+4 & x \\ x & x & x+4\end{vmatrix}$

$C_1 \to C_1 + C_2 + C_3$

$$= \begin{vmatrix}3x+4 & x & x \\ 3x+4 & x+4 & x \\ 3x+4 & x & x+4\end{vmatrix}$$

Taking $(3x+4)$ common from C_1

$$= (3x+4)\begin{vmatrix}1 & x & x \\ 1 & x+4 & x \\ 1 & x & x+4\end{vmatrix}$$

$R_1 \to R_1 - R_2,\ R_2 \to R_2 - R_3$

$$= (3x+4)\begin{vmatrix}0 & -4 & 0 \\ 0 & 4 & -4 \\ 1 & x & x+4\end{vmatrix}$$

Expanding along C_1

$$= (3x+4)\left[1\begin{vmatrix}-4 & 0 \\ 4 & -4\end{vmatrix}\right] = (3x+4)\,(16-0) = 16(3x+4)$$

Q6. $\begin{vmatrix}a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b\end{vmatrix}$

Sol. Let $\Delta = \begin{vmatrix}a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b\end{vmatrix}$

$R_1 \to R_1 + R_2 + R_3$

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

Taking $(a+b+c)$ common from R_1

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$C_1 \to C_1 - C_2$, $C_2 \to C_2 - C_3$

$$= (a+b+c)\begin{vmatrix} 0 & 0 & 1 \\ b+c+a & -(b+c+a) & 2b \\ 0 & a+b+c & c-a-b \end{vmatrix}$$

Taking $(b+c+a)$ from C_1 and C_2

$$= (a+b+c)^3\begin{vmatrix} 0 & 0 & 1 \\ 1 & -1 & 2b \\ 0 & 1 & c-a-b \end{vmatrix}$$

Expanding along R_1

$$= (a+b+c)^3\left[1\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix}\right] = (a+b+c)^3.$$

Using the properties of determinants in Exercises 7 to 9, prove that:

Q7. $\begin{vmatrix} y^2z^2 & yz & y+z \\ z^2x^2 & zx & z+x \\ x^2y^2 & xy & x+y \end{vmatrix} = 0$

Sol. L.H.S. $= \begin{vmatrix} y^2z^2 & yz & y+z \\ z^2x^2 & zx & z+x \\ x^2y^2 & xy & x+y \end{vmatrix}$

$R_1 \to xR_1$, $R_2 \to yR_2$, $R_3 \to zR_3$ and dividing the determinant by xyz.

$$= \frac{1}{xyz}\begin{vmatrix} xy^2z^2 & xyz & xy+zx \\ yz^2x^2 & yzx & yz+xy \\ zx^2y^2 & zxy & zx+zy \end{vmatrix}$$

Taking xyz common from C_1 and C_2

$$= \frac{xyz \cdot xyz}{xyz}\begin{vmatrix} yz & 1 & xy+zx \\ zx & 1 & yz+xy \\ xy & 1 & zx+zy \end{vmatrix}$$

$C_3 \to C_3 + C_1$

$$= xyz\begin{vmatrix} yz & 1 & xy+yz+zx \\ zx & 1 & xy+yz+zx \\ xy & 1 & xy+yz+zx \end{vmatrix}$$

Taking $(xy+yz+zx)$ common from C_3

$$= (xyz)(xy+yz+zx)\begin{vmatrix} yz & 1 & 1 \\ zx & 1 & 1 \\ xy & 1 & 1 \end{vmatrix}$$

$$= (xyz)(xy+yz+zx)\begin{vmatrix} yz & 1 & 1 \\ zx & 1 & 1 \\ xy & 1 & 1 \end{vmatrix} = 0$$

[$\because$ C_2 and C_3 are identical]

L.H.S. = R.H.S. Hence proved.

Q8. $\begin{vmatrix} y+z & z & y \\ z & z+x & x \\ y & x & x+y \end{vmatrix} = 4xyz$

Sol. L.H.S. $= \begin{vmatrix} y+z & z & y \\ z & z+x & x \\ y & x & x+y \end{vmatrix}$

$C_1 \to C_1 - (C_2 + C_3)$

$$= \begin{vmatrix} 0 & z & y \\ -2x & z+x & x \\ -2x & x & x+y \end{vmatrix}$$

Taking -2 common from C_1

$$= -2\begin{vmatrix} 0 & z & y \\ x & z+x & x \\ x & x & x+y \end{vmatrix}$$

$R_2 \to R_2 - R_3$

$$= -2\begin{vmatrix} 0 & z & y \\ 0 & z & -y \\ x & x & x+y \end{vmatrix}$$

Expanding along C_1

$$= -2\left[x\left|-zy - zy\right|\right] = -2(-2xyz) = 4xyz \text{ R.H.S.}$$

L.H.S. = R.H.S.

Hence, proved.

Q9. $\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a-1)^3$

Sol. L.H.S.= $\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix}$

$R_1 \to R_1 - R_2,\ R_2 \to R_2 - R_3$

$$= \begin{vmatrix} a^2-1 & a-1 & 0 \\ 2a-2 & a-1 & 0 \\ 3 & 3 & 1 \end{vmatrix} = \begin{vmatrix} (a+1)(a-1) & a-1 & 0 \\ 2(a-1) & a-1 & 0 \\ 3 & 3 & 1 \end{vmatrix}$$

Taking $(a-1)$ common from C_1 and C_2

$$= (a-1)(a-1)\begin{vmatrix} a+1 & 1 & 0 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{vmatrix}$$

Expanding along C_3

$$= (a-1)^2\left[1\begin{vmatrix} a+1 & 1 \\ 2 & 1 \end{vmatrix}\right]$$

$$= (a-1)^2(a+1-2) = (a-1)^2(a-1) = (a-1)^3 \text{ R.H.S.}$$

L.H.S.= R.H.S.

Hence, proved.

Q10. If $A + B + C = 0$, then prove that

$$\begin{vmatrix} 1 & \cos C & \cos B \\ \cos C & 1 & \cos A \\ \cos B & \cos A & 1 \end{vmatrix} = 0$$

Sol. L.H.S. = $\begin{vmatrix} 1 & \cos C & \cos B \\ \cos C & 1 & \cos A \\ \cos B & \cos A & 1 \end{vmatrix}$

Expanding along C_1

$$= 1\begin{vmatrix} 1 & \cos A \\ \cos A & 1 \end{vmatrix} - \cos C\begin{vmatrix} \cos C & \cos B \\ \cos A & 1 \end{vmatrix} + \cos B\begin{vmatrix} \cos C & \cos B \\ 1 & \cos A \end{vmatrix}$$

$= 1(1-\cos^2 A) - \cos C(\cos C - \cos A \cos B)$
$+ \cos B (\cos A \cos C - \cos B)$
$= \sin^2 A - \cos^2 C + \cos A \cos B \cos C$
$+ \cos A \cos B \cos C - \cos^2 B$
$= \sin^2 A - \cos^2 B - \cos^2 C + 2 \cos A \cos B \cos C$
$= -\cos (A+B) \cdot \cos (A-B) - \cos^2 C + 2 \cos A \cos B \cos C$
$[\because \sin^2 A - \cos^2 B = -\cos (A+B) \cdot \cos (A-B)]$
$= -\cos (-C) \cdot \cos (A-B) + \cos C (2 \cos A \cos B - \cos C)$
$[\because A+B+C=0]$
$= -\cos C(\cos A \cos B + \sin A \sin B)$
$+ \cos C(2 \cos A \cos B - \cos C)$
$= -\cos C(\cos A \cos B + \sin A \sin B - 2 \cos A \cos B + \cos C)$
$= -\cos C(-\cos A \cos B + \sin A \sin B + \cos C)$
$= \cos C(\cos A \cos B - \sin A \sin B - \cos C)$
$= \cos C[\cos(A+B) - \cos C]$
$= \cos C[\cos (-C) - \cos C]$ $[\because A+B=-C]$
$= \cos C[\cos C - \cos C] = \cos C \cdot 0 = 0$ R.H.S.

L.H.S. = R.H.S.

Hence, proved.

Q11. If the coordinates of the vertices of an equilateral triangle with sides of length 'a' are (x_1, y_1), (x_2, y_2) and (x_3, y_3), then

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 = \frac{3a^4}{4}.$$

Sol. Area of triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3)

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Let $\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \Rightarrow \Delta^2 = \frac{1}{4}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2$

But area of equilateral triangle whose side is 'a' $= \frac{\sqrt{3}}{4}a^2$

$$\therefore \quad \left(\frac{\sqrt{3}}{4}a^2\right)^2 = \frac{1}{4}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2$$

$$\Rightarrow \frac{3}{16}a^4 = \frac{1}{4}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 \quad \Rightarrow \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 = \frac{3}{16}a^4 \times 4 = \frac{3}{4}a^4$$

Hence, proved.

Q12. Find the value of θ satisfying $\begin{bmatrix} 1 & 1 & \sin 3\theta \\ -4 & 3 & \cos 2\theta \\ 7 & -7 & -2 \end{bmatrix} = 0$.

Sol. Let $A = \begin{bmatrix} 1 & 1 & \sin 3\theta \\ -4 & 3 & \cos 2\theta \\ 7 & -7 & -2 \end{bmatrix} = 0$

$$|A| = \begin{vmatrix} 1 & 1 & \sin 3\theta \\ -4 & 3 & \cos 2\theta \\ 7 & -7 & -2 \end{vmatrix} = 0$$

$C_1 \to C_1 - C_2$

$$\Rightarrow \begin{vmatrix} 0 & 1 & \sin 3\theta \\ -7 & 3 & \cos 2\theta \\ 14 & -7 & -2 \end{vmatrix} = 0$$

Taking 7 common from C_1

$$\Rightarrow 7\begin{vmatrix} 0 & 1 & \sin 3\theta \\ -1 & 3 & \cos 2\theta \\ 2 & -7 & -2 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 0 & 1 & \sin 3\theta \\ -1 & 3 & \cos 2\theta \\ 2 & -7 & -2 \end{vmatrix} = 0$$

Expanding along C_1

$$\Rightarrow 1\begin{vmatrix} 1 & \sin 3\theta \\ -7 & -2 \end{vmatrix} + 2\begin{vmatrix} 1 & \sin 3\theta \\ 3 & \cos 2\theta \end{vmatrix} = 0$$

$$\Rightarrow -2 + 7\sin 3\theta + 2(\cos 2\theta - 3\sin 3\theta) = 0$$

$$\Rightarrow -2 + 7\sin 3\theta + 2\cos 2\theta - 6\sin 3\theta = 0$$

$$\Rightarrow -2 + 2\cos 2\theta + \sin 3\theta = 0$$

$$\Rightarrow -2 + 2(1 - 2\sin^2\theta) + 3\sin\theta - 4\sin^3\theta = 0$$

$$\Rightarrow -2 + 2 - 4\sin^2\theta + 3\sin\theta - 4\sin^3\theta = 0$$

$$\Rightarrow \quad -4\sin^3\theta - 4\sin^2\theta + 3\sin\theta = 0$$
$$\Rightarrow \quad -\sin\theta(4\sin^2\theta + 4\sin\theta - 3) = 0$$
$$\sin\theta = 0 \quad \text{or} \quad 4\sin^2\theta + 4\sin\theta - 3 = 0$$
$$\therefore \quad \theta = n\pi \quad \text{or} \quad 4\sin^2\theta + 6\sin\theta - 2\sin\theta - 3 = 0$$

when $n \in I$

$$\Rightarrow \quad 2\sin\theta(2\sin\theta + 3) - 1(2\sin\theta + 3) = 0$$
$$\Rightarrow \quad (2\sin\theta + 3)(2\sin\theta - 1) = 0$$
$$\Rightarrow \quad 2\sin\theta + 3 = 0 \quad \text{or} \quad 2\sin\theta - 1 = 0$$
$$\sin\theta = \frac{-3}{2} \quad \text{or} \quad \sin\theta = \frac{1}{2}$$

$\sin\theta = \dfrac{-3}{2}$ is not possible as $-1 \le x \le 1$

$$\therefore \quad \sin\theta = \frac{1}{2} \quad \Rightarrow \quad \sin\theta = \sin\frac{\pi}{6} \quad \Rightarrow \quad \theta = n\pi + (-1)^n \cdot \frac{\pi}{6}$$

Hence, $\theta = n\pi$ or $n\pi + (-1)^n \dfrac{\pi}{6}$

Q13. If $\begin{bmatrix} 4-x & 4+x & 4+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{bmatrix} = 0$, then find values of x.

Sol. Let
$$A = \begin{bmatrix} 4-x & 4+x & 4+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{bmatrix} = 0$$

$$|A| = \begin{vmatrix} 4-x & 4+x & 4+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{vmatrix} = 0$$

$R_1 \to R_1 + R_2 + R_3$

$$\Rightarrow \quad \begin{vmatrix} 12+x & 12+x & 12+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{vmatrix} = 0$$

Taking $(12+x)$ common from R_1,

$$\Rightarrow \quad (12+x)\begin{vmatrix} 1 & 1 & 1 \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{vmatrix} = 0$$

$C_1 \to C_1 - C_2,\ C_2 \to C_2 - C_3$

$$(12+x)\begin{vmatrix} 0 & 0 & 1 \\ 2x & -2x & 4+x \\ 0 & 2x & 4-x \end{vmatrix} = 0$$

Expanding along R_1

$$\Rightarrow \qquad (12+x)\left[1\cdot\begin{vmatrix} 2x & -2x \\ 0 & 2x \end{vmatrix}\right] = 0$$

$$(12+x)(4x^2-0) = 0 \Rightarrow 12+x=0 \quad \text{or} \quad 4x^2=0$$

$\Rightarrow \quad x = -12 \quad \text{or} \quad x = 0$

Q14. If $a_1, a_2, a_3, \ldots, a_r$ are in G.P., then prove that the determinant

$$\begin{vmatrix} a_{r+1} & a_{r+5} & a_{r+9} \\ a_{r+7} & a_{r+11} & a_{r+15} \\ a_{r+11} & a_{r+17} & a_{r+21} \end{vmatrix} \text{ is independent of } r.$$

Sol. If $a_1, a_2, a_3, \ldots a_r$ be the terms of G.P., then

$$a_n = AR^{n-1}$$

(where A is the first term and R is the common ratio of the G.P.)

$\therefore \quad a_{r+1} = AR^{r+1-1} = AR^r;\ a_{r+5} = AR^{r+5-1} = AR^{r+4}$

$a_{r+9} = AR^{r+9-1} = AR^{r+8};\ a_{r+7} = AR^{r+7-1} = AR^{r+6}$

$a_{r+11} = AR^{r+11-1} = AR^{r+10};\ a_{r+15} = AR^{r+15-1} = AR^{r+14}$

$a_{r+17} = AR^{r+17-1} = AR^{r+16};\ a_{r+21} = AR^{r+21-1} = AR^{r+20}$

$\therefore$ The determinant becomes

$$\begin{vmatrix} AR^r & AR^{r+4} & AR^{r+8} \\ AR^{r+6} & AR^{r+10} & AR^{r+14} \\ AR^{r+10} & AR^{r+16} & AR^{r+20} \end{vmatrix}$$

Taking AR^r, AR^{r+6} and AR^{r+10} common from R_1, R_2 and R_3 respectively.

$$AR^r \cdot AR^{r+6} \cdot AR^{r+10}\begin{vmatrix} 1 & R^4 & R^8 \\ 1 & R^4 & R^8 \\ 1 & R^6 & R^{10} \end{vmatrix}$$

$$= AR^r \cdot AR^{r+6} \cdot AR^{r+10}\,|0|$$

[$\because$ R_1 and R_2 are identical rows]

$= 0$

Hence, the given determinant is independent of r.

Q15. Show that the points $(a+5, a-4)$, $(a-2, a+3)$ and (a, a) do not lie on a straight line for any value of a.

Sol. If the given points lie on a straight line, then the area of the triangle formed by joining the points pairwise is zero.

So, $\begin{vmatrix} a+5 & a-4 & 1 \\ a-2 & a+3 & 1 \\ a & a & 1 \end{vmatrix}$

$R_1 \to R_1 - R_2, R_2 \to R_2 - R_3$

$$\Rightarrow \qquad \begin{vmatrix} 7 & -7 & 0 \\ -2 & 3 & 0 \\ a & a & 1 \end{vmatrix}$$

Expanding along C_3

$$1 \cdot \begin{vmatrix} 7 & -7 \\ -2 & 3 \end{vmatrix} = 21 - 14 = 7 \text{ units}$$

As $7 \neq 0$. Hence,, the three points do not lie on a straight line for any value of a.

Q16. Show that the ΔABC is an isosceles triangle if the determinant

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1+\cos A & 1+\cos B & 1+\cos C \\ \cos^2 A + \cos A & \cos^2 B + \cos B & \cos^2 C + \cos C \end{vmatrix} = 0.$$

Sol. $\begin{vmatrix} 1 & 1 & 1 \\ 1+\cos A & 1+\cos B & 1+\cos C \\ \cos^2 A + \cos A & \cos^2 B + \cos B & \cos^2 C + \cos C \end{vmatrix} = 0$

$C_1 \to C_1 - C_2, C_2 \to C_2 - C_3$

$$\Rightarrow \begin{vmatrix} 0 & 0 & 1 \\ \cos A - \cos B & \cos B - \cos C & 1+\cos C \\ \begin{matrix}\cos^2 A + \cos A \\ -\cos^2 B - \cos B\end{matrix} & \begin{matrix}\cos^2 B + \cos B \\ -\cos^2 C - \cos C\end{matrix} & \cos^2 C + \cos C \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 0 & 0 & 1 \\ \cos A - \cos B & \cos B - \cos C & 1+\cos C \\ \begin{matrix}\cos^2 A - \cos^2 B \\ +\cos A - \cos B\end{matrix} & \begin{matrix}\cos^2 B - \cos^2 C \\ +\cos B - \cos C\end{matrix} & \cos^2 C + \cos C \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 0 & 0 & 1 \\ \cos A - \cos B & \cos B - \cos C & 1+\cos C \\ \begin{matrix}(\cos A + \cos B) \times \\ (\cos A - \cos B) \\ +(\cos A - \cos B)\end{matrix} & \begin{matrix}(\cos B + \cos C) \times \\ (\cos B - \cos C) \\ +(\cos B + \cos C)\end{matrix} & \cos^2 C + \cos C \end{vmatrix} = 0$$

Taking (cos A – cos B) and (cos B – cos C) common from C_1 and C_2 respectively.

$$\Rightarrow \quad (\cos A - \cos B)(\cos B - \cos C)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & 1+\cos C \\ \cos A + \cos B + 1 & \cos B + \cos C + 1 & \cos^2 C + \cos C \end{vmatrix} = 0$$

Expanding along R_1

$$\Rightarrow \quad (\cos A - \cos B)(\cos B - \cos C)\left[1\begin{vmatrix} 1 & 1 \\ \cos A + \cos B + 1 & \cos B + \cos C + 1 \end{vmatrix}\right] = 0$$

$$\Rightarrow \quad (\cos A - \cos B)(\cos B - \cos C)\begin{bmatrix} (\cos B + \cos C + 1) - \\ (\cos A + \cos B + 1) \end{bmatrix} = 0$$

$$\Rightarrow \quad (\cos A - \cos B)(\cos B - \cos C)[\cos B + \cos C + 1 - \cos A - \cos B - 1] = 0$$

$$\Rightarrow \quad (\cos A - \cos B)(\cos B - \cos C)(\cos C - \cos A) = 0$$

$\cos A - \cos B = 0$ or $\cos B - \cos C = 0$

or $\cos C - \cos A = 0$

$\Rightarrow \cos A = \cos B$ or $\cos B = \cos C$ or $\cos C = \cos A$

$\Rightarrow \angle A = \angle C$ or $\angle B = \angle C \Rightarrow \angle A = \angle B$

Hence, ΔABC is an isosceles triangle.

Q17. Find A^{-1} if $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and show that $A^{-1} = \dfrac{A^2 - 3I}{2}$.

Sol. Here, $\quad A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$$|A| = 0\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + 1\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$$

$= 0 - 1(0 - 1) + 1(1 - 0)$

$= 1 + 1 = 2 \neq 0$ (non-singular matrix.)

Now, co-factors,

$$a_{11} = +\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1, \quad a_{12} = -\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 1, \quad a_{13} = +\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$$

$$a_{21} = -\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 1, \quad a_{22} = +\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1, \quad a_{23} = -\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = 1$$

$$a_{31} = +\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1, \quad a_{32} = -\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = 1, \quad a_{33} = +\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$$

$$\text{Adj}(A) = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}' = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{|A|}\text{Adj}(A) = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

Now, $A^2 = A \cdot A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 0+1+1 & 0+0+1 & 0+1+0 \\ 0+0+1 & 1+0+1 & 1+0+0 \\ 0+1+0 & 1+0+0 & 1+1+0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Hence, $A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

Now, we have to prove that $A^{-1} = \dfrac{A^2 - 3I}{2}$

$$\text{R.H.S.} = \frac{\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}{2}$$

$$= \frac{\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}}{2} = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$= A^{-1} = \text{L.H.S.}$

Hence, proved.

LONG ANSWER TYPE QUESTIONS

Q18. If $A = \begin{bmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 0 & -1 & 1 \end{bmatrix}$, find A^{-1}. Using A^{-1}, solve the system of linear equations $x - 2y = 10$, $2x - y - z = 8$, $-2y + z = 7$.

Sol. Given that

$$A = \begin{bmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 0 & -1 & 1 \end{bmatrix}$$

$$|A| = 1\begin{vmatrix} -1 & -2 \\ -1 & 1 \end{vmatrix} - 2\begin{vmatrix} -2 & -2 \\ 0 & 1 \end{vmatrix} + 0\begin{bmatrix} -2 & -1 \\ 0 & -1 \end{bmatrix}$$

$$= 1(-1-2) - 2(-2-0) + 0$$

$$= -3 + 4 = 1 \neq 0 \text{ (non-singular matrix.)}$$

Now co-factors,

$$a_{11} = +\begin{vmatrix} -1 & -2 \\ -1 & 1 \end{vmatrix} = -3,\ a_{12} = -\begin{vmatrix} -2 & -2 \\ 0 & 1 \end{vmatrix} = 2,\ a_{13} = +\begin{vmatrix} -2 & -1 \\ 0 & -1 \end{vmatrix} = 2$$

$$a_{21} = -\begin{vmatrix} 2 & 0 \\ -1 & 1 \end{vmatrix} = -2,\ a_{22} = +\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1,\ a_{23} = -\begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} = 1$$

$$a_{31} = +\begin{vmatrix} 2 & 0 \\ -1 & -2 \end{vmatrix} = -4,\ a_{32} = -\begin{vmatrix} 1 & 0 \\ -2 & -2 \end{vmatrix} = 2,\ a_{33} = +\begin{vmatrix} 1 & 2 \\ -2 & -1 \end{vmatrix} = 3$$

$$\text{Adj}(A) = \begin{bmatrix} -3 & 2 & 2 \\ -2 & 1 & 1 \\ -4 & 2 & 3 \end{bmatrix}' = \begin{bmatrix} -3 & -2 & -4 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{|A|}\text{Adj}(A) = \frac{1}{1}\begin{bmatrix} -3 & -2 & -4 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}$$

$$\Rightarrow \quad A^{-1} = \begin{bmatrix} -3 & -2 & -4 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}$$

Now, the system of linear equations is given by $x - 2y = 10$, $2x - y - z = 8$ and $-2y + z = 7$, which is in the form of $CX = D$.

$$\begin{bmatrix} 1 & -2 & 0 \\ 2 & -1 & -1 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix}$$

where $C = \begin{bmatrix} 1 & -2 & 0 \\ 2 & -1 & -1 \\ 0 & -2 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $D = \begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix}$

$\because \quad (A^T)^{-1} = (A^{-1})^T$

$$\therefore \quad C^T = \begin{bmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 0 & -1 & 1 \end{bmatrix} = A$$

$$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = C^{-1}D \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 & 2 & 2 \\ -2 & 1 & 1 \\ -4 & 2 & 3 \end{bmatrix}\begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -30+16+14 \\ -20+8+7 \\ -40+16+21 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ -3 \end{bmatrix}$$

Hence, $x = 0, y = -5$ and $z = -3$

Q19. Using matrix method, solve the system of equation $3x + 2y - 2z = 3, x + 2y + 3z = 6, 2x - y + z = 2.$

Sol. Given that

$$3x + 2y - 2z = 3$$
$$x + 2y + 3z = 6$$
$$2x - y + z = 2$$

Let $\quad A = \begin{bmatrix} 3 & 2 & -2 \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$

$$|A| = 3\begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} - 2\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}$$

$$= 3(2+3) - 2(1-6) - 2(-1-4)$$

$$= 15 + 10 + 10 = 35 \neq 0 \text{ non-singular matrix}$$

Now, co-factors,

$$a_{11} = +\begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} = 5,\ a_{12} = -\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = 5,\ a_{13} = +\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = -5$$

$$a_{21} = -\begin{vmatrix} 2 & -2 \\ -1 & 1 \end{vmatrix} = 0,\ a_{22} = +\begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} = 7,\ a_{23} = -\begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix} = 7$$

$$a_{31} = +\begin{vmatrix} 2 & -2 \\ 2 & 3 \end{vmatrix} = 10,\ a_{32} = -\begin{vmatrix} 3 & -2 \\ 1 & 3 \end{vmatrix} = -11,\ a_{33} = +\begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} = 4$$

$$\text{Adj }(A) = \begin{bmatrix} 5 & 5 & -5 \\ 0 & 7 & 7 \\ 10 & -11 & 4 \end{bmatrix}' = \begin{bmatrix} 5 & 0 & 10 \\ 5 & 7 & -11 \\ -5 & 7 & 4 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{|A|}\text{Adj}(A) = \frac{1}{35}\begin{bmatrix} 5 & 0 & 10 \\ 5 & 7 & -11 \\ -5 & 7 & 4 \end{bmatrix}$$

Now, $\quad X = A^{-1}B$

$$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{35}\begin{bmatrix} 5 & 0 & 10 \\ 5 & 7 & -11 \\ -5 & 7 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} = \frac{1}{35}\begin{bmatrix} 15+0+20 \\ 15+42-22 \\ -15+42+8 \end{bmatrix} = \frac{1}{35}\begin{bmatrix} 35 \\ 35 \\ 35 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Hence, $x = 1, y = 1$ and $z = 1$.

Q20. If $A = \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$, then find BA and use this to solve the system of equations $y + 2z = 7, x - y = 3$ and $2x + 3y + 4z = 17$.

Sol. We have, $A = \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$

$$BA = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2+4+0 & 2-2+0 & -4+4+0 \\ 4-12+8 & 4+6-4 & -8-12+20 \\ 0-4+4 & 0+2-2 & 0-4+10 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} = 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 6\,I$$

$$\therefore \quad B^{-1} = \frac{1}{6}A = \frac{1}{6}\begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$$

The given equations can be re-write as,

$x-y=3$, $2x+3y+4z=17$ and $y+2z=7$

$$\therefore \quad \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 6+34-28 \\ -12+34-28 \\ 6-17+35 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 12 \\ -6 \\ 24 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$$

Hence, $x=2$, $y=-1$ and $z=4$

Q21. If $a+b+c \neq 0$ and $\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} = 0$, then prove that $a=b=c$.

Sol. Given that: $a+b+c \neq 0$ and $\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} = 0$

$C_1 \rightarrow C_1 + C_2 + C_3$

$$\Rightarrow \quad \begin{bmatrix} a+b+c & b & c \\ a+b+c & c & a \\ a+b+c & a & b \end{bmatrix} = 0$$

$$\Rightarrow \quad (a+b+c) \begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix} = 0 \quad \text{(Taking } a+b+c \text{ common from } C_1\text{)}$$

$$\Rightarrow \quad a+b+c \neq 0 \quad \therefore \begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix} = 0$$

$R_1 \rightarrow R_1 - R_2$ and $R_2 \rightarrow R_2 - R_3$

$$\Rightarrow \quad \begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 1 & a & b \end{vmatrix} = 0$$

Expanding along C_1

$$\Rightarrow \quad 1\begin{vmatrix} b-c & c-a \\ c-a & a-b \end{vmatrix} = 0$$

$$\Rightarrow \quad (b-c)(a-b)-(c-a)^2 = 0$$

$$\Rightarrow \quad ab-b^2-ac+bc-c^2-a^2+2ac = 0$$

$$\Rightarrow \quad -a^2-b^2-c^2+ab+bc+ac = 0$$

$$\Rightarrow \quad a^2+b^2+c^2-ab-bc-ac = 0$$

$$\Rightarrow \quad 2a^2+2b^2+2c^2-2ab-2bc-2ac = 0$$

(Multiplying both sides by 2)

$$\Rightarrow \quad (a^2+b^2-2ab)+(b^2+c^2-2bc)+(a^2+c^2-2ac) = 0$$

$$\Rightarrow \quad (a-b)^2+(b-c)^2+(a-c)^2 = 0$$

It is only possible when $(a-b)^2 = (b-c)^2 = (a-c)^2 = 0$

$\therefore \quad a = b = c$ Hence, proved.

Q22. Prove that $\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ac-b^2 \end{vmatrix}$ is divisible by $a+b+c$ and find the quotient.

Sol. Let $\Delta = \begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ac-b^2 \end{vmatrix}$

$C_1 \to C_1 + C_2 + C_3$

$$\Rightarrow \quad \begin{vmatrix} ab+bc+ac-a^2-b^2-c^2 & ca-b^2 & ab-c^2 \\ ab+bc+ac-a^2-b^2-c^2 & ab-c^2 & bc-a^2 \\ ab+bc+ac-a^2-b^2-c^2 & bc-a^2 & ac-b^2 \end{vmatrix}$$

Taking $ab+bc+ac-a^2-b^2-c^2$ common from C_1

$$(ab+bc+ac-a^2-b^2-c^2)\begin{vmatrix} 1 & ca-b^2 & ab-c^2 \\ 1 & ab-c^2 & bc-a^2 \\ 1 & bc-a^2 & ac-b^2 \end{vmatrix}$$

$R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$

$$\Rightarrow (ab+bc+ac-a^2-b^2-c^2)\begin{vmatrix} 0 & ca-b^2-ab+c^2 & ab-c^2-bc+a^2 \\ 0 & ab-c^2-bc+a^2 & bc-a^2-ac+b^2 \\ 1 & bc-a^2 & ac-b^2 \end{vmatrix}$$

$$\Rightarrow (ab+bc+ac-a^2-b^2-c^2)\begin{vmatrix} 0 & a(c-b)+(c+b)(c-b) & b(a-c)+(a+c)(a-c) \\ 0 & b(a-c)+(a+c)(a-c) & c(b-a)+(b+a)(b-a) \\ 1 & bc-a^2 & ac-b^2 \end{vmatrix}$$

$$\Rightarrow (ab+bc+ac-a^2-b^2-c^2)\begin{vmatrix} 0 & (c-b)(a+b+c) & (a-c)(a+b+c) \\ 0 & (a-c)(a+b+c) & (b-a)(a+b+c) \\ 1 & bc-a^2 & ac-b^2 \end{vmatrix}$$

$$\Rightarrow (ab+bc+ac-a^2-b^2-c^2)(a+b+c)(a+b+c)\begin{vmatrix} 0 & c-b & a-c \\ 0 & a-c & b-a \\ 1 & bc-a^2 & ac-b^2 \end{vmatrix}$$

$$\Rightarrow (a+b+c)^2(ab+bc+ac-a^2-b^2-c^2)\begin{vmatrix} 0 & c-b & a-c \\ 0 & a-c & b-a \\ 1 & bc-a^2 & ac-b^2 \end{vmatrix}$$

Expanding along C_1

$$\Rightarrow (a+b+c)^2(ab+bc+ac-a^2-b^2-c^2)\left[1\begin{vmatrix} c-b & a-c \\ a-c & b-a \end{vmatrix}\right]$$

$$\Rightarrow (a+b+c)^2(ab+bc+ac-a^2-b^2-c^2)[(c-b)(b-a)-(a-c)^2]$$

$$\Rightarrow (a+b+c)^2(ab+bc+ac-a^2-b^2-c^2)(bc-ca-b^2+ab-a^2-c^2+2ac)$$

$$\Rightarrow (a+b+c)^2(ab+bc+ac-a^2-b^2-c^2)(ab+bc+ca-a^2-b^2-c^2)$$

$$\Rightarrow (a+b+c)^2(ab+bc+ac-a^2-b^2-c^2)^2$$

$$\Rightarrow (a+b+c)(a+b+c)(a^2+b^2+c^2-ab-bc-ac)^2$$

Hence, the given determinant is divisible by a + b + c and the quotient is

$$(a+b+c)(a^2+b^2+c^2-ab-bc-ac)^2$$

$$\Rightarrow (a+b+c)(a^2+b^2+c^2-ab-bc-ac)(a^2+b^2+c^2-ab-bc-ac)$$

$$\Rightarrow (a^3+b^3+c^3-3abc)(a^2+b^2+c^2-ab-bc-ac)$$

$$\Rightarrow \quad -(a^3+b^3+c^3-3abc)(2a^2+2b^2+2c^2-2ab-2bc-2ac)$$

$$\Rightarrow \quad \frac{1}{2}(a^3+b^3+c^3-3abc)[(a-b)^2+(b-c)^2+(a-c)^2]$$

Q23. If $x+y+z=0$, prove that $\begin{vmatrix} xa & yb & zc \\ yc & za & xb \\ zb & xc & ya \end{vmatrix} = xyz\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$

Sol. L.H.S.

Let $\quad \Delta = \begin{vmatrix} xa & yb & zc \\ yc & za & xb \\ zb & xc & ya \end{vmatrix}$

Expanding along R_1

$$\Rightarrow \quad xa\begin{vmatrix} za & xb \\ xc & ya \end{vmatrix} - yb\begin{vmatrix} yc & xb \\ zb & ya \end{vmatrix} + zc\begin{vmatrix} yc & za \\ zb & xc \end{vmatrix}$$

$$\Rightarrow \quad xa(yza^2-x^2bc)-yb(y^2ac-xzb^2)+zc(xyc^2-z^2ab)$$

$$\Rightarrow \quad xyza^3-x^3abc-y^3abc+xyzb^3+xyzc^3-z^3abc$$

$$\Rightarrow \quad xyz(a^3+b^3+c^3)-abc(x^3+y^3+z^3)$$

$$\Rightarrow \quad xyz(a^3+b^3+c^3)-abc(3xyz)$$

$$[(\because \quad x+y+z=0)\ (\therefore \quad x^3+y^3+z^3=3xyz)]$$

$$\Rightarrow \quad xyz(a^3+b^3+c^3-3abc)$$

R.H.S. $\quad xyz\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$

$R_1 \to R_1+R_2+R_3$

$$\Rightarrow \quad xyz\begin{vmatrix} a+b+c & a+b+c & a+b+c \\ c & a & b \\ b & c & a \end{vmatrix}$$

$$\Rightarrow \quad xyz(a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ c & a & b \\ b & c & a \end{vmatrix}$$ (Taking $a+b+c$ common from R_1)

$C_1 \to C_1-C_2,\ C_2 \to C_2-C_3$

$$\Rightarrow \quad xyz(a+b+c)\begin{vmatrix} 0 & 0 & 1 \\ c-a & a-b & b \\ b-c & c-a & a \end{vmatrix}$$

Expanding along R_1

$$\Rightarrow \quad xyz(a+b+c)\left[1\begin{vmatrix} c-a & a-b \\ b-c & c-a \end{vmatrix}\right]$$

$$\Rightarrow \quad xyz(a+b+c)\left[(c-a)^2-(b-c)(a-b)\right]$$

$$\Rightarrow \quad xyz(a+b+c)(c^2+a^2-2ca-ab+b^2+ac-bc)$$

$$\Rightarrow \quad xyz(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$

$$\Rightarrow \quad xyz(a^3+b^3+c^3-3abc)$$

$$[a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)]$$

L.H.S. = R.H.S.

Hence, proved.

OBJECTIVE TYPE QUESTIONS (M.C.Q.)

Choose the correct answer from given four options in each of the Exercises from 24 to 37.

Q24. If $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix}=\begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}$, then the value of x is

(*a*) 3 (*b*) ± 3 (*c*) ± 6 (*d*) 6

Sol. Given that

$$\Rightarrow \quad \begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix}=\begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}$$

$$\Rightarrow \quad 2x^2-40=18+14 \Rightarrow 2x^2=32+40$$

$$\Rightarrow \quad 2x^2=72 \Rightarrow x^2=36$$

$$\therefore \quad x=\pm 6$$

Hence, the correct option is (*c*).

Q25. The value of determinant $\begin{vmatrix} a-b & b+c & a \\ b-a & c+a & b \\ c-a & a+b & c \end{vmatrix}$ is

(*a*) $a^3+b^3+c^3$ (*b*) $3bc$

(*c*) $a^3+b^3+c^3-3abc$ (*d*) None of these

Sol. Here, we have $\begin{vmatrix} a-b & b+c & a \\ b-a & c+a & b \\ c-a & a+b & c \end{vmatrix}$

$C_2 \to C_2 + C_3$

$$\Rightarrow \begin{vmatrix} a-b & a+b+c & a \\ b-a & a+b+c & b \\ c-a & a+b+c & c \end{vmatrix}$$

$$\Rightarrow \quad (a+b+c)\begin{vmatrix} a-b & 1 & a \\ b-a & 1 & b \\ c-a & 1 & c \end{vmatrix}$$ (Taking $a+b+c$ common from C_2)

$R_1 \to R_1 - R_2, R_2 \to R_2 - R_3$

$$\Rightarrow \quad (a+b+c)\begin{vmatrix} 2(a-b) & 0 & a-b \\ b-c & 0 & b-c \\ c-a & 1 & c \end{vmatrix}$$

Taking $(a-b)$ and $(b-c)$ common from R_1 and R_2 respectively

$$\Rightarrow \quad (a+b+c)(a-b)(b-c)\begin{vmatrix} 2 & 0 & 1 \\ 1 & 0 & 1 \\ c-a & 1 & c \end{vmatrix}$$

Expanding along C_2

$$\Rightarrow \quad (a+b+c)(a-b)(b-c)\left[-1\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}\right]$$

$$\Rightarrow \quad (a+b+c)(a-b)(b-c)(-1)$$

$$\Rightarrow \quad (a+b+c)(a-b)(c-b)$$

Hence, the correct option is (*d*).

Q26. The area of a triangle with vertices $(-3, 0)$, $(3, 0)$ and $(0, k)$ is 9 sq units. Then, the value of k will be

(*a*) 9 (*b*) 3 (*c*) -9 (*d*) 6

Sol. Area of triangle with vertices $(x_1\, y_1)$, $(x_2\, y_2)$ and (x_3, y_3) will be:

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \Rightarrow \Delta = \frac{1}{2}\begin{vmatrix} -3 & 0 & 1 \\ 3 & 0 & 1 \\ 0 & k & 1 \end{vmatrix}$$

$$\Rightarrow \quad = \frac{1}{2}\left[-3\begin{vmatrix} 0 & 1 \\ k & 1 \end{vmatrix} - 0\begin{vmatrix} 3 & 1 \\ 0 & 1 \end{vmatrix} + 1\begin{vmatrix} 3 & 0 \\ 0 & k \end{vmatrix}\right]$$

$$\Rightarrow \quad = \frac{1}{2}[-3(-k) - 0 + 1(3k)]$$

$$\Rightarrow \quad = \frac{1}{2}(3k+3k) \quad \Rightarrow \quad \frac{1}{2}(6k) = 3k$$

$$3k = 9 \quad \Rightarrow \quad k = 3$$

Hence, the correct option is (*b*).

Q27. The determinant $\begin{vmatrix} b^2-ab & b-c & bc-ac \\ ab-a^2 & a-b & b^2-ab \\ bc-ac & c-a & ab-a^2 \end{vmatrix}$ equals

(*a*) $abc(b-c)(c-a)(a-b)$ (*b*) $(b-c)(c-a)(a-b)$
(*c*) $(a+b+c)(b-c)(c-a)(a-b)$ (*d*) None of these

Sol. Let
$$\Delta = \begin{vmatrix} b^2-ab & b-c & bc-ac \\ ab-a^2 & a-b & b^2-ab \\ bc-ac & c-a & ab-a^2 \end{vmatrix}$$

$$= \begin{vmatrix} b(b-a) & b-c & c(b-a) \\ a(b-a) & a-b & b(b-a) \\ c(b-a) & c-a & a(b-a) \end{vmatrix}$$ (Taking $(b-a)$ common from C_1 and C_3)

$$= (b-a)^2 \begin{vmatrix} b & b-c & c \\ a & a-b & b \\ c & c-a & a \end{vmatrix}$$

$C_1 \to C_1 - C_3$

$$= (a-b)^2 \begin{vmatrix} b-c & b-c & c \\ a-b & a-b & b \\ c-a & c-a & a \end{vmatrix}$$ (C_1 and C_2 are identical columns.)

$$= (a-b)^2 \cdot 0$$
$$= 0$$

Hence, the correct option is (*d*).

Q28. The number of distinct real roots of $\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$ in the interval $-\frac{\pi}{4} \le x \le \frac{\pi}{4}$ is

(*a*) 0 (*b*) 2 (*c*) 1 (*d*) 3

Sol. Given that

$$\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$$

$C_1 \to C_1 + C_2 + C_3$

$$\Rightarrow \quad \begin{vmatrix} 2\cos x + \sin x & \cos x & \cos x \\ 2\cos x + \sin x & \sin x & \cos x \\ 2\cos x + \sin x & \cos x & \sin x \end{vmatrix} = 0$$

Taking $2\cos x + \sin x$ common from C_1

$$\Rightarrow \quad (2\cos x + \sin x)\begin{vmatrix} 1 & \cos x & \cos x \\ 1 & \sin x & \cos x \\ 1 & \cos x & \sin x \end{vmatrix} = 0$$

$R_1 \to R_1 - R_2, R_2 \to R_2 - R_3$

$$\Rightarrow (2\cos x + \sin x)\begin{vmatrix} 0 & \cos x - \sin x & 0 \\ 0 & \sin x - \cos x & \cos x - \sin x \\ 1 & \cos x & \sin x \end{vmatrix} = 0$$

$$\Rightarrow (2\cos x + \sin x)\left[1\begin{vmatrix} \cos x - \sin x & 0 \\ \sin x - \cos x & \cos x - \sin x \end{vmatrix}\right]$$

$$\Rightarrow \quad (2\cos x + \sin x)(\cos x - \sin x)^2 = 0$$

$2\cos x + \sin x = 0$	$(\cos x - \sin x)^2 = 0$
$2 + \tan x = 0$	$\cos x - \sin x = 0$
$\therefore \quad \tan x = -2$	$\Rightarrow \quad \tan x = 1$
But $\quad -\frac{\pi}{4} \le x \le \frac{\pi}{4}$	$\Rightarrow \quad \tan x = \tan\frac{\pi}{4}$
	$\therefore \quad x = \frac{\pi}{4} \in \left[\frac{-\pi}{4}, \frac{\pi}{4}\right]$
So, x has no solution.	So, it will have only one real root.

Hence, the correct option is (*c*).

Q29. If A, B and C are angles of a triangle, then the determinant $\begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix}$ is equal to

(*a*) 0 (*b*) -1 (*c*) 1 (*d*) None of these

Sol. Let $\Delta = \begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix}$

$C_1 \to aC_1 + bC_2 + cC_3$

$$\Rightarrow \begin{vmatrix} -a + b\cos C + c\cos B & \cos C & \cos B \\ a\cos C - b + c\operatorname{Cos} A & -1 & \cos A \\ a\cos B + b\cos A - C & \cos A & -1 \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} -a+a & \cos C & \cos B \\ -b+b & -1 & \cos A \\ -c+c & \cos A & -1 \end{vmatrix} \qquad \left[\begin{array}{l} \because \text{ From projection formula} \\ a = b\cos C + c\cos B \\ b = a\cos C + c\cos A \\ c = b\cos A + a\cos B \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} 0 & \cos C & \cos B \\ 0 & -1 & \cos A \\ 0 & \cos A & -1 \end{bmatrix} = 0$$

Hence, the correct option is (*a*).

Q30. Let $f(t) = \begin{bmatrix} \cos t & t & 1 \\ 2\sin t & t & 2t \\ \sin t & t & t \end{bmatrix}$, then $\lim\limits_{t \to 0} \dfrac{f(t)}{t^2}$ is equal to

(*a*) 0 (*b*) -1 (*c*) 2 (*d*) 3

Sol. We have $f(t) = \begin{bmatrix} \cos t & t & 1 \\ 2\sin t & t & 2t \\ \sin t & t & t \end{bmatrix}$

Expanding along R_1

$$= \cos t \begin{vmatrix} t & 2t \\ t & t \end{vmatrix} - t \begin{vmatrix} 2\sin t & 2t \\ \sin t & t \end{vmatrix} + 1 \begin{vmatrix} 2\sin t & t \\ \sin t & t \end{vmatrix}$$

$$= \cos t(t^2 - 2t^2) - t(2t\sin t - 2t\sin t) + (2t\sin t - t\sin t)$$

$$= -t^2\cos t + t\sin t$$

$$\therefore \quad \frac{f(t)}{t^2} = \frac{-t^2\cos t + t\sin t}{t^2}$$

$$\Rightarrow \quad \frac{f(t)}{t^2} = -\cos t + \frac{\sin t}{t}$$

$$\Rightarrow \quad \lim_{t \to 0} \frac{f(t)}{t^2} = \lim_{t \to 0}(-\cos t) + \lim_{t \to 0} \frac{\sin t}{t} = -1 + 1 = 0$$

Hence, the correct option is (*a*).

Q31. The maximum value of

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+\sin\theta & 1 \\ 1+\cos\theta & 1 & 1 \end{vmatrix} \text{ is} \qquad (\theta \text{ is real number})$$

(*a*) $\dfrac{1}{2}$ (*b*) $\dfrac{\sqrt{3}}{2}$ (*c*) $\sqrt{2}$ (*d*) $\dfrac{2\sqrt{3}}{4}$

Sol. Given that: $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+\sin\theta & 1 \\ 1+\cos\theta & 1 & 1 \end{vmatrix}$

$C_1 \to C_1 - C_2$, $C_2 \to C_2 - C_3$

$$= \begin{vmatrix} 0 & 0 & 1 \\ -\sin\theta & \sin\theta & 1 \\ \cos\theta & 0 & 1 \end{vmatrix}$$

Expanding along R_1

$$= 1\begin{vmatrix} -\sin\theta & \sin\theta \\ \cos\theta & 0 \end{vmatrix} = -\sin\theta\cos\theta$$

$$\Rightarrow \qquad = -\frac{1}{2}\cdot 2\sin\theta\cos\theta = -\frac{1}{2}\sin 2\theta$$

but maximum value of $\sin 2\theta = 1 \Rightarrow \left|-\frac{1}{2}\cdot 1\right| = \frac{1}{2}$

Hence, the correct option is (*a*).

Q32. If $f(x) = \begin{vmatrix} 0 & x-a & x-b \\ x+a & 0 & x-c \\ x+b & x+c & 0 \end{vmatrix}$, then

(*a*) $f(a) = 0$ (*b*) $f(b) = 0$ (*c*) $f(0) = 0$ (*d*) $f(1) = 0$

Sol. Given that: $f(x) = \begin{vmatrix} 0 & x-a & x-b \\ x+a & 0 & x-c \\ x+b & x+c & 0 \end{vmatrix}$

$$f(a) = \begin{vmatrix} 0 & 0 & a-b \\ 2a & 0 & a-c \\ a+b & a+c & 0 \end{vmatrix}$$

Expanding along $R_1 = (a-b)\begin{vmatrix} 2a & 0 \\ a+b & a+c \end{vmatrix}$

$$= (a-b)\,[2a(a+c)] = (a-b)\cdot 2a\cdot(a+c) \neq 0$$

$$f(b) = \begin{vmatrix} 0 & b-a & 0 \\ b+a & 0 & b-c \\ 2b & b+c & 0 \end{vmatrix}$$

Expanding along R_1

$$-(b-a)\begin{vmatrix} b+a & b-c \\ 2b & 0 \end{vmatrix}$$

$$= -(b-a)\,[(-2b)(b-c)] = 2b(b-a)(b-c) \neq 0$$

$$f(0) = \begin{vmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{vmatrix}$$

Expanding along $R_1 = a\begin{vmatrix} a & -c \\ b & 0 \end{vmatrix} - b\begin{vmatrix} a & 0 \\ b & c \end{vmatrix}$

$$= a(bc) - b(ac) = abc - abc = 0$$

Hence, the correct option is (*c*).

Q33. If $A = \begin{bmatrix} 2 & \lambda & -3 \\ 0 & 2 & 5 \\ 1 & 1 & 3 \end{bmatrix}$, then A^{-1} exists if

(*a*) $\lambda = 2$ (*b*) $\lambda \neq 2$ (*c*) $\lambda \neq -2$ (*d*) None of these

Sol. We have,

$$A = \begin{bmatrix} 2 & \lambda & -3 \\ 0 & 2 & 5 \\ 1 & 1 & 3 \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} 2 & \lambda & -3 \\ 0 & 2 & 5 \\ 1 & 1 & 3 \end{vmatrix}$$

Expanding along $R_1 = 2\begin{vmatrix} 2 & 5 \\ 1 & 3 \end{vmatrix} - \lambda\begin{vmatrix} 0 & 5 \\ 1 & 3 \end{vmatrix} - 3\begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix}$

$$= 2(6-5) - \lambda(0-5) - 3(0-2)$$

$$= 2 + 5\lambda + 6 = 8 + 5\lambda$$

If A^{-1} exists then $|A| \neq 0$

$\therefore$ $8 + 5\lambda \neq 0$ so $\lambda \neq \dfrac{-8}{5}$

Hence, the correct option is (*d*).

Q34. If A and B are invertible matrices, then which of the following is not correct?

(*a*) $\text{adj } A = |A| \cdot A^{-1}$ (*b*) $\det (A)^{-1} = [\det(A)]^{-1}$

(*c*) $(AB)^{-1} = B^{-1}A^{-1}$ (*d*) $(A+B)^{-1} = B^{-1} + A^{-1}$

Sol. If A and B are two invertible matrices then

(*a*) $\text{adj } A = |A| \cdot A^{-1}$ is correct

(*b*) $\det (A)^{-1} = [\det(A)]^{-1} = \dfrac{1}{\det(A)}$ is correct

(*c*) Also, $(AB)^{-1} = B^{-1}A^{-1}$ is correct

(*d*) $(A+B)^{-1} = \dfrac{1}{|A+B|} \cdot \text{adj}(A+B)$

$\therefore$ $(A+B)^{-1} \neq B^{-1} + A^{-1}$

Hence, the correct option is (*d*).

Q35. If *x*, *y*, *z* are all different from zero an $\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix} = 0$, then the value of $x^{-1}+y^{-1}+z^{-1}$ is

(a) xyz (b) $x^{-1}y^{-1}z^{-1}$ (c) $-x-y-z$ (d) -1

Sol. Given that

$$\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix}=0$$

Taking x, y and z common from R_1, R_2 and R_3 respectively.

$$\Rightarrow \quad xyz\begin{vmatrix} \frac{1}{x}+1 & \frac{1}{x} & \frac{1}{x} \\ \frac{1}{y} & \frac{1}{y}+1 & \frac{1}{y} \\ \frac{1}{z} & \frac{1}{z} & \frac{1}{z}+1 \end{vmatrix}=0$$

$R_1 \to R_1+R_2+R_3$

$$\Rightarrow \quad xyz\begin{vmatrix} \frac{1}{x}+\frac{1}{y}+\frac{1}{z}+1 & \frac{1}{x}+\frac{1}{y}+\frac{1}{z}+1 & \frac{1}{x}+\frac{1}{y}+\frac{1}{z}+1 \\ \frac{1}{y} & \frac{1}{y}+1 & \frac{1}{y} \\ \frac{1}{z} & \frac{1}{z} & \frac{1}{z}+1 \end{vmatrix}=0$$

Taking $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+1$ common from R_1

$$\Rightarrow \quad xyz\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+1\right)\begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{y} & \frac{1}{y}+1 & \frac{1}{y} \\ \frac{1}{z} & \frac{1}{z} & \frac{1}{z}+1 \end{vmatrix}=0$$

$C_1 \to C_1-C_2$, $C_2 \to C_2-C_3$

$$\Rightarrow \quad xyz\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+1\right)\begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & \frac{1}{y} \\ 0 & -1 & \frac{1}{z}+1 \end{vmatrix}=0$$

Expanding along R_1

$$\Rightarrow \quad xyz\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+1\right)\left[1\begin{vmatrix}-1 & 1\\ 0 & -1\end{vmatrix}\right] = 0$$

$$\Rightarrow \quad xyz\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+1\right)(1) = 0$$

$$\Rightarrow \quad \frac{1}{x}+\frac{1}{y}+\frac{1}{z}+1=0 \text{ and } xyz \neq 0 \qquad (x \neq y \neq z \neq 0)$$

$$\therefore \quad x^{-1}+y^{-1}+z^{-1} = -1$$

Hence, the correct option is (*d*).

Q36. The value of the determinant $\begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$ is

(*a*) $9x^2(x+y)$ (b) $9y^2(x+y)$ (c) $3y^2(x+y)$ (d) $7x^2(x+y)$

Sol. Let $\Delta = \begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$

$C_1 \to C_1 + C_2 + C_3$

$$= \begin{vmatrix} 3x+3y & x+y & x+2y \\ 3x+3y & x & x+y \\ 3x+3y & x+2y & x \end{vmatrix}$$

$$= (3x+3y)\begin{vmatrix} 1 & x+y & x+2y \\ 1 & x & x+y \\ 1 & x+2y & x \end{vmatrix}$$

[Taking $(3x+3y)$ common from C_1]

$R_1 \to R_1 - R_2$, $R_2 \to R_2 - R_3$

$$\Rightarrow \quad 3(x+y)\begin{vmatrix} 0 & y & y \\ 0 & -2y & y \\ 1 & x+2y & x \end{vmatrix}$$

Expanding along C_1

$$\Rightarrow \quad 3(x+y)\left[1\begin{vmatrix} y & y \\ -2y & y \end{vmatrix}\right]$$

$\Rightarrow \quad 3(x+y)(y^2+2y^2) \Rightarrow 3(x+y)(3y^2) \Rightarrow 9y^2(x+y)$

Hence, the correct option is (*b*).

Q37. There are two values of '*a*' which makes determinant, $\Delta = \begin{vmatrix} 1 & -2 & 5 \\ 2 & a & -1 \\ 0 & 4 & 2a \end{vmatrix} = 86$, then sum of these numbers is

(*a*) 4 (*b*) 5 (*c*) -4 (*d*) 9

Sol. Given that, $\Delta = \begin{vmatrix} 1 & -2 & 5 \\ 2 & a & -1 \\ 0 & 4 & 2a \end{vmatrix} = 86$

Expanding along C_1

$$\Rightarrow 1\begin{vmatrix} a & -1 \\ 4 & 2a \end{vmatrix} - 2\begin{vmatrix} -2 & 5 \\ 4 & 2a \end{vmatrix} + 0\begin{vmatrix} -2 & 5 \\ a & -1 \end{vmatrix} = 86$$

$$\Rightarrow \quad (2a^2+4) - 2(-4a-20) = 86$$

$$\Rightarrow \quad 2a^2+4+8a+40 = 86$$

$$\Rightarrow \quad 2a^2+8a+4+40-86 = 0$$

$$\Rightarrow \quad 2a^2+8a-42 = 0$$

$$\Rightarrow \quad a^2+4a-21 = 0$$

$$\Rightarrow \quad a^2+7a-3a-21 = 0$$

$$\Rightarrow \quad a(a+7)-3(a+7) = 0$$

$$\Rightarrow \quad (a-3)(a+7) = 0$$

$\therefore \quad a = 3, -7$

Required sum of the two numbers $= 3-7 = -4$.

Hence, the correct option is (*c*).

Fill in the Blanks

Q38. If A is a matrix of order 3×3, then $|3A| =$ ________ .

Sol. We know that for a matrix of order 3×3,

$$|KA| = K^3 |A|$$

$$\therefore \quad |3A| = 3^3|A| = 27|A|$$

Q39. If A is invertible matrix of order 3×3, then $|A^{-1}|$ ________ .

Sol. We know that for an invertible matrix A of any order, $|A^{-1}| = \dfrac{1}{|A|}$.

Q40. If $x, y, z \in R$, then the value of determinant

$$\begin{vmatrix} (2^x+2^{-x})^2 & (2^x-2^{-x})^2 & 1 \\ (3^x+3^{-x})^2 & (3^x-3^{-x})^2 & 1 \\ (4^x+4^{-x})^2 & (4^x-4^{-x})^2 & 1 \end{vmatrix} \text{ is equal to } \_\_\_\_\_\_\_\_ .$$

Sol. We have, $\begin{vmatrix} (2^x+2^{-x})^2 & (2^x-2^{-x})^2 & 1 \\ (3^x+3^{-x})^2 & (3^x-3^{-x})^2 & 1 \\ (4^x+4^{-x})^2 & (4^x-4^{-x})^2 & 1 \end{vmatrix}$

$C_1 \to C_1 - C_2$

$$\Rightarrow \begin{vmatrix} (2^x+2^{-x})^2-(2^x-2^{-x})^2 & (2^x-2^{-x})^2 & 1 \\ (3^x+3^{-x})^2-(3^x-3^{-x})^2 & (3^x-3^{-x})^2 & 1 \\ (4^x+4^{-x})^2-(4^x-4^{-x})^2 & (4^x-4^{-x})^2 & 1 \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} 4\cdot 2^x\cdot 2^{-x} & (2^x-2^{-x})^2 & 1 \\ 4\cdot 3^x\cdot 3^{-x} & (3^x-3^{-x})^2 & 1 \\ 4\cdot 4^x\cdot 4^{-x} & (4^x-4^{-x})^2 & 1 \end{vmatrix} \quad \text{[applying } (a+b)^2-(a-b)^2=4ab]$$

$$\Rightarrow \begin{vmatrix} 4 & (2^x-2^{-x})^2 & 1 \\ 4 & (3^x-3^{-x})^2 & 1 \\ 4 & (4^x-4^{-x})^2 & 1 \end{vmatrix}$$

$$\Rightarrow 4\begin{vmatrix} 1 & (2^x-2^{-x})^2 & 1 \\ 1 & (3^x-3^{-x})^2 & 1 \\ 1 & (4^x-4^{-x})^2 & 1 \end{vmatrix} \quad \text{(Taking 4 common from } C_1)$$

$$\Rightarrow 4\cdot 0 = 0 \quad (\because \ C_1 \text{ and } C_3 \text{ are identical columns})$$

Q41. If $\cos 2\theta = 0$, then $\begin{vmatrix} 0 & \cos\theta & \sin\theta \\ \cos\theta & \sin\theta & 0 \\ \sin\theta & 0 & \cos\theta \end{vmatrix}^2 = \_\_\_\_\_\_\_\_ .$

Sol. Given that: $\cos 2\theta = 0$

$$\Rightarrow \cos 2\theta = \cos\frac{\pi}{2} \Rightarrow 2\theta = \frac{\pi}{2}$$

$$\therefore \theta = \frac{\pi}{4}$$

The determinant can be written as

$$\Rightarrow \begin{vmatrix} 0 & \cos\frac{\pi}{4} & \sin\frac{\pi}{4} \\ \cos\frac{\pi}{4} & \sin\frac{\pi}{4} & 0 \\ \sin\frac{\pi}{4} & 0 & \cos\frac{\pi}{4} \end{vmatrix}^2 \Rightarrow \begin{vmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{vmatrix}^2$$

$$\Rightarrow \left[\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix}\right]^2 \quad \left(\text{Taking } \frac{1}{\sqrt{2}} \text{ common from } C_1, C_2 \text{ and } C_3\right)$$

Expanding along C_1,

$$\Rightarrow \left[\frac{1}{2\sqrt{2}}\left|-1\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}+1\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}\right|\right]^2 \Rightarrow \left[\frac{1}{2\sqrt{2}}\left|-1(1)+1(0-1)\right|\right]^2$$

$$\Rightarrow \left[\frac{1}{2\sqrt{2}}\left|-1-1\right|\right]^2 \Rightarrow \frac{1}{8}\cdot(4) = \frac{1}{2}$$

Q42. If A is a matrix of order 3 × 3, then $(A^2)^{-1}$ = ________ .

Sol. For any square matrix A, $(A^2)^{-1} = \mathbf{(A^{-1})^2}$.

Q43. If A is a matrix of order 3 × 3, then the number of minors in the determinants of A are ________ .

Sol. The order of a matrix is 3 × 3

$\therefore$ Total number of elements = 3 × 3 = 9

Hence, the number of minors in the determinant is **9**.

Q44. The sum of the products of elements of any row with the co-factors of corresponding elements is equal to ________ .

Sol. The sum of the products of elements of any row with the co-factors of corresponding elements is equal to **the value of the determinant of the given matrix.**

$$\text{Let } \Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Expanding along R_1

$$a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\Rightarrow \quad a_{11}M_{11} + a_{12}M_{12} + a_{13}M_{13}$$

(where M_{11}, M_{12} and M_{13} are the minors of the corresponding elements)

Q45. If $x = -9$ is a root of $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$, then other two roots are ________.

Sol. We have, $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$

Expanding along R_1

$$\Rightarrow \quad x\begin{vmatrix} x & 2 \\ 6 & x \end{vmatrix} - 3\begin{vmatrix} 2 & 2 \\ 7 & x \end{vmatrix} + 7\begin{vmatrix} 2 & x \\ 7 & 6 \end{vmatrix} = 0$$

$$\Rightarrow \quad x(x^2 - 12) - 3(2x - 14) + 7(12 - 7x) = 0$$

$$\Rightarrow \quad x^3 - 12x - 6x + 42 + 84 - 49x = 0$$

$$\Rightarrow \quad x^3 - 67x + 126 = 0 \qquad \text{...(1)}$$

The roots of the equation may be the factors of 126 *i.e.*, $2 \times 7 \times 9$

9 is given the root of the determinant put $x = 2$ in eq. (1)

$$(2)^3 - 67 \times 2 + 126 \quad \Rightarrow \quad 8 - 134 + 126 = 0$$

Hence, $x = 2$ is the other root.

Now, put $x = 7$ in eq. (1)

$$(7)^3 - 67(7) + 126 \quad \Rightarrow \quad 343 - 469 + 126 = 0$$

Hence, $x = 7$ is also the other root of the determinant.

Q46. $\begin{vmatrix} 0 & xyz & x - z \\ y - x & 0 & y - z \\ z - x & z - y & 0 \end{vmatrix} =$ ________.

Sol. Let $\Delta = \begin{vmatrix} 0 & xyz & x - z \\ y - x & 0 & y - z \\ z - x & z - y & 0 \end{vmatrix}$

$C_1 \rightarrow C_1 - C_3$

$$= \begin{vmatrix} z - x & xyz & x - z \\ z - x & 0 & y - z \\ z - x & z - y & 0 \end{vmatrix}$$

Taking $(z - x)$ common from C_1

$$= (z - x)\begin{vmatrix} 1 & xyz & x - z \\ 1 & 0 & y - z \\ 1 & z - y & 0 \end{vmatrix}$$

$R_1 \to R_1 - R_2, R_2 \to R_2 - R_3$

$$= (z-x)\begin{vmatrix} 0 & xyz & x-y \\ 0 & y-z & y-z \\ 1 & z-y & 0 \end{vmatrix}$$

Taking $(y-z)$ common from R_2

$$= (z-x)(y-z)\begin{vmatrix} 0 & xyz & x-y \\ 0 & 1 & 1 \\ 1 & z-y & 0 \end{vmatrix}$$

Expanding along C_1

$$= (z-x)(y-z)\left[1\begin{vmatrix} xyz & x-y \\ 1 & 1 \end{vmatrix}\right]$$

$$= (z-x)(y-z)(xyz-x+y) = (y-z)(z-x)(y-x+xyz)$$

Q47. If $f(x) = \begin{vmatrix} (1+x)^{17} & (1+x)^{19} & (1+x)^{23} \\ (1+x)^{23} & (1+x)^{29} & (1+x)^{34} \\ (1+x)^{41} & (1+x)^{43} & (1+x)^{47} \end{vmatrix} = A + Bx + Cx^2 + \cdots$

then A = ________.

Sol. Given that

$$\begin{vmatrix} (1+x)^{17} & (1+x)^{19} & (1+x)^{23} \\ (1+x)^{23} & (1+x)^{29} & (1+x)^{34} \\ (1+x)^{41} & (1+x)^{43} & (1+x)^{47} \end{vmatrix} = A + Bx + Cx^2 + \cdots$$

Taking $(1+x)^{17}$, $(1+x)^{23}$ and $(1+x)^{41}$ common from R_1, R_2 and R_3 respectively

$$(1+x)^{17}\cdot(1+x)^{23}\cdot(1+x)^{41}\begin{vmatrix} 1 & (1+x)^2 & (1+x)^6 \\ 1 & (1+x)^6 & (1+x)^{11} \\ 1 & (1+x)^2 & (1+x)^6 \end{vmatrix}$$

$\Rightarrow \quad (1+x)^{17}\cdot(1+x)^{23}\cdot(1+x)^{41}\cdot 0 \qquad$ (R_1 and R_3 are identical)

$\therefore \quad 0 = A + Bx + Cx^2 + \ldots$

By comparing the like terms, we get **A = 0.**

State True or False for the statements of the following Exercises:

Q48. $(A^3)^{-1} = (A^{-1})^3$, where A is a square matrix and $|A| \neq 0$.

Sol. Since $(A^K)^{-1} = (A^{-1})^K$ where $K \in N$

So, $\qquad (A^3)^{-1} = (A^{-1})^3$ is true

Q49. $(aA)^{-1} = \frac{1}{a} A^{-1}$, where a is any real number and A is a square matrix.

Sol. If A is a non-singular square matrix, then for any non-zero scalar 'a', aA is invertible.

$$\therefore \quad (aA)\cdot\left(\frac{1}{a}A^{-1}\right) = a\cdot\frac{1}{a}\cdot A\cdot A^{-1} = I$$

So, (aA) is inverse of $\left(\frac{1}{a}A^{-1}\right)$

$$\Rightarrow \quad (aA)^{-1} = \frac{1}{a}A^{-1} \text{ is true.}$$

Q50. $|A^{-1}| \neq |A|^{-1}$, where A is a non-singular matrix.

Sol. False.

Since $|A^{-1}| = |A|^{-1}$ for a non-singular matrix.

Q51. If A and B are matrices of order 3 and $|A| = 5, |B| = 3$ then $|3AB| = 27 \times 5 \times 3 = 405$

Sol. True.

$$|3AB| = 3^3|AB| = 27\,|A|\,|B| = 27 \times 5 \times 3 \qquad [\because |KA| = K^n|A|]$$

Q52. If the value of a third order determinant is 12, then the value of the determinant formed by replacing each element by its co-factor will be 144.

Sol. True.

Since $|A| = 12$

If A is a square matrix of order n

then $\quad |\text{Adj } A| = |A|^{n-1}$

$$\therefore \quad |\text{Adj } A| = |A|^{3-1} = |A|^2 = (12)^2 = 144 \qquad [n = 3]$$

Q53. $\begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix} = 0$, where a, b, c are in A.P.

Sol. True.

Let $$\Delta = \begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix}$$

$R_2 \to 2R_2 - (R_1 + R_3)$

$$= \begin{vmatrix} x+1 & x+2 & x+a \\ 0 & 0 & 2b-(a+c) \\ x+3 & x+4 & x+c \end{vmatrix}$$

a, b, c are in A.P.

$\therefore b-a=c-b \quad \Rightarrow \quad 2b=a+c$

$$= \begin{vmatrix} x+1 & x+2 & x+a \\ 0 & 0 & 0 \\ x+3 & x+4 & x+c \end{vmatrix} = 0$$

Q54. $|\text{adj } A| = |A|^2$, where A is a square matrix of order two.

Sol. False.

Since $|\text{adj } A| = |A|^{n-1}$ where n is the order of the square matrix.

Q55. The determinant $\begin{vmatrix} \sin A & \cos A & \sin A + \cos B \\ \sin B & \cos A & \sin B + \cos B \\ \sin C & \cos A & \sin C + \cos B \end{vmatrix}$ is equal to zero.

Sol. True.

Let $\Delta = \begin{vmatrix} \sin A & \cos A & \sin A + \cos B \\ \sin B & \cos A & \sin B + \cos B \\ \sin C & \cos A & \sin C + \cos B \end{vmatrix}$

Splitting up C_3

$$= \begin{vmatrix} \sin A & \cos A & \sin A \\ \sin B & \cos A & \sin B \\ \sin C & \cos A & \sin C \end{vmatrix} + \begin{vmatrix} \sin A & \cos A & \cos B \\ \sin B & \cos A & \cos B \\ \sin C & \cos A & \cos B \end{vmatrix}$$

$$= 0 + \begin{vmatrix} \sin A & \cos A & \cos B \\ \sin B & \cos A & \cos B \\ \sin C & \cos A & \cos B \end{vmatrix} \qquad [\because C_1 \text{ and } C_3 \text{ are identical}]$$

$$= \cos A \cos B \begin{vmatrix} \sin A & 1 & 1 \\ \sin B & 1 & 1 \\ \sin C & 1 & 1 \end{vmatrix}$$

[Taking cos A and cos B common from C_2 and C_3 respectively]

$= \cos A \cos B\ (0)$ $\qquad [\because C_2 \text{ and } C_3 \text{ are identical}]$

$= 0$

Q56. If the determinant $\begin{vmatrix} x+a & p+u & l+f \\ y+b & q+v & m+g \\ z+c & r+w & n+h \end{vmatrix}$ splits into exactly K determinants of order 3, each element of which contains only one term, then the value of K is 8.

Sol. True.

Let $\qquad \Delta = \begin{vmatrix} x+a & p+u & l+f \\ y+b & q+v & m+g \\ z+c & r+w & n+h \end{vmatrix}$

Splitting up C_1

$$\Rightarrow \begin{vmatrix} x & p+u & l+f \\ y & q+v & m+g \\ z & r+w & n+h \end{vmatrix} + \begin{vmatrix} a & p+u & l+f \\ b & q+v & m+g \\ c & r+w & n+h \end{vmatrix}$$

Splitting up C_2 in both determinants

$$\Rightarrow \begin{vmatrix} x & p & l+f \\ y & q & m+g \\ z & r & n+h \end{vmatrix} + \begin{vmatrix} x & u & l+f \\ y & v & m+g \\ z & w & n+h \end{vmatrix} + \begin{vmatrix} a & p & l+f \\ b & q & m+g \\ c & r & n+h \end{vmatrix}$$

$$+ \begin{vmatrix} a & u & l+f \\ b & v & m+g \\ c & w & n+h \end{vmatrix}$$

Similarly by splitting C_3 in each determinant, we will get 8 determinants.

Q57. Let $\Delta = \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$ 16

then $\Delta_1 = \begin{vmatrix} p+x & a+x & a+p \\ q+y & b+y & b+q \\ r+z & c+z & c+r \end{vmatrix} = 32$

Sol. True.

Given that: $\Delta = \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} = 16$

L.H.S. $\Delta_1 = \begin{vmatrix} p+x & a+x & a+p \\ q+y & b+y & b+q \\ r+z & c+z & c+r \end{vmatrix}$

$C_1 \rightarrow C_1 + C_2 + C_3$

$$= \begin{vmatrix} 2p+2x+2a & a+x & a+p \\ 2q+2y+2b & b+y & b+q \\ 2r+2z+2c & c+z & c+r \end{vmatrix}$$

$$= 2\begin{vmatrix} p+x+a & a+x & a+p \\ q+y+b & b+y & b+q \\ r+z+c & c+z & c+r \end{vmatrix}$$ [Taking 2 common from C_1]

$$C_1 \to C_1 - C_2 = 2\begin{vmatrix} p & a+x & a+p \\ q & b+y & b+q \\ r & c+z & c+r \end{vmatrix}$$

$$C_3 \to C_3 - C_1{}^d = 2\begin{vmatrix} p & a+x & a \\ q & b+y & b \\ r & c+z & c \end{vmatrix}$$

Splitting up C_2

$$= 2\begin{vmatrix} p & a & a \\ q & b & b \\ r & c & c \end{vmatrix} + 2\begin{vmatrix} p & x & a \\ q & y & b \\ r & z & c \end{vmatrix} = 2(0) + 2\begin{vmatrix} p & x & a \\ q & y & b \\ r & z & c \end{vmatrix}$$

$$= 2\begin{vmatrix} p & x & a \\ q & y & b \\ r & z & c \end{vmatrix} \Rightarrow 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} \qquad (C_1 \leftrightarrow C_3 \text{ and } C_2 \leftrightarrow C_3)$$

$= 2 \times 16 = 32$

Q58. The maximum value of $\begin{vmatrix} 1 & 1 & 1 \\ 1 & (1+\sin\theta) & 1 \\ 1 & 1 & 1+\cos\theta \end{vmatrix}$ is $\frac{1}{2}$.

Sol. True.

Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & (1+\sin\theta) & 1 \\ 1 & 1 & 1+\cos\theta \end{vmatrix}$

$C_1 \to C_1 - C_2,\ C_2 \to C_2 - C_3$

$$= \begin{vmatrix} 0 & 0 & 1 \\ -\sin\theta & \sin\theta & 1 \\ 0 & -\cos\theta & 1+\cos\theta \end{vmatrix}$$

Expanding along C_3

$$= 1\begin{vmatrix} -\sin\theta & \sin\theta \\ 0 & -\cos\theta \end{vmatrix} = \sin\theta\cos\theta - 0 = \sin\theta\cos\theta$$

$$= \frac{1}{2}.2\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$$

$$= \frac{1}{2} \times 1 \qquad [\text{Maximum value of } \sin 2\theta = 1]$$

$$= \frac{1}{2}$$

❑❑❑

5 Continuity and Differentiability

5.3 EXERCISE

SHORT ANSWER TYPE QUESTIONS

Q1. Examine the continuity of the function
$f(x) = x^3 + 2x^2 - 1$ at $x = 1$

Sol. We know that $y = f(x)$ will be continuous at $x = a$ if

$$\lim_{x \to a^-} f(x) = \lim_{x \to a} f(x) = \lim_{x \to a^+} f(x)$$

Given: $f(x) = x^3 + 2x^2 - 1$

$$\lim_{x \to 1^-} f(x) = \lim_{h \to 0} (1+h)^3 + 2(1+h)^2 - 1 = 1 + 2 - 1 = 2$$

$$\lim_{x \to 1} f(x) = (1)^3 + 2(1)^2 - 1$$
$$= 1 + 2 - 1 = 2$$

$$\lim_{x \to 1^+} f(x) = \lim_{\to} (1+h)^3 + 2(1+h)^2 - 1$$
$$= 1 + 2 - 1 = 2$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1} f(x) = \lim_{x \to 1^+} f(x) = 2.$$

Hence, $f(x)$ is continuous at $x = 1$.

Find which of the functions in Exercises 2 to 10 is continuous or discontinuous at the indicated points:

Q2. $f(x) = \begin{cases} 3x + 5, \text{ if } x \geq 2 \\ x^2, \text{ if } x < 2 \end{cases}$ at $x = 2$

Sol.
$$\lim_{x \to 2^+} f(x) = 3x + 5$$
$$= \lim_{h \to 0} 3(2+h) + 5 = 11$$

$$\lim_{x \to 2} f(x) = 3x + 5 = 3(2) + 5 = 11$$

$$\lim_{x \to 2^-} f(x) = x^2 = \lim_{h \to 0} (2-h)^2$$
$$= \lim_{h \to 0} (2)^2 + h^2 - 4h = (2)^2 = 4$$

Since $\lim_{x \to 2^-} f(x) = \lim_{x \to 2} f(x) \neq \lim_{x \to 2} f(x)$

Hence $f(x)$ is discontinuous at $x = 2$.

Q3. $f(x) = \begin{cases} \dfrac{1 - \cos 2x}{x^2}, \text{ if } x \neq 0 \\ 5, \quad \text{if } x = 0 \end{cases}$ at $x = 0$.

Sol. $\lim\limits_{x\to 0^-} f(x) = \dfrac{1-\cos 2x}{x^2}$

$$= \lim_{h\to 0}\frac{1-\cos 2(0-h)}{(0-h)^2} = \lim_{h\to 0}\frac{1-\cos(-2h)}{h^2}$$

$$= \lim_{h\to 0}\frac{1-\cos 2h}{h^2}$$

$$= \lim_{h\to 0}\frac{2\sin^2 h}{h^2} \qquad \left[\because 1-\cos\theta = 2\sin^2\frac{\theta}{2}\right]$$

$$= \lim_{h\to 0}\frac{2\sin h}{h}.\frac{\sin h}{h} = 2.1.1 = 2 \qquad \left[\lim_{x\to 0}\frac{\sin x}{x} = 1\right]$$

$$\lim_{x\to 0^+} f(x) = \frac{1-\cos 2x}{x^2}$$

$$= \lim_{h\to 0}\frac{1-\cos 2(0+h)}{(0+h)^2} = \lim_{h\to 0}\frac{1-\cos 2h}{h^2}$$

$$= \lim_{h\to 0}\frac{2\sin^2 h}{h^2} = \frac{2\sin h}{h}.\frac{\sin h}{h} = 2.1.1 = 2$$

$$\lim_{x\to 0} f(x) = 5$$

As $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} f(x) \neq \lim\limits_{x\to 0} f(x)$

$\therefore f(x)$ is discontinuous at $x = 0$.

Q4. $f(x) = \begin{cases} \dfrac{2x^2-3x-2}{x-2}, & \text{if } x \neq 2 \\ 5, & \text{if } x = 2 \end{cases}$ at $x = 2$

Sol. $f(x) = \dfrac{2x^2-3x-2}{x-2}$

$$= \frac{2x^2-4x+x-2}{x-2} = \frac{2x(x-2)+1(x-2)}{x-2}$$

$$= \frac{(2x+1)(x-2)}{x-2} = 2x+1$$

$$\lim_{x\to 2^-} f(x) = 2x+1$$

$$= \lim_{h\to 0} 2(2-h)+1 = 4+1 = 5$$

$$\lim_{x\to 2^+} f(x) = 2x+1$$

$$= \lim_{h\to 0} 2(2+h)+1 = 4+1 = 5$$

$$\lim_{x\to 2} f(x) = 5$$

As $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2} f(x) = 5$

Hence, $f(x)$ is continuous at $x = 2$.

Q5. $f(x) = \begin{cases} \dfrac{|x-4|}{2(x-4)}, & \text{if } x \neq 4 \\ 0, & \text{if } x = 4 \end{cases}$ at $x = 4$

Sol. $\lim\limits_{x \to 4^-} f(x) = \dfrac{|x-4|}{2(x-4)}$ $\left[\begin{array}{l} \text{for } x < 4, |x-4| = -(x-4) \\ \text{for } x > 4, |x-4| = (x-4) \end{array}\right]$

$$= \lim_{h \to 0} \frac{-[4-h-4]}{2[4-h-4]} = \lim_{h \to 0} \frac{h}{-2h} = -\frac{1}{2}$$

$$\lim_{x \to 4^+} f(x) = \frac{|x-4|}{2(x-4)} = \lim_{h \to 0} \frac{[4+h-4]}{2[4+h-4]} = \frac{1}{2}$$

$$\lim_{x \to 4} f(x) = 0$$

$\therefore \lim\limits_{x \to 4^-} f(x) \neq \lim\limits_{x \to 4^+} f(x) \neq \lim\limits_{x \to 4} f(x)$

Hence, $f(x)$ is discontinuous at $x = 4$.

Q6. $f(x) = \begin{cases} |x| \cos \dfrac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ at $x = 0$

Sol. $\lim\limits_{x \to 0^-} f(x) = |x| \cos \dfrac{1}{x}$

$$= \lim_{h \to 0} |0-h| \cos \frac{1}{(0-h)} = \lim_{h \to 0} h \cos \frac{1}{h}$$

$= 0$ $\left[\because \cos \dfrac{1}{x} \text{ oscillate between } -1 \text{ and } 1\right]$

$$\lim_{x \to 0^-} f(x) = |x| \cos \frac{1}{x}$$

$$= \lim_{h \to 0} |0+h| \cos \frac{1}{(0+h)} = \lim_{h \to 0} h . \cos \frac{1}{h} = 0$$

$$\lim_{x \to 0} f(x) = 0$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = \lim_{x \to 0} f(x) = 0$$

Hence, $f(x)$ is continuous at $x = 0$.

Q7. $f(x) = \begin{cases} |x-a| \sin \dfrac{1}{x-a}, & \text{if } x \neq 0 \\ 0, & \text{if } x = a \end{cases}$ at $x = a$.

Sol. $\lim\limits_{x\to a^-} f(x) = |x-a| \sin \dfrac{1}{x-a}$

$$= \lim_{h\to 0} |a-h-a| . \sin \frac{1}{a-h-a} = \lim_{h\to 0} h . \sin \frac{1}{-h}$$

$$= \lim_{h\to 0} - h . \sin \frac{1}{h} \qquad [\because \sin(-\theta) = -\sin\theta]$$

$= 0 \times$ [a number oscillating between -1 and 1]

$= 0$

$\lim\limits_{x\to a^+} f(x) = |x-a| \sin \dfrac{1}{x-a}$

$$= \lim_{h\to 0} |a+h-a| . \sin \frac{1}{a+h-a} = \lim_{h\to 0} h . \sin \frac{1}{h}$$

$= 0 \times$ [a number oscillating between -1 and 1]

$\lim\limits_{x\to a} f(x) = 0$

As $\lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a} f(x) = 0$

Hence, $f(x)$ is continuous at $x = a$.

Q8. $f(x) = \begin{cases} \dfrac{e^{1/x}}{1+e^{1/x}}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ at $x = 0$

Sol. $\lim\limits_{x\to 0^-} f(x) = \dfrac{e^{1/x}}{1+e^{1/x}}$

$$= \lim_{h\to 0} \frac{e^{\frac{1}{0-h}}}{1+e^{\frac{1}{0-h}}} = \lim_{h\to 0} \frac{e^{-1/h}}{1+e^{-1/h}}$$

$$= \lim_{h\to 0} \frac{1}{e^{1/h}\left(1-e^{-1/h}\right)} = \lim_{h\to 0} \frac{1}{e^{1/h}-1} = \frac{1}{e^{1/0}-1}$$

$$= \frac{1}{e^{\infty}-1} = \frac{1}{0-1} = -1 \qquad [\because e^{\infty} = 0]$$

$\lim\limits_{x\to 0^+} f(x) = \dfrac{e^{1/x}}{1+e^{1/x}}$

$$= \lim_{h\to 0} \frac{e^{\frac{1}{0+h}}}{1+e^{\frac{1}{0+h}}} = \lim_{h\to 0} \frac{e^{1/h}}{1+e^{1/h}}$$

$$= \lim_{h\to 0}\frac{1}{e^{-1/h}(1+e^{1/h})} = \lim_{h\to 0}\frac{1}{e^{-1/h}+1}$$

$$= \frac{1}{e^{-\infty}+1} = \frac{1}{0+1} = 1 \qquad [e^{-\infty}=0]$$

$\lim_{x\to 0} f(x) = 0$

As $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x) \neq \lim_{x\to 0} f(x)$

Hence, $f(x)$ is discontinuous at $x = 0$.

Q9. $f(x) = \begin{cases} \dfrac{x^2}{2}, & \text{if } 0 \le x \le 1 \\ 2x^2 - 3x + \dfrac{3}{2}, & \text{if } 1 < x \le 2 \end{cases}$ at $x = 1$.

Sol. $\lim_{x\to 1^-} f(x) = \dfrac{x^2}{2} = \lim_{h\to 0}\dfrac{(1-h)^2}{2} = \dfrac{1}{2}$

$\lim_{x\to 1} f(x) = \dfrac{x^2}{2} = \dfrac{(1)^2}{2} = \dfrac{1}{2}$

$\lim_{x\to 1^+} f(x) = 2x^2 - 3x + \dfrac{3}{2} = 2(1)^2 - 3(1) + \dfrac{3}{2} = 2 - 3 + \dfrac{3}{2} = \dfrac{1}{2}$

As $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1} f(x) = \dfrac{1}{2}$

Hence, $f(x)$ is continuous at $x = 1$.

Q10. $f(x) = |x| + |x-1|$ at $x = 1$.

Sol. $\lim_{x\to 1^-} f(x) = |x| + |x-1| = \lim_{h\to 0}|1-h| + |1-h-1|$

$= |1-0| + |1-0-1| = 1 + 0 = 1$

$\lim_{x\to 1^+} f(x) = |x| + |x-1|$

$= \lim_{h\to 0}|1+h| + |1+h-1| = 1 + 0 = 1$

$\lim_{x\to 1} f(x) = |x| + |x-1| = |1| + |1-1| = 1 + 0 = 1$

As $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1} f(x)$

Hence, $f(x)$ is continuous at $x = 1$.

Find the value of k in each of the Exercises 11 to 14 so that the function f is continuous at the indicated point:

Q11. $f(x) = \begin{cases} 3x - 8, & \text{if } x \le 5 \\ 2k, & \text{if } x > 5 \end{cases}$ at $x = 5$

Sol. $\lim\limits_{x \to 5^-} f(x) = 3x - 8$

$= \lim\limits_{h \to 0} 3(5-h) - 8 = 15 - 8 = 7$

$\lim\limits_{x \to 5^+} f(x) = 2k$

As the function is continuous at $x = 5$

$\therefore \lim\limits_{x \to 5^-} f(x) = \lim\limits_{x \to 5^+} f(x)$

$\therefore \quad 7 = 2k \Rightarrow k = \dfrac{7}{2}$

Hence, the value of k is $\dfrac{7}{2}$.

Q12. $f(x) = \begin{cases} \dfrac{2^{x+2} - 16}{4^x - 16}, & \text{if } x \neq 2 \\ k, & \text{if } x = 2 \end{cases}$ at $x = 2$

Sol. $f(x) = \dfrac{2^{x+2} - 16}{4^x - 16} = \dfrac{2^2.2^x - 16}{(2^x)^2 - (4)^2} = \dfrac{4(2^x - 4)}{(2^x - 4)(2^x + 4)}$

$f(x) = \dfrac{4}{2^x + 4}$

$\lim\limits_{x \to 2^-} f(x) = \lim\limits_{h \to 0} \dfrac{4}{2^{2-h} + 4} = \dfrac{4}{2^2 + 4} = \dfrac{4}{4+4} = \dfrac{4}{8} = \dfrac{1}{2}$

$\lim\limits_{x \to 2} f(x) = k$

As the function is continuous at $x = 2$.

$\therefore \lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2} f(x)$

$\therefore \quad k = \dfrac{1}{2}$

Hence, value of k is $\dfrac{1}{2}$.

Q13. $f(x) = \begin{cases} \dfrac{\sqrt{1+kx} - \sqrt{1-kx}}{x}, & \text{if } -1 \leq x < 0 \\ \dfrac{2x+1}{x-1}, & \text{if } 0 \leq x \leq 1 \end{cases}$ at $x = 0$

Sol. $\lim\limits_{x \to 0^-} f(x) = \dfrac{\sqrt{1+kx} - \sqrt{1-kx}}{x}$

$= \lim\limits_{x \to 0^-} \dfrac{\sqrt{1+kx} - \sqrt{1-kx}}{x} \times \dfrac{\sqrt{1+kx} + \sqrt{1-kx}}{\sqrt{1+kx} + \sqrt{1-kx}}$

$$= \lim_{x\to 0^-} \frac{(1+kx)-(1-kx)}{x\left[\sqrt{1+kx}+\sqrt{1-kx}\right]}$$

$$= \lim_{x\to 0^-} \frac{1+kx-1+kx}{x\left[\sqrt{1+kx}+\sqrt{1-kx}\right]}$$

$$= \lim_{x\to 0^-} \frac{2kx}{x\left[\sqrt{1+kx}+\sqrt{1-kx}\right]}$$

$$= \lim_{x\to 0^-} \frac{2k}{\sqrt{1+kx}+\sqrt{1-kx}}$$

$$= \lim_{h\to 0} \frac{2k}{\sqrt{1+k(0-h)}+\sqrt{1-k(0-h)}}$$

$$= \frac{2k}{\sqrt{1}+\sqrt{1}} = \frac{2k}{2} = k$$

$$\lim_{x\to 0} f(x) = \frac{2x+1}{x-1} = \frac{2(0)+1}{0-1} = \frac{1}{-1} = -1$$

As the function is continuous at $x = 0$.

$$\therefore \lim_{x\to 0^-} f(x) = \lim_{x\to 0} f(x)$$

$$k = -1$$

Hence, the value of k is -1.

Q14. $f(x) = \begin{cases} \dfrac{1-\cos kx}{x \sin x}, & \text{if } x \neq 0 \\ \dfrac{1}{2}, & \text{if } x = 0 \end{cases}$ at $x = 0$

Sol. $\lim_{x\to 0^-} f(x) = \dfrac{1-\cos kx}{x \sin x}$

$$= \lim_{h\to 0} \frac{1-\cos k(0-h)}{(0-h)\sin(0-h)} = \lim_{h\to 0} \frac{1-\cos(-kh)}{-h.\sin(-h)}$$

$$= \lim_{h\to 0} \frac{1-\cos kh}{h \sin h} \qquad \left[\because \begin{array}{l} \sin(-\theta) = -\sin\theta \\ \cos(-\theta) = \cos\theta \end{array}\right]$$

$$= \lim_{h\to 0} \frac{2\sin^2 \frac{kh}{2}}{h \sin h}$$

$$= \lim_{\substack{h\to 0 \\ kh\to 0}} \frac{2\sin\frac{kh}{2}}{\frac{kh}{2}} \times \frac{kh}{2} \times \frac{\sin\frac{kh}{2}}{\frac{kh}{2}} \times \frac{kh}{2} . \frac{1}{h.\frac{\sin h}{h}.h}$$

$$= 2.1.\frac{kh}{2}.1.\frac{kh}{2}.\frac{1}{h^2}.1 \qquad \left[\lim_{h\to 0}\frac{\sin h}{h}=1 \text{ and } \lim_{kh\to 0}\frac{\sin kh}{kh}=1\right]$$

$$= \frac{k^2}{2}$$

$$\lim_{x\to 0} f(x) = \frac{1}{2}$$

As $\lim_{x\to 0^-} f(x) = \lim_{x\to 0} f(x)$

$$\therefore \quad \frac{k^2}{2} = \frac{1}{2}$$

$$\Rightarrow \quad k^2 = 1 \Rightarrow k = \pm 1$$

Hence, the value of k is ± 1.

Q15. Prove that the function f defined by

$$f(x) = \begin{cases} \dfrac{x}{|x|+2x^2}, & x \neq 0 \\ k, & x = 0 \end{cases}$$

remains discontinuous at $x = 0$, regardless the choice of k.

Sol.
$$\lim_{x\to 0^-} f(x) = \frac{x}{|x|+2x^2} = \lim_{h\to 0}\frac{0-h}{|0-h|+2(0-h)^2}$$

$$= \lim_{h\to 0}\frac{-h}{h+2h^2} = \lim_{h\to 0}\frac{-h}{h(1+2h)}$$

$$= \lim_{h\to 0}\frac{-1}{1+2h} = \frac{-1}{1+2(0)} = -1$$

$$\lim_{x\to 0^+} f(x) = \frac{x}{|x|+2x^2} = \lim_{h\to 0}\frac{0+h}{|0+h|+2(0+h)^2}$$

$$= \lim_{h\to 0}\frac{h}{h+2h^2} = \lim_{h\to 0}\frac{h}{h(1+2h)} = \frac{1}{1+0} = 1$$

$$\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$$

Hence, $f(x)$ is discontinuous at $x = 0$ regardless the choice of k.

Q16. Find the values of a and b such that the function f defined by

$$f(x) = \begin{cases} \dfrac{x-4}{|x-4|} + a, & \text{if } x < 4 \\ a + b, & \text{if } x = 4 \\ \dfrac{x-4}{|x-4|} + b, & \text{if } x > 4 \end{cases}$$

is a continuous function at $x = 4$.

Sol. $\lim\limits_{x \to 4^-} f(x) = \dfrac{x-4}{|x-4|} + a$

$= \lim\limits_{h \to 0} \dfrac{4-h-4}{|4-h-4|} + a = \lim\limits_{h \to 0} \dfrac{-h}{h} + a = -1 + a$

$\lim\limits_{x \to 4} f(x) = a + b$

$\lim\limits_{x \to 4^+} f(x) = \dfrac{x-4}{|x-4|} + b$

$= \lim\limits_{h \to 0} \dfrac{4+h-4}{|4+h-4|} + b = \lim\limits_{h \to 0} \dfrac{h}{h} + b = 1 + b$

As the function is continuous at $x = 4$.

$\therefore \quad \lim\limits_{\to} f(x) = \lim\limits_{x \to 4} f(x) = \lim\limits_{x \to 4^+} f(x)$

$-1 + a = a + b = 1 + b$

$\therefore \quad -1 + a = a + b \Rightarrow b = -1$

$1 + b = a + b \Rightarrow a = 1$

Hence, the value of $a = 1$ and $b = -1$.

Q17. Given the function $f(x) = \dfrac{1}{x+2}$. Find the point of discontinuity of the composite function $y = f[f(x)]$.

Sol. $f(x) = \dfrac{1}{x+2}$

$f[f(x)] = \dfrac{1}{f(x)+2} = \dfrac{1}{\dfrac{1}{x+2}+2} = \dfrac{1}{\dfrac{1+2x+4}{x+2}} = \dfrac{x+2}{2x+5}$

$\therefore \quad f[f(x)] = \dfrac{x+2}{2x+5}$

This function will not be defined and continuous where

$2x + 5 = 0 \Rightarrow x = \dfrac{-5}{2}$.

Hence, $x = \dfrac{-5}{2}$ is the point of discontinuity.

Q18. Find all the points of discontinuity of the function

$f(t) = \dfrac{1}{t^2+t-2}$, where $t = \dfrac{1}{x-1}$.

Sol. We have $f(t) = \dfrac{1}{t^2+t-2}$

$\Rightarrow \quad f(t) = \dfrac{1}{\dfrac{1}{(x-1)^2} + \dfrac{1}{(x-1)} - 2} \quad \left[\text{putting } t = \dfrac{1}{x-1}\right]$

$$= \frac{1}{\frac{1+x-1-2(x-1)^2}{(x-1)^2}} = \frac{(x-1)^2}{x-2x^2-2+4x}$$

$$= \frac{(x-1)^2}{-2x^2+5x-2} = \frac{(x-1)^2}{-(2x^2-5x+2)}$$

$$= \frac{(x-1)^2}{-[2x^2-4x-x+2]} = \frac{(x-1)^2}{-[2x(x-2)-1(x-2)]}$$

$$= \frac{(x-1)^2}{-(x-2)(2x-1)} = \frac{(x-1)^2}{(2-x)(2x-1)}$$

So, if $f(t)$ is discontinuous, then $2-x=0 \quad \therefore x=2$

and $2x-1=0 \quad \therefore x=\frac{1}{2}$

Hence, the required points of discontinuity are 2 and $\frac{1}{2}$.

Q19. Show that the function $f(x) = |\sin x + \cos x|$ is continuous at $x=\pi$.

Sol. Given that $f(x) = |\sin x + \cos x|$ at $x = \pi$

Put $g(x) = \sin x + \cos x$ and $h(x) = |x|$

$\therefore \quad h[g(x)] = h(\sin x + \cos x) = |\sin x + \cos x|$

Now, $g(x) = \sin x + \cos x$ is a continuous function since $\sin x$ and $\cos x$ are two continuous functions at $x=\pi$.

We know that every modulus function is a continuous function everywhere.

Hence, $f(x) = |\sin x + \cos x|$ is continuous function at $x=\pi$.

Q20. Examine the differentiability of f, where f is defined by

$$f(x) = \begin{cases} x[x], & \text{if } 0 \le x < 2 \\ (x-1)x, & \text{if } 2 \le x < 3 \end{cases} \quad \text{at } x=2.$$

Sol. We know that a function f is differentiable at a point 'a' in its domain if

$$Lf'(c) = Rf'(c)$$

where $Lf'(c) = \lim_{h\to 0} \frac{f(a-h)-f(a)}{-h}$ and

$$Rf'(c) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

Here, $f(x) = \begin{cases} x[x], \text{if } 0 \le x < 2 \\ (x-1)x, \text{if } 2 \le x < 2 \end{cases}$ at $x=2$.

at $x=2$

$$Lf'(c) = \lim_{h\to 0}\frac{f(2-h)-f(2)}{-h} = \lim_{h\to 0}\frac{(2-h)[2-h]-(2-1)2}{-h}$$

$$= \lim_{h\to 0}\frac{(2-h).1-2}{-h} \qquad [\because [2-h]=1]$$

$$= \lim_{h\to 0}\frac{2-h-2}{-h} = 1$$

$$Rf'(c) = \lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$$

$$= \lim_{h\to 0}\frac{(2+h-1)(2+h)-(2-1).2}{h}$$

$$= \lim_{h\to 0}\frac{(1+h)(2+h)-2}{h} = \lim_{h\to 0}\frac{2+h+2h+h^2-2}{h}$$

$$= \lim_{h\to 0}\frac{3h+h^2}{h} = \lim_{h\to 0}\frac{h(3+h)}{h} = 3$$

$$Lf'(2) \neq Rf'(2)$$

Hence, $f(x)$ is not differentiable at $x = 2$.

Q21. Examine the differentiability of f, where f is defined by

$$f(x) = \begin{cases} x^2 \sin\dfrac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases} \quad \text{at } x = 0.$$

Sol. Given that:

$$f(x) = \begin{cases} x^2 \sin\dfrac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases} \quad \text{at } x = 0$$

For differentiability we know that:

$$Lf'(c) = Rf'(c)$$

$$\therefore\ Lf'(0) = \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}$$

$$= \lim_{h\to 0}\frac{(0-h)^2 \sin\dfrac{1}{(0-h)} - 0}{-h} = \frac{h^2 .\sin\left(-\dfrac{1}{h}\right)}{-h}$$

$$= h.\sin\left(\frac{1}{h}\right) = 0 \times \left[-1 \leq \sin\left(\frac{1}{h}\right) \leq 1\right]$$

$$= 0$$

$$Rf'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{(0+h)^2 \sin\left(\dfrac{1}{0+h}\right) - 0}{h}$$

$$= \lim_{h \to 0} \frac{h^2 \sin\left(\frac{1}{h}\right)}{h} = \lim_{h \to 0} h.\sin\left(\frac{1}{h}\right)$$

$$= 0 \times \left[-1 \le \sin\left(\frac{1}{h}\right) \le 1\right] = 0$$

So, $Lf'(0) = Rf'(0) = 0$

Hence, $f(x)$ is differentiable at $x = 0$.

Q22. Examine the differentiability of f, where f is defined by

$$f(x) = \begin{cases} 1 + x, \text{ if } x \le 2 \\ 5 - x, \text{ if } x > 2 \end{cases} \quad \text{at } x = 2.$$

Sol. $f(x)$ is differentiable at $x = 2$ if

$$Lf'(2) = Rf'(2)$$

$$\therefore \quad Lf'(2) = \lim_{h \to 0} \frac{f(2-h) - f(2)}{-h}$$

$$= \lim_{h \to 0} \frac{(1+2-h) - (1+2)}{-h} = \lim_{h \to 0} \frac{3-h-3}{-h} = \frac{-h}{-h} = 1$$

$$Rf'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \to 0} \frac{[5-(2+h)] - (1+2)}{h} = \lim_{h \to 0} \frac{3-h-3}{h}$$

$$= \frac{-h}{h} = -1$$

So, $\quad Lf'(2) \ne Rf'(2)$

Hence, $f(x)$ is not differentiable at $x = 2$.

Q23. Show that $f(x) = |x - 5|$ is continuous but not differentiable at $x = 5$.

Sol. We have $f(x) = |x - 5|$

$$\Rightarrow \quad f(x) = \begin{cases} -(x-5) \text{ if } x - 5 < 0 \text{ or } x < 5 \\ \quad x - 5 \text{ if } x - 5 > 0 \text{ or } x > 5 \end{cases}$$

For continuity at $x = 5$

L.H.L. $\lim_{h \to 5^-} f(x) = -(x-5)$

$$= \lim_{h \to 0} -(5-h-5) = \lim_{h \to 0} h = 0$$

R.H.L. $\lim_{x \to 5^+} f(x) = x - 5$

$$= \lim_{h \to 0} (5+h-5) = \lim_{h \to 0} h = 0$$

L.H.L. = R.H.L.

So, $f(x)$ is continuous at $x = 5$.

Now, for differentiability

$$Lf'(5) = \lim_{h\to 0} \frac{f(5-h)-f(5)}{-h}$$

$$= \lim_{h\to 0} \frac{-(5-h-5)-(5-5)}{-h} = \lim_{h\to 0} \frac{h}{-h} = -1$$

$$Rf'(5) = \lim_{h\to 0} \frac{f(5+h)-f(5)}{h}$$

$$= \lim_{h\to 0} \frac{(5+h-5)-(5-5)}{h} = \lim_{h\to 0} \frac{h-0}{h} = 1$$

$\because \quad Lf'(5) \neq Rf'(5)$

Hence, $f(x)$ is not differentiable at $x = 5$.

Q24. A function $f : R \to R$ satisfies the equation $f(x + y) = f(x).f(y)$ $\forall\, x, y \in R, f(x) \neq 0$. Suppose that the function is differentiable at $x = 0$ and $f'(0) = 2$. Prove that $f'(x) = 2f(x)$.

Sol. Given that: $f : R \to R$ satisfies the equation $f(x + y) = f(x).f(y)$ $\forall\, x, y \in R, f(x) \neq 0$.

Let us take any point $x = 0$ at which the function $f(x)$ is differentiable.

$$\therefore \quad f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$$

$$2 = \lim_{h\to 0} \frac{f(0).f(h)-f(0)}{h} \quad [\because f(0) = f(h)] \qquad ...(i)$$

$$\Rightarrow \quad 2 = \lim_{h\to 0} \frac{f(0)[f(h)-1]}{h}$$

$$\text{Now} \quad f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x).f(h)-f(x)}{h} \quad [\because f(x+y) = f(x).f(y)]$$

$$= \lim_{h\to 0} \frac{f(x)[f(h)-1]}{h} = 2f(x) \qquad \text{from eqn. } (i)$$

Hence, $f'(x) = 2f(x)$.

Differentiate each of the following w.r.t. x (Exercises 25 to 43):

Q25. $2^{\cos^2 x}$

Sol. Let $\quad y = 2^{\cos^2 x}$

Taking log on both sides, we get

$$\log y = \log 2^{\cos^2 x} \Rightarrow \log y = \cos^2 x \,.\, \log 2$$

Differentiating both sides w.r.t. x

$$\Rightarrow \quad \frac{1}{y}.\frac{dy}{dx} = \log 2.\frac{d}{dx}\cos^2 x$$

$$\Rightarrow \quad \frac{1}{y}.\frac{dy}{dx} = \log 2\left[2\cos x.\frac{d}{dx}\cos x\right]$$

$$\Rightarrow \quad \frac{1}{y}.\frac{dy}{dx} = \log 2\ [2\cos x(-\sin x)]$$

$$\Rightarrow \quad \frac{1}{y}.\frac{dy}{dx} = \log 2\ (-\sin 2x)$$

$$\frac{dy}{dx} = -y.\log 2\sin 2x$$

Hence, $\dfrac{dy}{dx} = -2^{\cos^2 x}(\log 2\sin 2x)$

Q26. $\dfrac{8^x}{x^8}$

Sol. Let $\quad y = \dfrac{8^x}{x^8}$

Taking log on both sides, we get, $\log y = \log \dfrac{8^x}{x^8}$

$$\Rightarrow \quad \log y = \log 8^x - \log x^8 \Rightarrow \log y = x\log 8 - 8\log x$$

Differentiating both sides w.r.t. x

$$\Rightarrow \quad \frac{1}{y}.\frac{dy}{dx} = \log 8.1 - \frac{8}{x} \quad \Rightarrow \frac{dy}{dx} = y\left[\log 8 - \frac{8}{x}\right]$$

Hence, $\dfrac{dy}{dx} = \dfrac{8^x}{x^8}\left[\log 8 - \dfrac{8}{x}\right]$

Q27. $\log\left(x + \sqrt{x^2 + a}\right)$

Sol. Let $\quad y = \log\left(x + \sqrt{x^2 + a}\right)$

Differentiating both sides w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\log\left(x + \sqrt{x^2 + a}\right)$$

$$= \frac{1}{x + \sqrt{x^2 + a}}.\frac{d}{dx}\left(x + \sqrt{x^2 + a}\right)$$

$$= \frac{1}{x + \sqrt{x^2 + a}}.\left[1 + \frac{1}{2\sqrt{x^2 + a}} \times \frac{d}{dx}(x^2 + a)\right]$$

$$= \frac{1}{x+\sqrt{x^2+a}} \cdot \left[1 + \frac{1}{2\sqrt{x^2+a}} \cdot 2x\right]$$

$$= \frac{1}{x+\sqrt{x^2+a}} \cdot \left[1 + \frac{x}{\sqrt{x^2+a}}\right]$$

$$= \frac{1}{x+\sqrt{x^2+a}} \cdot \left(\frac{\sqrt{x^2+a}+x}{\sqrt{x^2+a}}\right) = \frac{1}{\sqrt{x^2+a}}$$

Hence, $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{x^2+a}}$.

Q28. $\log[\log(\log x^5)]$

Sol. Let $y = \log[\log(\log x^5)]$

Differentiating both sides w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\log[\log(\log x^5)]$$

$$= \frac{1}{\log(\log x^5)} \times \frac{d}{dx}\log(\log x^5)$$

$$= \frac{1}{\log(\log x^5)} \times \frac{1}{\log(x^5)} \times \frac{d}{dx}\log x^5$$

$$= \frac{1}{\log(\log x^5)} \cdot \frac{1}{\log(x^5)} \cdot \frac{1}{x^5} \cdot \frac{d}{dx}x^5$$

$$= \frac{1}{\log(\log x^5)} \cdot \frac{1}{\log(x^5)} \cdot \frac{1}{x^5} \cdot 5x^4$$

$$= \frac{5}{x\log(x^5).\log(\log x^5)}$$

Hence, $\dfrac{dy}{dx} = \dfrac{5}{x\log(x^5).\log(\log x^5)}$.

Q29. $\sin\sqrt{x} + \cos^2\sqrt{x}$

Sol. Let $y = \sin\sqrt{x} + \cos^2\sqrt{x}$

Differentiating both sides w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sin\sqrt{x}\right) + \frac{d}{dx}\left(\cos^2\sqrt{x}\right)$$

$$= \cos\sqrt{x} \cdot \frac{d}{dx}\left(\sqrt{x}\right) + 2\cos\sqrt{x} \cdot \frac{d}{dx}\left(\cos\sqrt{x}\right)$$

$$= \cos\sqrt{x}\cdot\frac{1}{2\sqrt{x}} + 2\cos\sqrt{x}\left(-\sin\sqrt{x}\right)\cdot\frac{d}{dx}\sqrt{x}$$

$$= \frac{1}{2\sqrt{x}}\cdot\cos\sqrt{x} - 2\cos\sqrt{x}\cdot\sin\sqrt{x}\cdot\frac{1}{2\sqrt{x}}$$

$$= \frac{\cos\sqrt{x}}{2\sqrt{x}} - \frac{\sin 2\sqrt{x}}{2\sqrt{x}}$$

Hence, $\dfrac{dy}{dx} = \dfrac{\cos\sqrt{x}}{2\sqrt{x}} - \dfrac{\sin 2\sqrt{x}}{2\sqrt{x}}$.

Q30. $\sin^n (ax^2 + bx + c)$

Sol. Let $y = \sin^n (ax^2 + bx + c)$

Differentiating both sides w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\sin^n(ax^2 + bx + c)$$

$$= n.\sin^{n-1}(ax^2 + bx + c).\frac{d}{dx}\sin(ax^2 + bx + c)$$

$$= n.\sin^{n-1}(ax^2 + bx + c).\cos(ax^2 + bx + c).\frac{d}{dx}(ax^2 + bx + c)$$

$$= n.\sin^{n-1}(ax^2 + bx + c).\cos(ax^2 + bx + c).(2ax + b)$$

Hence, $\dfrac{dy}{dx} = n(2ax + b).\sin^{n-1}(ax^2 + bx + c).\cos(ax^2 + bx + c)$

Q31. $\cos(\tan\sqrt{x+1})$

Sol. Let $y = \cos\left(\tan\sqrt{x+1}\right)$

Differentiating both sides w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\cos\left(\tan\sqrt{x+1}\right)$$

$$= -\sin\left(\tan\sqrt{x+1}\right).\frac{d}{dx}\left(\tan\sqrt{x+1}\right)$$

$$= -\sin\left(\tan\sqrt{x+1}\right).\sec^2\sqrt{x+1}.\frac{d}{dx}\sqrt{x+1}$$

$$= -\sin\left(\tan\sqrt{x+1}\right).\sec^2\sqrt{x+1}.\frac{1}{2\sqrt{x+1}}.1$$

Hence, $\dfrac{dy}{dx} = -\dfrac{1}{2\sqrt{x+1}}\sin\left(\tan\sqrt{x+1}\right).\sec^2\sqrt{x+1}$

Q32. $\sin x^2 + \sin^2 x + \sin^2(x^2)$

Sol. Let $y = \sin x^2 + \sin^2 x + \sin^2(x^2)$

Differentiating both sides w.r.t. x,

$$\frac{dy}{dx} = \frac{d}{dx}\sin(x^2) + \frac{d}{dx}\sin^2 x + \frac{d}{dx}\sin^2(x^2)$$

$$= \cos x^2 . \frac{d}{dx}(x^2) + 2\sin x . \frac{d}{dx}(\sin x) + 2\sin(x^2)\frac{d}{dx}\sin(x^2)$$

$$= \cos x^2 . 2x + 2\sin x . \cos x + 2\sin x^2 . \cos x^2 . \frac{d}{dx}(x^2)$$

$$= 2x . \cos x^2 + \sin 2x + 2\sin x^2 . \cos x^2 . 2x$$

Hence, $\frac{dy}{dx} = 2x . \cos x^2 + \sin 2x + 2x \sin 2x^2$

Q33. $\sin^{-1}\left(\frac{1}{\sqrt{x+1}}\right)$

Sol. Let $y = \sin^{-1}\left(\frac{1}{\sqrt{x+1}}\right)$

Differentiating both sides w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\sin^{-1}\left(\frac{1}{\sqrt{x+1}}\right) = \frac{1}{\sqrt{1-\left(\frac{1}{\sqrt{x+1}}\right)^2}} . \frac{d}{dx}\left(\frac{1}{\sqrt{x+1}}\right)$$

$$= \frac{1}{\sqrt{1-\frac{1}{x+1}}} . \frac{d}{dx}(x+1)^{-1/2}$$

$$= \frac{1}{\sqrt{\frac{x+1-1}{x+1}}} . \frac{-1}{2}(x+1)^{-3/2} . \frac{d}{dx}(x+1)$$

$$= \frac{\sqrt{x+1}}{\sqrt{x}} . \frac{-1}{2}(x+1)^{-3/2} . 1$$

$$= \frac{-1}{2} . \frac{\sqrt{x+1}}{\sqrt{x}} . \frac{1}{(x+1)^{3/2}} = -\frac{1}{2\sqrt{x}(x+1)}$$

Hence, $\frac{dy}{dx} = -\frac{1}{2\sqrt{x}(x+1)}$

Q34. $(\sin x)^{\cos x}$

Sol. Let $y = (\sin x)^{\cos x}$

Taking log on both sides,

$$\log y = \log(\sin x)^{\cos x}$$

$\Rightarrow \quad \log y = \cos x . \log(\sin x)$ $\quad [\because \log x^y = y \log x]$

Differentiating both sides w.r.t. x,

$$\frac{1}{y}.\frac{dy}{dx} = \frac{d}{dx}\cos x.\log(\sin x)$$

$$\Rightarrow \quad \frac{1}{y}.\frac{dy}{dx} = \cos x.\frac{d}{dx}\log(\sin x) + \log(\sin x).\frac{d}{dx}\cos x$$

$$\Rightarrow \quad \frac{1}{y}.\frac{dy}{dx} = \cos x.\frac{1}{\sin x}.\frac{d}{dx}(\sin x) + \log(\sin x).(-\sin x)$$

$$\Rightarrow \quad \frac{1}{y}.\frac{dy}{dx} = \cot x\,.\cos x - \sin x\,.\log(\sin x)$$

$$\frac{dy}{dx} = y[\cot x.\cos x - \sin x.\log(\sin x)]$$

Hence, $$\frac{dy}{dx} = (\sin x)^{\cos x}\left[\frac{\cos^2 x}{\sin x} - \sin x.\log(\sin x)\right]$$

Q35. $\sin^m x\,.\cos^n x$

Sol. Let $\qquad y = \sin^m x\,.\cos^n x$

Differentiating both sides w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}(\sin^m x.\cos^n x)$$

$$= \sin^m x.\frac{d}{dx}(\cos^n x) + \cos^n x.\frac{d}{dx}\sin^m x$$

$$= \sin^m x.n.\cos^{n-1} x\frac{d}{dx}(\cos x) + \cos^n x.m.\sin^{m-1} x$$
$$\frac{d}{dx}(\sin x)$$

$$= n.\sin^m x.\cos^{n-1} x.(-\sin x) + m.\cos^n x.\sin^{m-1} x.\cos x$$
$$= -n.\sin^{m+1} x.\cos^{n-1} x + m\cos^{n+1} x.\sin^{m-1} x$$

$$= \sin^m x.\cos^n x\left[-n\frac{\sin x}{\cos x} + m.\frac{\cos x}{\sin x}\right]$$

Hence, $$\frac{dy}{dx} = \sin^m x.\cos^n x[-n\tan x + m.\cot x]$$

Q36. $(x+1)^2(x+2)^3(x+3)^4$

Sol. Let $\qquad y = (x+1)^2(x+2)^3(x+3)^4$

Taking log on both sides,

$$\log y = \log[(x+1)^2.(x+2)^3.(x+3)^4]$$

$$\Rightarrow \quad \log y = \log(x+1)^2 + \log(x+2)^3 + \log(x+3)^4$$

$$[\because \log xy = \log x + \log y]$$

$$\Rightarrow \quad \log y = 2\log(x+1) + 3\log(x+2) + 4\log(x+3)$$
$$[\because \log x^y = y\log x]$$

Differentiating both sides w.r.t. x,

$$\frac{1}{y}.\frac{dy}{dx} = 2.\frac{d}{dx}\log(x+1) + 3.\frac{d}{dx}\log(x+2) + 4.\frac{d}{dx}\log(x+3)$$

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx} = 2.\frac{1}{x+1} + 3.\frac{1}{x+2} + 4.\frac{1}{x+3}$$

$$\Rightarrow \quad \frac{dy}{dx} = y\left[\frac{2}{x+1} + \frac{3}{x+2} + \frac{4}{x+3}\right]$$

$$\Rightarrow \quad \frac{dy}{dx} = (x+1)^2(x+2)^3(x+3)^4\left[\frac{2}{x+1} + \frac{3}{x+2} + \frac{4}{x+3}\right]$$

$$= (x+1)^2(x+2)^3(x+3)^4$$
$$\left[\frac{2(x+2)(x+3) + 3(x+1)(x+3) + 4(x+1)(x+2)}{(x+1)(x+2)(x+3)}\right]$$

$$= (x+1)(x+2)^2(x+3)^3(2x^2 + 10x + 12 + 3x^2 + 12x + 9$$
$$+ 4x^2 + 12x + 8)$$

$$= (x+1)(x+2)^2(x+3)^3(9x^2 + 34x + 29)$$

Hence, $\dfrac{dy}{dx} = (x+1)(x+2)^2(x+3)^3(9x^2 + 34x + 29)$

Q37. $\cos^{-1}\left(\dfrac{\sin x + \cos x}{\sqrt{2}}\right), \ -\dfrac{\pi}{4} < x < \dfrac{\pi}{4}$

Sol. Let $y = \cos^{-1}\left(\dfrac{\sin x + \cos x}{\sqrt{2}}\right)$

$$= \cos^{-1}\left[\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right]$$

$$= \cos^{-1}\left[\sin\frac{\pi}{4}\sin x + \cos\frac{\pi}{4}.\cos x\right] = \cos^{-1}\left[\cos\left(\frac{\pi}{4} - x\right)\right]$$

$$y = \frac{\pi}{4} - x \qquad \left[\because -\frac{\pi}{4} < x < \frac{\pi}{4}\right]$$

Differentiating both sides w.r.t. x

$$\frac{dy}{dx} = -1$$

Q38. $\tan^{-1}\left[\sqrt{\dfrac{1-\cos x}{1+\cos x}}\right], \ -\dfrac{\pi}{4} < x < \dfrac{\pi}{4}$

Sol. Let $$y = \tan^{-1}\left[\sqrt{\frac{1-\cos x}{1+\cos x}}\right]$$

$$= \tan^{-1}\left[\sqrt{\frac{2\sin^2 x/2}{2\cos^2 x/2}}\right] \quad \left[\because \begin{array}{l} 1-\cos x = 2\sin^2 x/2 \\ 1+\cos x = 2\cos^2 x/2 \end{array}\right]$$

$$= \tan^{-1}\left[\frac{\sin x/2}{\cos x/2}\right] = \tan^{-1}\left[\tan\frac{x}{2}\right]$$

$$\therefore \quad y = \frac{x}{2}$$

Differentiating both sides w.r.t. x

$$\frac{dy}{dx} = \frac{1}{2}\frac{d}{dx}(x) = \frac{1}{2}.1 = \frac{1}{2}$$

Hence, $$\frac{dy}{dx} = \frac{1}{2}$$

Q39. $\tan^{-1}(\sec x + \tan x)$, $\frac{-\pi}{2} < x < \frac{\pi}{2}$

Sol. Let $$y = \tan^{-1}(\sec x + \tan x)$$

Differentiating both sides w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}[\tan^{-1}(\sec x + \tan x)]$$

$$= \frac{1}{1+(\sec x + \tan x)^2}.\frac{d}{dx}(\sec x + \tan x)$$

$$= \frac{1}{1+\sec^2 x + \tan^2 x + 2\sec x \tan x}.(\sec x \tan x + \sec^2 x)$$

$$= \frac{1}{(1+\tan^2 x) + \sec^2 x + 2\sec x \tan x}.\sec x(\tan x + \sec x)$$

$$= \frac{1}{\sec^2 x + \sec^2 x + 2\sec x \tan x}.\sec x(\tan x + \sec x)$$

$$= \frac{1}{2\sec^2 x + 2\sec x \tan x}.\sec x(\tan x + \sec x)$$

$$= \frac{1}{2\sec x(\sec x + \tan x)}.\sec x(\tan x + \sec x) = \frac{1}{2}$$

Hence, $$\frac{dy}{dx} = \frac{1}{2}$$

Alternate solution

Let $y = \tan^{-1}(\sec x + \tan x)$, $\frac{-\pi}{2} < x < \frac{\pi}{2}$

$$= \tan^{-1}\left(\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right) = \tan^{-1}\left(\frac{1 + \sin x}{\cos x}\right)$$

$$= \tan^{-1}\left[\frac{\cos^2 x/2 + \sin^2 x/2 + 2\sin x/2 \cos x/2}{\cos^2 x/2 - \sin^2 x/2}\right]$$

$$\left[\begin{array}{l}\because \sin 2x = 2\sin x \cos x \\ \cos 2x = \cos^2 x - \sin^2 x\end{array}\right]$$

$$= \tan^{-1}\left[\frac{(\cos x/2 + \sin x/2)^2}{(\cos x/2 + \sin x/2)(\cos x/2 - \sin x/2)}\right]$$

$$= \tan^{-1}\left[\frac{\cos x/2 + \sin x/2}{\cos x/2 - \sin x/2}\right]$$

$$= \tan^{-1}\left[\frac{1 + \tan x/2}{1 - \tan x/2}\right]$$ [Dividing the Nr. and Den. by $\cos x/2$]

$$= \tan^{-1}\left[\frac{\tan \pi/4 + \tan x/2}{1 - \tan \pi/4 . \tan x/2}\right] = \tan^{-1}\left[\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right]$$

$$\therefore \quad y = \frac{\pi}{4} + \frac{x}{2}$$

Differentiating both sides w.r.t. x

$$\frac{dy}{dx} = \frac{1}{2}\frac{d}{dx}(x) = \frac{1}{2}.1 = \frac{1}{2}$$

Hence, $\frac{dy}{dx} = \frac{1}{2}$.

Q40. $\tan^{-1}\left(\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right)$, $\frac{-\pi}{2} < x < \frac{\pi}{2}$ and $\frac{a}{b}\tan x > -1$.

Sol. Let $$y = \tan^{-1}\left(\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right)$$

$$\Rightarrow \quad y = \tan^{-1}\left[\frac{\frac{a\cos x}{b\cos x} - \frac{b\sin x}{b\cos x}}{\frac{b\cos x}{b\cos x} + \frac{a\sin x}{b\cos x}}\right]$$

$$\Rightarrow \qquad y = \tan^{-1}\left[\frac{\frac{a}{b} - \tan x}{1 + \frac{a}{b}\tan x}\right]$$

$$\Rightarrow \qquad y = \tan^{-1}\frac{a}{b} - \tan^{-1}(\tan x)$$

$$\left[\because \tan^{-1}\left(\frac{x-y}{1+xy}\right) = \tan^{-1}x - \tan^{-1}y\right]$$

$$\Rightarrow \qquad y = \tan^{-1}\frac{a}{b} - x$$

Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\tan^{-1}\frac{a}{b}\right) - \frac{d}{dx}(x) = 0 - 1 = -1$$

Hence, $\dfrac{dy}{dx} = -1.$

Q41. $\sec^{-1}\left(\dfrac{1}{4x^3 - 3x}\right),\ 0 < x < \dfrac{1}{\sqrt{2}}.$

Sol. Let $\qquad y = \sec^{-1}\left(\dfrac{1}{4x^3 - 3x}\right)$

Put $x = \cos\theta \quad \therefore \theta = \cos^{-1}x$

$$y = \sec^{-1}\left(\frac{1}{4\cos^3\theta - 3\cos\theta}\right)$$

$$\Rightarrow \qquad y = \sec^{-1}\left(\frac{1}{\cos 3\theta}\right) \qquad [\because \cos 3\theta = 4\cos^3\theta - 3\cos\theta]$$

$$\Rightarrow \qquad y = \sec^{-1}(\sec 3\theta) \Rightarrow y = 3\theta$$

$$y = 3\cos^{-1}x$$

Differentiating both sides w.r.t. x

$$\frac{dy}{dx} = 3.\frac{d}{dx}\cos^{-1}x = 3\left(\frac{-1}{\sqrt{1-x^2}}\right) = \frac{-3}{\sqrt{1-x^2}}$$

Hence, $\dfrac{dy}{dx} = \dfrac{-3}{\sqrt{1-x^2}}.$

Q42. $\tan^{-1}\left(\dfrac{3a^2x - x^3}{a^3 - 3ax^2}\right),\ \dfrac{-1}{\sqrt{3}} < \dfrac{x}{a} < \dfrac{1}{\sqrt{3}}.$

Sol. Let $\qquad y = \tan^{-1}\left[\dfrac{3a^2x - x^3}{a^3 - 3ax^2}\right]$

Put $x = a\tan\theta \quad \therefore \theta = \tan^{-1}\dfrac{x}{a}$

$$y = \tan^{-1}\left[\frac{3a^2.a\tan\theta - a^3\tan^3\theta}{a^3 - 3a.a^2\tan^2\theta}\right]$$

$$\Rightarrow \quad y = \tan^{-1}\left[\frac{3a^3\tan\theta - a^3\tan^3\theta}{a^3 - 3a^3\tan^2\theta}\right]$$

$$\Rightarrow \quad y = \tan^{-1}\left[\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\right]$$

$$\Rightarrow \quad y = \tan^{-1}[\tan 3\theta] \quad \left[\because \tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\right]$$

$$\Rightarrow \quad y = 3\theta \Rightarrow y = 3\tan^{-1}\frac{x}{a}$$

Differentiating both sides w.r.t. x

$$\frac{dy}{dx} = 3.\frac{d}{dx}\left(\tan^{-1}\frac{x}{a}\right)$$

$$= 3.\frac{1}{1+\frac{x^2}{a^2}}.\frac{d}{dx}.\left(\frac{x}{a}\right) = 3.\frac{a^2}{a^2+x^2}.\frac{1}{a} = \frac{3a}{a^2+x^2}$$

Hence, $\dfrac{dy}{dx} = \dfrac{3a}{a^2+x^2}$.

Q43. $\tan^{-1}\left(\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right), -1 < x < 1, x \neq 0.$

Sol. Let $\quad y = \tan^{-1}\left(\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)$

Putting $x^2 = \cos 2\theta \quad \therefore \theta = \dfrac{1}{2}\cos^{-1}x^2$

$$y = \tan^{-1}\left(\frac{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}\right)$$

$$\Rightarrow \quad y = \tan^{-1}\left(\frac{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}\right)$$

$$\Rightarrow \quad y = \tan\left(\frac{\sqrt{2}\cos\theta+\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta-\sqrt{2}\sin\theta}\right)$$

$$\Rightarrow \qquad y = \tan^{-1}\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right)$$

$$\Rightarrow \qquad y = \tan^{-1}\left[\frac{\frac{\cos\theta}{\cos\theta} + \frac{\sin\theta}{\cos\theta}}{\frac{\cos\theta}{\cos\theta} - \frac{\sin\theta}{\cos\theta}}\right]$$

$$\Rightarrow \qquad y = \tan^{-1}\left[\frac{1 + \tan\theta}{1 - \tan\theta}\right]$$

$$\Rightarrow \qquad y = \tan^{-1}\left[\frac{\tan\frac{\pi}{4} + \tan\theta}{1 - \tan\frac{\pi}{4}.\tan\theta}\right]$$

$$\Rightarrow \qquad y = \tan^{-1}\left[\tan\left(\frac{\pi}{4} + \theta\right)\right]$$

$$\Rightarrow \qquad y = \frac{\pi}{4} + \theta \quad \Rightarrow y = \frac{\pi}{4} + \frac{1}{2}\cos^{-1}x^2$$

Differentiating both sides w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{4}\right) + \frac{1}{2}\frac{d}{dx}(\cos^{-1}x^2)$$

$$= 0 + \frac{1}{2} \times \frac{-1}{\sqrt{1-x^4}}.\frac{d}{dx}(x^2) = \frac{-1.2x}{2\sqrt{1-x^4}} = -\frac{x}{\sqrt{1-x^4}}$$

Hence, $\dfrac{dy}{dx} = -\dfrac{x}{\sqrt{1-x^4}}$.

Find $\dfrac{dy}{dx}$ of each of the functions expressed in parametric form in Exercises from 44 to 48:

Q44. $x = t + \dfrac{1}{t},\ y = t - \dfrac{1}{t}$

Sol. Given that:

$$x = t + \frac{1}{t},\ y = t - \frac{1}{t}$$

Differentiating both the given parametric functions w.r.t. t

$$\frac{dx}{dt} = 1 - \frac{1}{t^2},\ \frac{dy}{dt} = 1 + \frac{1}{t^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1+\frac{1}{t^2}}{1-\frac{1}{t^2}} = \frac{t^2+1}{t^2-1}$$

Hence, $\frac{dy}{dx} = \frac{t^2+1}{t^2-1}$.

Q45. $x = e^{\theta}\left(\theta + \frac{1}{\theta}\right), y = e^{-\theta}\left(\theta - \frac{1}{\theta}\right)$

Sol. Given that:

$$x = e^{\theta}\left(\theta + \frac{1}{\theta}\right), \quad y = e^{-\theta}\left(\theta - \frac{1}{\theta}\right)$$

Differentiating both the parametric functions w.r.t. θ.

$$\frac{dx}{d\theta} = e^{\theta}\left(1 - \frac{1}{\theta^2}\right) + \left(\theta + \frac{1}{\theta}\right).e^{\theta}$$

$$\frac{dx}{d\theta} = e^{\theta}\left(1 - \frac{1}{\theta^2} + \theta + \frac{1}{\theta}\right) \Rightarrow e^{\theta}\left(\frac{\theta^2 - 1 + \theta^3 + \theta}{\theta^2}\right)$$

$$= \frac{e^{\theta}(\theta^3 + \theta^2 + \theta - 1)}{\theta^2}$$

$$y = e^{-\theta}\left(\theta - \frac{1}{\theta}\right)$$

$$\frac{dy}{d\theta} = e^{-\theta}\left(1 + \frac{1}{\theta^2}\right) + \left(\theta - \frac{1}{\theta}\right).(-e^{-\theta})$$

$$\frac{dy}{d\theta} = e^{-\theta}\left(1 + \frac{1}{\theta^2} - \theta + \frac{1}{\theta}\right) \Rightarrow e^{-\theta}\left(\frac{\theta^2 + 1 - \theta^3 + \theta}{\theta^2}\right)$$

$$= e^{-\theta}\frac{(-\theta^3 + \theta^2 + \theta + 1)}{\theta^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{e^{-\theta}\left(\frac{-\theta^3 + \theta^2 + \theta + 1}{\theta^2}\right)}{e^{\theta}\left(\frac{\theta^3 + \theta^2 + \theta - 1}{\theta^2}\right)}$$

$$= e^{-2\theta}\left(\frac{-\theta^3 + \theta^2 + \theta + 1}{\theta^3 + \theta^2 + \theta - 1}\right)$$

Hence, $\frac{dy}{dx} = e^{-2\theta}\left(\frac{-\theta^3+\theta^2+\theta+1}{\theta^3+\theta^2+\theta-1}\right)$.

Q46. $x = 3\cos\theta - 2\cos^3\theta,\ y = 3\sin\theta - 2\sin^3\theta$.

Sol. Given that: $x = 3\cos\theta - 2\cos^3\theta$ and $y = 3\sin\theta - 2\sin^3\theta$.

Differentiating both the parametric functions w.r.t. θ

$$\frac{dx}{d\theta} = -3\sin\theta - 6\cos^2\theta.\frac{d}{d\theta}(\cos\theta)$$
$$= -3\sin\theta - 6\cos^2\theta.(-\sin\theta)$$
$$= -3\sin\theta + 6\cos^2\theta.\sin\theta$$
$$\frac{dy}{d\theta} = 3\cos\theta - 6\sin^2\theta.\frac{d}{d\theta}(\sin\theta)$$
$$= 3\cos\theta - 6\sin^2\theta.\cos\theta$$
$$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3\cos\theta - 6\sin^2\theta\cos\theta}{-3\sin\theta+6\cos^2\theta.\sin\theta}$$
$$\Rightarrow \quad \frac{dy}{dx} = \frac{\cos\theta(3-6\sin^2\theta)}{\sin\theta(-3+6\cos^2\theta)} = \frac{\cos\theta\left[3-6(1-\cos^2\theta)\right]}{\sin\theta\left[-3+6\cos^2\theta\right]}$$
$$= \cot\theta\left(\frac{3-6+6\cos^2\theta}{-3+6\cos^2\theta}\right) = \cot\theta\left(\frac{-3+6\cos^2\theta}{-3+6\cos^2\theta}\right)$$
$$= \cot\theta$$

Hence, $\frac{dy}{dx} = \cot\theta$.

Q47. $\sin x = \frac{2t}{1+t^2},\ \tan y = \frac{2t}{1-t^2}$

Sol. Given that $\sin x = \frac{2t}{1+t^2}$ and $\tan y = \frac{2t}{1-t^2}$

$\therefore$ Taking $\sin x = \frac{2t}{1+t^2}$

Differentiating both sides w.r.t t, we get

$$\cos x.\frac{dx}{dt} = \frac{(1+t^2).\frac{d}{dt}(2t) - 2t.\frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$
$$\Rightarrow \quad \cos x.\frac{dx}{dt} = \frac{2(1+t^2)-2t.2t}{(1+t^2)^2}$$
$$\Rightarrow \quad \frac{dx}{dt} = \frac{2+2t^2-4t^2}{(1+t^2)^2}\times\frac{1}{\cos x}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{2-2t^2}{(1+t^2)^2} \times \frac{1}{\sqrt{1-\sin^2 x}}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{2(1-t^2)}{(1+t^2)^2} \times \frac{1}{\sqrt{1-\left(\frac{2t}{1+t^2}\right)^2}}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{2(1-t^2)}{(1+t^2)^2} \times \frac{1}{\sqrt{\frac{(1+t^2)^2-4t^2}{(1+t^2)^2}}}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{2(1-t^2)}{(1+t^2)^2} \times \frac{1+t^2}{\sqrt{1+t^4+2t^2-4t^2}}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{2(1-t^2)}{(1 \quad {}^2)^2} \times \frac{(1+t^2)}{\sqrt{1+t^4-2t^2}}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{2(1-t^2)}{(1+t^2)} \times \frac{1}{\sqrt{(1-t^2)^2}}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{2(1-t^2)}{(1+t^2)} \times \frac{1}{(1-t^2)} \quad \Rightarrow \quad \frac{dx}{dt} = \frac{2}{1+t^2}$$

Now taking, $\tan y = \frac{2}{1-t^2}$

Differentiating both sides w.r.t, t, we get

$$\frac{d}{dt}(\tan y) = \frac{d}{dt}\left(\frac{2t}{1-t^2}\right)$$

$$\Rightarrow \quad \sec^2 y \frac{dy}{dt} = \frac{(1-t^2)\,.\,\frac{d}{dt}(2t) - 2t\,.\,\frac{d}{dt}(1-t^2)}{(1-t^2)^2}$$

$$\Rightarrow \quad \sec^2 y \frac{dy}{dt} = \frac{(1-t^2)\,.\,2 - 2t\,.\,(-2t)}{(1-t^2)^2}$$

$$\Rightarrow \quad \sec^2 y \frac{dy}{dt} = \frac{2-2t^2+4t^2}{(1-t^2)^2}$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{2+2t^2}{(1-t^2)^2} \times \frac{1}{\sec^2 y}$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{2(1+t^2)}{(1-t^2)^2} \times \frac{1}{1+\tan^2 y}$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{2(1+t^2)}{(1-t^2)^2} \times \frac{1}{1+\left(\frac{2t}{1-t^2}\right)^2}$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{2(1+t^2)}{(1-t^2)^2} \times \frac{1}{\frac{(1-t^2)^2+4t^2}{(1-t^2)^2}}$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{2(1+t^2)}{(1-t^2)^2} \times \frac{(1-t^2)^2}{1+t^2+2t^2+4t^2}$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{2(1+t^2)}{(1-t^2)^2} \times \frac{(1-t^2)^2}{1+t^4+2t^2}$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{2(1+t^2)}{(1-t^2)^2} \times \frac{(1-t^2)^2}{(1+t^2)^2} \quad \Rightarrow \quad \frac{dy}{dt} = \frac{2}{1+t^2}$$

$$\therefore \quad \frac{dy}{dt} = \frac{dy/dt}{dx/dt} = \frac{\frac{2}{1+t^2}}{\frac{2}{1+t^2}} = 1$$

Hence $\frac{dy}{dt} = 1$

Q48. $x = \frac{1+\log t}{t^2},\ y = \frac{3+2\log t}{t}$.

Sol. Given that: $x = \frac{1+\log t}{t^2}, y = \frac{3+2\log t}{t}$.

Differentiating both the parametric functions w.r.t. t

$$\frac{dx}{dt} = \frac{t^2 . \frac{d}{dt}(1+\log t) - (1+\log t).\frac{d}{dt}(t^2)}{t^4}$$

$$= \frac{t^2 . \left(\frac{1}{t}\right) - (1+\log t).2t}{t^4} = \frac{t-(1+\log t).2t}{t^4}$$

$$= \frac{t[1-2-2\log t]}{t^4} = \frac{-(1+2\log t)}{t^3}$$

$$y = \frac{3+2\log t}{t}$$

$$\frac{dy}{dt} = \frac{t.\frac{d}{dt}(3+2\log t)-(3+2\log t).\frac{d}{dt}(t)}{t^2}$$

$$= \frac{t(2/t)-(3+2\log t).1}{t^2}$$

$$= \frac{2-3-2\log t}{t^2} = \frac{-(1+2\log t)}{t^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{-(1+2\log t)}{t^2}}{\frac{-(1+2\log t)}{t^3}} = \frac{t^3}{t^2} = t$$

Hence, $\frac{dy}{dx} = t.$

Q49. If $x = e^{\cos 2t}$ and $y = e^{\sin 2t}$, prove that $\frac{dy}{dx} = \frac{-y\log x}{x\log y}$.

Sol. Given that: $x = e^{\cos 2t}$ and $y = e^{\sin 2t}$

$\Rightarrow \cos 2t = \log x$ and $\sin 2t = \log y$.

Differentiating both the parametric functions w.r.t. t

$$\frac{dx}{dt} = e^{\cos 2t}.\frac{d}{dt}(\cos 2t) = e^{\cos 2t}(-\sin 2t).\frac{d}{dt}(2t)$$

$$= -e^{\cos 2t}.\sin 2t.2 = -2e^{\cos 2t}.\sin 2t$$

Now $\quad y = e^{\sin 2t}$

$$\frac{dy}{dt} = e^{\sin 2t}.\frac{d}{dt}(\sin 2t) = e^{\sin 2t}.\cos 2t.\frac{d}{dt}(2t)$$

$$= e^{\sin 2t}.\cos 2t.2 = 2e^{\sin 2t}.\cos 2t$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2e^{\sin 2t}.\cos 2t}{-2e^{\cos 2t}.\sin 2t} = \frac{e^{\sin 2t}.\cos 2t}{-e^{\cos 2t}.\sin 2t} = \frac{y\cos 2t}{-x\sin 2t}$$

$$= \frac{y\log x}{-x\log y} \qquad \left[\because \begin{array}{l}\cos 2t = \log x\\ \sin 2t = \log y\end{array}\right]$$

Hence, $\frac{dy}{dx} = -\frac{y\log x}{x\log y}$.

Q50. If $x = a\sin 2t\,(1+\cos 2t)$ and $y = b\cos 2t\,(1-\cos 2t)$, show that $\left(\frac{dy}{dx}\right)_{at\ t=\frac{\pi}{4}} = \frac{b}{a}$.

Sol. Given that: $x = a\sin 2t\,(1+\cos 2t)$ and $y = b\cos 2t\,(1-\cos 2t)$.

Differentiating both the parametric functions w.r.t. t

$$\frac{dx}{dt} = a\left[\sin 2t.\frac{d}{dt}(1+\cos 2t) + (1+\cos 2t).\frac{d}{dt}\sin 2t\right]$$
$$= a[\sin 2t.(-\sin 2t).2 + (1+\cos 2t)(\cos 2t).2]$$
$$= a[-2\sin^2 2t + 2\cos 2t + 2\cos^2 2t]$$
$$= a[2(\cos^2 2t - \sin^2 2t) + 2\cos 2t]$$
$$= a[2\cos 4t + 2\cos 2t] \qquad [\because \cos 2x = \cos^2 x - \sin^2 x]$$
$$= 2a[\cos 4t + \cos 2t]$$
$$y = b\cos 2t\,(1-\cos 2t)$$
$$\frac{dy}{dt} = b\left[\cos 2t.\frac{d}{dt}(1-\cos 2t) + (1-\cos 2t).\frac{d}{dt}(\cos 2t)\right]$$
$$= b[\cos 2t.\sin 2t.2 + (1-\cos 2t).(-\sin 2t).2]$$
$$= b[2\sin 2t.\cos 2t - 2\sin 2t + 2\sin 2t\cos 2t]$$
$$= b[\sin 4t - 2\sin 2t + \sin 4t] \quad [\because \sin 2x = 2\sin x\cos x]$$
$$= b[2\sin 4t - 2\sin 2t] = 2b(\sin 4t - \sin 2t)$$
$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2b[\sin 4t - \sin 2t]}{2a[\cos 4t + \cos 2t]} = \frac{b}{a}\left[\frac{\sin 4t - \sin 2t}{\cos 4t + \cos 2t}\right]$$

Put $\quad t = \frac{\pi}{4}$

$$\therefore \left(\frac{dy}{dx}\right)_{at\ t=\frac{\pi}{4}} = \frac{b}{a}\left[\frac{\sin 4\left(\frac{\pi}{4}\right) - \sin 2.\left(\frac{\pi}{4}\right)}{\cos 4\left(\frac{\pi}{4}\right) + \cos 2.\left(\frac{\pi}{4}\right)}\right] = \frac{b}{a}\left[\frac{\sin\pi - \sin\frac{\pi}{2}}{\cos\pi + \cos\frac{\pi}{2}}\right]$$
$$= \frac{b}{a}\left[\frac{0-1}{-1+0}\right] = \frac{b}{a}\left(\frac{-1}{-1}\right) = \frac{b}{a}$$

Hence, $\left(\frac{dy}{dx}\right)_{at\ t=\frac{\pi}{4}} = \frac{b}{a}$.

Q51. If $x = 3\sin t - \sin 3t$, $y = 3\cos t - \cos 3t$, find $\frac{dy}{dx}$ at $t = \frac{\pi}{3}$.

Sol. Given that: $x = 3\sin t - \sin 3t$, $y = 3\cos t - \cos 3t$.
Differentiating both parametric functions w.r.t. t

$$\frac{dx}{dt} = 3\cos t - \cos 3t.3 = 3(\cos t - \cos 3t)$$
$$\frac{dy}{dt} = -3\sin t + \sin 3t.3 = 3(-\sin t + \sin 3t)$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3(-\sin t + \sin 3t)}{3(\cos t - \cos 3t)} = \frac{-\sin t + \sin 3t}{\cos t - \cos 3t}$$

Put $\quad t = \frac{\pi}{3}$

$$\frac{dy}{dx} = \frac{-\sin\frac{\pi}{3} + \sin 3\left(\frac{\pi}{3}\right)}{\cos\frac{\pi}{3} - \cos 3\left(\frac{\pi}{3}\right)}$$

$$= \frac{-\frac{\sqrt{3}}{2} + \sin\pi}{\frac{1}{2} - \cos\pi} = \frac{-\frac{\sqrt{3}}{2} + 0}{\frac{1}{2} - (-1)} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2} + 1} = \frac{-\frac{\sqrt{3}}{2}}{\frac{3}{2}} = \frac{-1}{\sqrt{3}}$$

Hence, $\frac{dy}{dx} = \frac{-1}{\sqrt{3}}$.

Q52. Differentiate $\frac{x}{\sin x}$ w.r.t. $\sin x$.

Sol. Let $\quad y = \frac{x}{\sin x} \quad$ and $\quad z = \sin x$.

Differentiating both the parametric functions w.r.t. x,

$$\frac{dy}{dx} = \frac{\sin x.\frac{d}{dx}(x) - x.\frac{d}{dx}(\sin x)}{(\sin x)^2}$$

$$= \frac{\sin x.1 - x.\cos x}{\sin^2 x} = \frac{\sin x - x\cos x}{\sin^2 x}$$

$$\frac{dz}{dx} = \cos x$$

$$\therefore \quad \frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{\frac{\sin x - x\cos x}{\sin^2 x}}{\cos x} = \frac{\sin x - x\cos x}{\sin^2 x\cos x}$$

$$= \frac{\sin x}{\sin^2 x\cos x} - \frac{x\cos x}{\sin^2 x\cos x}$$

$$= \frac{\tan x}{\sin^2 x} - \frac{x}{\sin^2 x} = \frac{\tan x - x}{\sin^2 x}$$

Hence, $\frac{dy}{dz} = \frac{\tan x - x}{\sin^2 x}$.

Q53. Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2} - 1}{x}\right)$ w.r.t. $\tan^{-1} x$, when $x \neq 0$.

Sol. Let $y = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ and $z = \tan^{-1} x$.

Put $x = \tan\theta$.

$\therefore \quad y = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$ and $z = \tan^{-1}(\tan\theta) = \theta$.

$$\Rightarrow \quad \tan\left(\frac{\sqrt{\sec\ \theta}-1}{\tan}\right) = \tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{\frac{1}{\cos\theta}-1}{\frac{\sin\theta}{\cos\theta}}\right) = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{2\sin^2\theta/2}{2\sin\theta/2\cos\theta/2}\right) = \tan^{-1}\left(\frac{\sin\theta/2}{\cos\theta/2}\right)$$

$$\Rightarrow \quad y = \tan^{-1}\left(\tan\frac{\theta}{2}\right) \Rightarrow y = \frac{\theta}{2}$$

Differentiating both parametric functions w.r.t. θ

$$\frac{dy}{d\theta} = \frac{1}{2}.\frac{d}{d\theta}(\theta) \quad \text{and} \quad \frac{dz}{d\theta} = \frac{d}{d\theta}(\theta)$$

$$= \frac{1}{2}.1 = \frac{1}{2} \quad \text{and} \quad \frac{dz}{d\theta} = 1$$

$$\therefore \quad \frac{dy}{dz} = \frac{dy/d\theta}{dz/d\theta} = \frac{1/2}{1} = \frac{1}{2}.$$

Find $\frac{dy}{dx}$ when x *and* y are connected by the relation given in each of the Exercises 54 to 57:

Q54. $\sin xy + \frac{x}{y} = x^2 - y$.

Sol. Given that: $\sin xy + \frac{x}{y} = x^2 - y$.

Differentiating both sides w.r.t. x

$$\frac{d}{dx}\sin(xy) + \frac{d}{dx}\left(\frac{x}{y}\right) = \frac{d}{dx}(x^2) - \frac{d}{dx}(y)$$

$$\Rightarrow \quad \cos xy.\frac{d}{dx}(xy) + \frac{y.\frac{d}{dx}x - x.\frac{dy}{dx}}{y^2} = 2x - \frac{dy}{dx}$$

$$\Rightarrow \quad \cos xy\left[x.\frac{dy}{dx}+y.1\right]+\frac{y.1}{y^2}-\frac{x}{y^2}.\frac{dy}{dx} = 2x-\frac{dy}{dx}$$

$$\Rightarrow x\cos xy.\frac{dy}{dx}+y\cos xy+\frac{1}{y}-\frac{x}{y^2}\frac{dy}{dx} = 2x-\frac{dy}{dx}$$

$$\Rightarrow \quad x\cos xy.\frac{dy}{dx}-\frac{x}{y^2}.\frac{dy}{dx}+\frac{dy}{dx} = -y\cos xy-\frac{1}{y}+2x$$

$$\Rightarrow \quad \left[x\cos xy-\frac{x}{y^2}+1\right]\frac{dy}{dx} = 2x-y\cos xy-\frac{1}{y}$$

$$\Rightarrow \quad \frac{\left[xy^2\cos xy-x+y^2\right]}{y^2}\frac{dy}{dx} = \frac{2xy-y^2\cos xy-1}{y}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2xy-y^2\cos xy-1}{y}\times\frac{y^2}{xy^2\cos xy-x+y^2}$$

$$= \frac{2xy^2-y^3\cos(xy)-y}{xy^2\cos(xy)-x+y^2}$$

Hence, $\dfrac{dy}{dx} = \dfrac{2xy^2-y^3\cos(xy)-y}{xy^2\cos(xy)-x+y^2}$.

Q55. $\sec(x+y)=xy$

Sol. Given that: $\sec(x+y)=xy$

Differentiating both sides w.r.t. x

$$\frac{d}{dx}\sec(x+y) = \frac{d}{dx}(xy)$$

$$\Rightarrow \quad \sec(x+y)\tan(x+y).\frac{d}{dx}(x+y) = x.\frac{dy}{dx}+y.1$$

$$\Rightarrow \quad \sec(x+y).\tan(x+y)\left(1+\frac{dy}{dx}\right) = x.\frac{dy}{dx}+y$$

$$\Rightarrow \sec(x+y).\tan(x+y)+\sec(x+y).\tan(x+y).\frac{dy}{dx} = x.\frac{dy}{dx}+y$$

$$\Rightarrow \quad \sec(x+y).\tan(x+y).\frac{dy}{dx}-x\frac{dy}{dx} = y-\sec(x+y).\tan(x+y)$$

$$\Rightarrow \quad [\sec(x+y).\tan(x+y)-x]\frac{dy}{dx} = y-\sec(x+y).\tan(x+y)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y-\sec(x+y).\tan(x+y)}{\sec(x+y).\tan(x+y)-x}$$

Hence, $\dfrac{dy}{dx} = \dfrac{y-\sec(x+y).\tan(x+y)}{\sec(x+y).\tan(x+y)-x}$.

Q56. $\tan^{-1}(x^2+y^2)=a$

Sol. Given that: $\tan^{-1}(x^2+y^2)=a$

$\Rightarrow \quad x^2+y^2 = \tan a.$

Differentiating both sides w.r.t. x.

$$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(\tan a)$$

$$\Rightarrow \quad 2x+2y.\frac{dy}{dx} = 0 \Rightarrow 2y.\frac{dy}{dx} = -2x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y}$$

Hence, $\dfrac{dy}{dx} = \dfrac{-x}{y}.$

Q57. $(x^2+y^2)^2=xy$

Sol. Given that: $(x^2+y^2)^2=xy$

$\Rightarrow \quad x^4+y^4+2x^2y^2 = xy$

Differentiating both sides w.r.t. x

$$\frac{d}{dx}(x^4)+\frac{d}{dx}(y^4)+2.\frac{d}{dx}(x^2y^2) = \frac{d}{dx}(xy)$$

$$\Rightarrow 4x^3+4y^3.\frac{dy}{dx}+2\left[x^2.2y.\frac{dy}{dx}+y^2.2x\right] = x\frac{dy}{dx}+y.1$$

$$\Rightarrow \quad 4x^3+4y^3.\frac{dy}{dx}+4x^2y.\frac{dy}{dx}+4xy^2 = x\frac{dy}{dx}+y$$

$$\Rightarrow \quad 4y^3\frac{dy}{dx}+4x^2y\frac{dy}{dx}-x\frac{dy}{dx} = y-4x^3-4xy^2$$

$$\Rightarrow \quad (4y^3+4x^2y-x)\frac{dy}{dx} = y-4x^3-4xy^2$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y-4x^3-4xy^2}{4y^3+4x^2y-x}$$

Hence, $\dfrac{dy}{dx} = \dfrac{y-4x^3-4xy^2}{4x^2y+4y^3-x}.$

Q58. If $ax^2+2hxy+by^2+2gx+2fy+c=0$, then show that $\dfrac{dy}{dx}.\dfrac{dx}{dy}=1.$

Sol. Given that: $ax^2+2hxy+by^2+2gx+2fy+c=0.$

Differentiating both sides w.r.t. x

$$\frac{d}{dx}(ax^2+2hxy+by^2+2gx+2fy+c) = \frac{d}{dx}(0)$$

$$\Rightarrow a.2x+2h\left(x.\frac{dy}{dx}+y.1\right)+b.2y.\frac{dy}{dx}+2g.1+2f.\frac{dy}{dx}+0=0$$

$$\Rightarrow 2ax+2hx.\frac{dy}{dx}+2hy+2by.\frac{dy}{dx}+2g+2f.\frac{dy}{dx}=0$$

$$\Rightarrow \quad 2hx.\frac{dy}{dx}+2by\frac{dy}{dx}+2f\frac{dy}{dx}=-2ax-2hy-2g$$

$$\Rightarrow \quad (2hx+2by+2f)\frac{dy}{dx}=-2(ax+hy+g)$$

$$\Rightarrow \quad 2(hx+by+f)\frac{dy}{dx}=-2(ax+hy+g)$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{-2(ax+hy+g)}{2(hx+by+f)}$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{-(ax+hy+g)}{(hx+by+f)}$$

Now, differentiating the given equation w.r.t. y.

$$\frac{d}{dy}(ax^2+2hxy+by^2+2gx+2fy+c)=\frac{d}{dy}(0)$$

$$\Rightarrow \quad 2ax.\frac{dx}{dy}+2h\left(y.\frac{dx}{dy}+x.1\right)+2by+2g.\frac{dx}{dy}+2f.1+0=0$$

$$\Rightarrow \quad 2ax.\frac{dx}{dy}+2hy.\frac{dx}{dy}+2hx+2by+2g.\frac{dx}{dy}+2f=0$$

$$\Rightarrow \quad 2ax\frac{dx}{dy}+2hy.\frac{dx}{dy}+2g.\frac{dx}{dy}=-2hx-2by-2f$$

$$\Rightarrow \quad (2ax+2hy+2g)\frac{dx}{dy}=-2hx-2by-2f$$

$$\Rightarrow \quad \frac{dx}{dy}=\frac{-2hx-2by-2f}{2ax+2hy+2g}$$

$$\Rightarrow \quad \frac{dx}{dy}=\frac{-2(hx+by+f)}{2(ax+hy+g)} \Rightarrow \frac{dx}{dy}=\frac{-(hx+by+f)}{(ax+hy+g)}$$

$$\therefore \quad \frac{dy}{dx}.\frac{dx}{dy}=\left[\frac{-(ax+hy+g)}{(hx+by+f)}\right]\left[\frac{-(hx+by+f)}{(ax+hy+g)}\right]=1$$

Hence, $\frac{dy}{dx}.\frac{dx}{dy}=1$. Hence, proved.

Q59. If $x = e^{x/y}$, prove that $\frac{dy}{dx} = \frac{x-y}{x \log x}$.

Sol. Given that: $x = e^{x/y}$

Taking log on both the sides,

$$\log x = \log e^{x/y}$$

$$\Rightarrow \quad \log x = \frac{x}{y}\log e \quad \Rightarrow \log x = \frac{x}{y} \quad [\because \log e = 1] \quad ...(i)$$

Differentiating both sides w.r.t. x

$$\frac{d}{dx}\log x = \frac{d}{dx}\left(\frac{x}{y}\right)$$

$$\Rightarrow \quad \frac{1}{x} = \frac{y.1 - x.\frac{dy}{dx}}{y^2}$$

$$\Rightarrow \quad y^2 = xy - x^2.\frac{dy}{dx} \quad \Rightarrow x^2.\frac{dy}{dx} = xy - y^2$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y(x-y)}{x^2} \quad \Rightarrow \frac{dy}{dx} = \frac{y}{x}.\left(\frac{x-y}{x}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\log x}.\left(\frac{x-y}{x}\right) \quad \left(\because \log x = \frac{x}{y} \text{ from eqn. } (i)\right)$$

Hence, $\frac{dy}{dx} = \frac{x-y}{x \log x}$.

Q60. If $y^x = e^{y-x}$, prove that $\frac{dy}{dx} = \frac{(1+\log y)^2}{\log y}$.

Sol. Given that: $y^x = e^{y-x}$

Taking log on both sides $\log y^x = \log e^{y-x}$

$$\Rightarrow \quad x \log y = (y - x) \log e$$

$$\Rightarrow \quad x \log y = y - x \quad [\because \log e = 1]$$

$$\Rightarrow \quad x \log y + x = y$$

$$\Rightarrow \quad x(\log y + 1) = y$$

$$\Rightarrow \quad x = \frac{y}{\log y + 1}.$$

Differentiating both sides w.r.t. y

$$\frac{dx}{dy} = \frac{d}{dy}\left(\frac{y}{\log y + 1}\right)$$

$$= \frac{(\log y + 1).1 - y.\frac{d}{dy}(\log y + 1)}{(\log y + 1)^2}$$

$$= \frac{\log y + 1 - y.\frac{1}{y}}{(\log y + 1)^2} = \frac{\log y + 1 - 1}{(\log y + 1)^2} = \frac{\log y}{(\log y + 1)^2}$$

We know that

$$\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{\frac{\log y}{(\log y + 1)^2}} = \frac{(\log y + 1)^2}{\log y}$$

Hence, $\frac{dy}{dx} = \frac{(\log y + 1)^2}{\log y}$.

Q61. If $y = (\cos x)^{(\cos x)^{(\cos x)\ldots\infty}}$, show that $\frac{dy}{dx} = \frac{y^2 \tan x}{y \log \cos x - 1}$.

Sol. Given that $y = (\cos x)^{(\cos x)^{(\cos x)\ldots\infty}}$

$\Rightarrow \quad y = (\cos x)^y \quad \left[\because y = (\cos x)^{(\cos x)^{(\cos x)\ldots\infty}}\right]$

Taking log on both sides $\log y = y.\log(\cos x)$

Differentiating both sides w.r.t. x

$$\frac{1}{y}.\frac{dy}{dx} = y.\frac{d}{dx}\log(\cos x) + \log(\cos x).\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{1}{y}.\frac{dy}{dx} = y.\frac{1}{\cos x}.\frac{d}{dx}(\cos x) + \log(\cos x).\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{1}{y}.\frac{dy}{dx} = y.\frac{1}{\cos x}.(-\sin x) + \log(\cos x).\frac{dy}{dx}$$

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx} - \log(\cos x)\frac{dy}{dx} = -y\tan x$$

$$\Rightarrow \left[\frac{1}{y} - \log(\cos x)\right]\frac{dy}{dx} = -y\tan x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-y\tan x}{\frac{1}{y} - \log(\cos x)} = \frac{y^2 \tan x}{y \log \cos x - 1}$$

Hence, $\frac{dy}{dx} = \frac{y^2 \tan x}{y \log \cos x - 1}$. Hence, proved.

Q62. If $x \sin(a + y) + \sin a \cos(a + y) = 0$, prove that $\frac{dy}{dx} = \frac{\sin^2(a + y)}{\sin a}$.

Sol. Given that: $x \sin(a + y) + \sin a \cos(a + y) = 0$

$$\Rightarrow \quad x \sin(a+y) = -\sin a \cos(a+y)$$

$$\Rightarrow \quad x = \frac{-\sin a . \cos(a+y)}{\sin(a+y)} \Rightarrow x = -\sin a . \cot(a+y)$$

Differentiating both sides w.r.t. y

$$\Rightarrow \quad \frac{dx}{dy} = -\sin a . \frac{d}{dy}\cot(a+y)$$

$$\Rightarrow \quad \frac{dx}{dy} = -\sin a[-\operatorname{cosec}^2(a+y)]$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{\sin a}{\sin^2(a+y)}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{\dfrac{\sin a}{\sin^2(a+y)}}$$

Hence, $\dfrac{dy}{dx} = \dfrac{\sin^2(a+y)}{\sin a}$. Hence proved.

Q63. If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$, prove that $\dfrac{dy}{dx} = \sqrt{\dfrac{1-y^2}{1-x^2}}$.

Sol. Given that: $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$

Put $x = \sin\theta$ and $y = \sin\phi$.

$\therefore$ $\theta = \sin^{-1} x$ and $\phi = \sin^{-1} y$

$$\sqrt{1-\sin^2\theta} + \sqrt{1-\sin^2\phi} = a(\sin\theta - \sin\phi)$$

$$\Rightarrow \quad \sqrt{\cos^2\theta} + \sqrt{\cos^2\phi} = a(\sin\theta - \sin\phi)$$

$$\Rightarrow \quad \cos\theta + \cos\phi = a(\sin\theta - \sin\phi)$$

$$\Rightarrow \quad \frac{\cos\theta + \cos\phi}{\sin\theta - \sin\phi} = a \Rightarrow \frac{2\cos\dfrac{\theta+\phi}{2}.\cos\dfrac{\theta-\phi}{2}}{2\cos\dfrac{\theta+\phi}{2}.\sin\dfrac{\theta-\phi}{2}} = a$$

$$\left[\begin{array}{l} \because \cos A + \cos B = 2\cos\dfrac{A+B}{2}.\cos\dfrac{A-B}{2} \\ \sin A - \sin B = 2\cos\dfrac{A+B}{2}.\sin\dfrac{A-B}{2} \end{array}\right]$$

$$\Rightarrow \quad \frac{\cos\left(\frac{\theta-\phi}{2}\right)}{\sin\left(\frac{\theta-\phi}{2}\right)} = a \quad \Rightarrow \quad \cot\left(\frac{\theta-\phi}{2}\right) = a$$

$$\Rightarrow \quad \frac{\theta-\phi}{2} = \cot^{-1} a \Rightarrow \theta - \phi = 2\cot^{-1} a$$

$$\Rightarrow \quad \sin^{-1} x - \sin^{-1} y = 2\cot^{-1} a$$

Differentiating both sides w.r.t. x

$$\frac{d}{dx}(\sin^{-1} x) - \frac{d}{dx}(\sin^{-1} y) = 2.\frac{d}{dx}\cot^{-1} a$$

$$\Rightarrow \quad \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-y^2}}.\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{1}{\sqrt{1-y^2}}.\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\therefore \quad \frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

Hence, $\quad \dfrac{dy}{dx} = \sqrt{\dfrac{1-y^2}{1-x^2}}.$

Q64. If $y = \tan^{-1} x$, find $\dfrac{d^2y}{dx^2}$ in terms of y alone.

Sol. Given that: $y = \tan^{-1} x \Rightarrow x = \tan y$

Differentiating both sides w.r.t. y

$$\frac{dx}{dy} = \sec^2 y \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sec^2 y} = \cos^2 y$$

Again differentiating both sides w.r.t. x

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\cos^2 y)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 2\cos y.\frac{d}{dx}(\cos y)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 2\cos y(-\sin y).\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -2\sin y \cos y \,.\, \cos^2 y$$

$$\therefore \quad \frac{d^2y}{dx^2} = -2\sin y \cos^3 y$$

Verify the Rolle's Theorem for each of the functions in Exercises 65 to 69:

Q65. $f(x) = x(x-1)^2$ in $[0, 1]$

Sol. Given that: $f(x) = x(x-1)^2$ in $[0, 1]$

(*i*) $f(x) = x(x-1)^2$, being an algebraic polynomial, is continuous in $[0, 1]$.

(*ii*)
$$\begin{aligned} f'(x) &= x.2(x-1) + (x-1)^2.1 \\ &= 2x^2 - 2x + x^2 + 1 - 2x \\ &= 3x^2 - 4x + 1 \text{ which exists in } (0, 1) \end{aligned}$$

(*iii*)
$$f(x) = x(x-1)^2$$
$$f(0) = 0(0-1)^2 = 0;\ f(1) = 1(1-1)^2 = 0$$
$$\Rightarrow \quad f(0) = f(1) = 0$$

As the above conditions are satisfied, then there must exist at least one point $c \in (0, 1)$ such that $f'(c) = 0$

$$\therefore \quad f'(c) = 3c^2 - 4c + 1 = 0 \Rightarrow 3c^2 - 3c - c + 1 = 0$$
$$\Rightarrow \quad 3c(c-1) - 1(c-1) = 0 \Rightarrow (c-1)(3c-1) = 0$$
$$\Rightarrow \quad c - 1 = 0 \Rightarrow c = 1$$
$$3c - 1 = 0 \Rightarrow 3c = 1 \quad \therefore c = \frac{1}{3} \in (0, 1)$$

Hence, Rolle's Theorem is verified.

Q66. $f(x) = \sin^4 x + \cos^4 x$ in $\left[0, \frac{\pi}{2}\right]$.

Sol. Given that: $f(x) = \sin^4 x + \cos^4 x$ in $\left[0, \frac{\pi}{2}\right]$

(*i*) $f(x) = \sin^4 x + \cos^4 x$, being sine and cosine functions, $f(x)$ is continuous function in $\left[0, \frac{\pi}{2}\right]$.

(*ii*)
$$\begin{aligned} f'(x) &= 4\sin^3 x.\cos x + 4\cos^3 x(-\sin x) \\ &= 4\sin^3 x.\cos x - 4\cos^3 x.\sin x \end{aligned}$$

$= 4 \sin x \cos x (\sin^2 x - \cos^2 x)$
$= -4 \sin x \cos x (\cos^2 x - \sin^2 x)$
$= -2.2 \sin x \cos x . \cos 2x \quad \left[\begin{array}{l} \because \cos 2x = \cos^2 x - \sin^2 x \\ \sin 2x = 2 \sin x \cos x \end{array}\right]$
$= -2 \sin 2x . \cos 2x$
$= -\sin 4x$ which exists in $\left(0, \frac{\pi}{2}\right)$.

So, $f(x)$ is differentiable in $\left(0, \frac{\pi}{2}\right)$.

(*iii*) $f(0) = \sin^4(0) + \cos^4(0) = 1$

$f\left(\frac{\pi}{2}\right) = \sin^4\left(\frac{\pi}{2}\right) + \cos^2\left(\frac{\pi}{2}\right) = 1$

$\therefore \quad f(0) = f\left(\frac{\pi}{2}\right) = 1$

As the above conditions are satisfied, there must exist at least one point $c \in \left(0, \frac{\pi}{2}\right)$ such that $f'(c) = 0$

$\Rightarrow \quad -\sin 4c = 0$
$\Rightarrow \quad \sin 4c = 0 \quad \Rightarrow \quad \sin 4c = \sin 0$
$\Rightarrow \quad 4c = n\pi$
$\therefore \quad c = \frac{n\pi}{4}, n \in I$

For $n = 1$, $\quad c = \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)$

Hence, the Rolle's Theorem is verified.

Q67. $f(x) = \log (x^2 + 2) - \log 3$ in $[-1, 1]$.

Sol. Given that: $f(x) = \log (x^2 + 2) - \log 3$ in $[-1, 1]$

(*i*) $f(x) = \log (x^2 + 2) - \log 3$, being a logarithm function, is continuous in $[-1, 1]$.

(*ii*) $f'(x) = \frac{1}{x^2+2} . 2x - 0 = \frac{2x}{x^2+2}$ which exists in $(-1, 1)$

So, $f(x)$ is differentiable in $(-1, 1)$.

(*iii*) $f(-1) = \log (1 + 2) - \log 3 \quad \Rightarrow \quad \log 3 - \log 3 = 0$

$f(1) = \log (1 + 2) - \log 3 \quad \Rightarrow \quad \log 3 - \log 3 = 0$

$\therefore f(-1) = f(1) = 0$

As the above conditions are satisfied, then there must exist atleast one point $c \in (-1, 1)$ such that $f'(c) = 0$.

$$\therefore \frac{2c}{c^2+2} = 0 \quad \Rightarrow \quad 2c = 0 \quad \therefore c = 0 \in (-1, 1)$$

Hence, Rolle's Theorem is verified.

Q68. $f(x) = x(x+3)\,e^{-x/2}$ in $[-3, 0]$.

Sol. Given that: $f(x) = x(x+3)\,e^{-x/2}$ in $[-3, 0]$

(*i*) Algebraic functions and exponential functions are continuous in their domains.

$\therefore f(x)$ is continuous in $[-3, 0]$

(*ii*)
$$f'(x) = x(x+3).\frac{d}{dx}e^{-x/2} + x.e^{-x/2}.\frac{d}{dx}(x+3) + (x+3).e^{-x/2}\frac{d}{dx}x$$
$$= x(x+3).e^{-x/2}.\left(-\frac{1}{2}\right) + x.e^{-x/2}.1 + (x+3).e^{-x/2}.1$$
$$= e^{-x/2}\left[\frac{-x(x+3)}{2} + x + x + 3\right]$$
$$= e^{-x/2}\left[\frac{-x(x+3)}{2} + 2x + 3\right] = e^{-x/2}\left[\frac{-x^2 - 3x + 4x + 6}{2}\right]$$
$$= e^{-x/2}\left[\frac{-x^2 + x + 6}{2}\right] \text{ which exists in } (-3, 0).$$

So, $f(x)$ is differentiable in $(-3, 0)$.

(*iii*) $f(-3) = (-3)(-3+3)\,e^{-3/2} = 0$

$f(0) = (0)(0+3)\,e^{-0/2} = 0$

$\therefore f(-3) = f(0) = 0$

As the above conditions are satisfied, then there must exist atleast one point $c \in (-3, 0)$ such that

$$f'(c) = 0 \Rightarrow e^{-c/2}\left[\frac{-c^2 + c + 6}{2}\right] = 0$$
$$\Rightarrow -\frac{e^{-c/2}}{2}[c^2 - c - 6] = 0$$
$$\Rightarrow -\frac{e^{-c/2}}{2}(c-3)(c+2) = 0$$
$$\Rightarrow e^{-c/2} \neq 0 \quad \therefore (c-3)(c+2) = 0$$

Which gives $c = 3, c = -2 \in (-3, 0)$.

Hence, Rolle's Theorem is verified.

Q69. $f(x) = \sqrt{4-x^2}$ in $[-2, 2]$.

Sol. Given that: $f(x) = \sqrt{4-x^2}$ in $[-2, 2]$

(*i*) Since algebraic polynomials are continuous,

$\therefore f(x)$ is continuous in $[-2, 2]$

(*ii*) $f'(x) = \dfrac{d}{dx}\sqrt{4-x^2} = \dfrac{1}{2\sqrt{4-x^2}} \times -2x = \dfrac{-x}{\sqrt{4-x^2}}$ which exists in $(-2, 2)$

So, $f'(x)$ is differentiable in $(-2, 2)$.

(*iii*)
$$f(-2) = \sqrt{4-(-2)^2} = \sqrt{4-4} = 0$$
$$f(2) = \sqrt{4-(2)^2} = \sqrt{4-4} = 0$$

So $f(-2) = f(2) = 0$

As the above conditions are satisfied, then there must exist atleast one point $c \in (-2, 2)$ such that

$$f'(c) = 0 \Rightarrow \frac{-c}{\sqrt{4-c^2}} = 0 \Rightarrow c = 0 \in (-2, 2)$$

Hence, Rolle's Theorem is verified.

Q70. Discuss the applicability of Rolle's Theorem on the function given by

$$f(x) = \begin{cases} x^2+1, \text{ if } 0 \le x \le 1 \\ 3-x, \text{ if } 1 \le x \le 2 \end{cases}$$

Sol. (*i*) $f(x)$ being an algebraic polynomial, is continuous everywhere.

(*ii*) $f(x)$ must be differentiable at $x = 1$

$$\text{L.H.L.} = \lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1}$$
$$= \lim_{x\to 1} \frac{(x^2+1)-(1+1)}{x-1}$$
$$= \lim_{x\to 1} \frac{x^2+1-2}{x-1} = \lim_{x\to 1} \frac{x^2-1}{x-1}$$
$$= \lim_{x\to 1} \frac{(x-1)(x+1)}{x-1} = \lim_{x\to 1}(x+1) = (1+1) = 2$$

and $$\text{R.H.L.} = \lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1}$$

$$= \lim_{x \to 1} \frac{(3-x)-(1+1)}{x-1}$$

$$= \lim_{x \to 1} \frac{(3-x)-2}{x-1} = \lim_{x \to 1} \frac{1-x}{x-1} = -1$$

$\therefore$ L.H.L. $\neq$ R.H.L.

So, $f(x)$ is not differentiable at $x = 1$.

Hence, Rolle's Theorem is not applicable in [0, 2].

Q71. Find the points on the curve $y = (\cos x - 1)$ in $[0, 2\pi]$, where the tangent is parallel to x-axis.

Sol. Given that: $y = \cos x - 1$ on $[0, 2\pi]$

We have to find a point c on the given curve $y = \cos x - 1$ on $[0, 2\pi]$ such that the tangent at $c \in [0, 2\pi]$ is parallel to x-axis i.e., $f'(c) = 0$ where $f'(c)$ is the slope of the tangent.

So, we have to verify the Rolle's Theorem.

(*i*) $y = \cos x - 1$ is the combination of cosine and constant functions. So, it is continuous on $[0, 2\pi]$.

(*ii*) $\dfrac{dy}{dx} = -\sin x$ which exists in $(0, 2\pi)$.

So, it is differentiable on $(0, 2\pi)$.

(*iii*) Let $f(x) = \cos x - 1$

$f(0) = \cos 0 - 1 = 1 - 1 = 0; f(2\pi) = \cos 2\pi - 1 = 1 - 1 = 0$

$\therefore \quad f(0) = f(2\pi) = 0$

As the above conditions are satisfied, then there lies a point $c \in (0, 2\pi)$ such that $f'(c) = 0$.

$\therefore -\sin c = 0 \quad \Rightarrow \quad \sin c = 0$

$\therefore c = n\pi, n \in I$

$\Rightarrow c = \pi \in (0, 2\pi)$

Hence, $c = \pi$ is the point on the curve in $(0, 2\pi)$ at which the tangent is parallel to x-axis.

Q72. Using Rolle's theorem, find the point on the curve $y = x(x-4)$, $x \in [0, 4]$, where the tangent is parallel to x-axis.

Sol. Given that: $y = x(x-4), x \in [0, 4]$

Let $\quad f(x) = x(x-4), x \in [0, 4]$

(*i*) $f(x)$ being an algebraic polynomial, is continuous function everywhere.

So, $f(x) = x(x-4)$ is continuous in [0, 4].

(*ii*) $\quad f'(x) = 2x - 4$ which exists in (0, 4).

So, $f(x)$ is differentiable.

(*iii*) $f(0) = 0(0-4) = 0$

$f(4) = 4(4-4) = 0$

So $f(0) = f(4) = 0$

As the above conditions are satisfied, then there must exist at least one point $c \in (0, 4)$ such that $f'(c) = 0$

$\therefore\ 2c - 4 = 0 \quad \Rightarrow \quad c = 2 \in (0, 4)$

Hence, $c = 2$ is the point in $(0, 4)$ on the given curve at which the tangent is parallel to the x-axis.

Verify mean value theorem for each of the functions given in Exercises 73 to 76.

Statement of Mean Value Theorem:

Let $f(x)$ be a real valued function defined on $[a, b]$ such that if

(*i*) $f(x)$ is continuous on $[a, b]$

(*ii*) $f(x)$ is differentiable on (a, b)

Then there is some $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Q73. $f(x) = \dfrac{1}{4x - 1}$ in [1, 4].

Sol. Given that: $f(x) = \dfrac{1}{4x - 1}$ in [1, 4].

(*i*) $f(x)$ is an algebraic function, so it is continuous in [1, 4].

(*ii*) $f'(x) = \dfrac{-4}{(4x-1)^2}$ which exists in (1, 4).

So, $f(x)$ is differentiable.

As the above conditions are satisfied then there must exist a point $c \in (1, 4)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\frac{-4}{(4c-1)^2} = \frac{\dfrac{1}{4(4)-1} - \dfrac{1}{4(1)-1}}{4-1}$$

$$\Rightarrow \quad \frac{-4}{(4c-1)^2} = \frac{\dfrac{1}{15} - \dfrac{1}{3}}{3} = \frac{1-5}{15 \times 3} = \frac{-4}{45} \quad = \frac{1}{(4c-1)^2} = \frac{1}{45}$$

$$\Rightarrow \quad (4c-1)^2 = 45$$

$$\Rightarrow \quad 4c - 1 = \pm 3\sqrt{5} \quad \Rightarrow 4c = +1 \pm 3\sqrt{5}$$

$$\Rightarrow \quad c = \frac{+1 \pm 3\sqrt{5}}{4}$$

$$\therefore \qquad c = \frac{+1 \pm 3\sqrt{5}}{4} \in (1, 4)$$

Hence, Mean Value Theorem is verified.

Q74. $f(x) = x^3 - 2x^2 - x + 3$ in $[0, 1]$.

Sol. Given that: $f(x) = x^3 - 2x^2 - x + 3$ in $[0, 1]$

(*i*) Being an algebraic polynomial, $f(x)$ is continuous in $[0, 1]$

(*ii*) $f'(x) = 3x^2 - 4x - 1$ which exists in $(0, 1)$.

So, $f(x)$ is differentiable.

As the above conditions are satisfied, then there must exist atleast one point $c \in (0, 1)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\Rightarrow \quad 3c^2 - 4c - 1 = \frac{[(1)^3 - 2(1)^2 - (1) + 3] - [0 - 0 - 0 + 3]}{1 - 0}$$

$$\Rightarrow \quad 3c^2 - 4c - 1 = \frac{(1 - 2 - 1 + 3) - (3)}{1}$$

$$\Rightarrow \quad 3c^2 - 4c - 1 = 1 - 3 \Rightarrow 3c^2 - 4c - 1 = -2$$

$$\Rightarrow \quad 3c^2 - 4c + 1 = 0 \Rightarrow 3c^2 - 3c - c + 1 = 0$$

$$\Rightarrow 3c(c - 1) - 1(c - 1) = 0 \Rightarrow (c - 1)(3c - 1) = 0$$

$$\Rightarrow \quad c - 1 = 0 \quad \therefore c = 1$$

$$3c - 1 = 0 \quad \therefore c = \frac{1}{3} \in (0, 1)$$

Hence, Mean Value Theorem is verified.

Q75. $f(x) = \sin x - \sin 2x$ in $[0, \pi]$.

Sol. Given that: $f(x) = \sin x - \sin 2x$ in $[0, \pi]$

(*i*) Since trigonometric functions are always continuous on their domain.

So, $f(x)$ is continuous on $[0, \pi]$.

(*ii*) $f'(x) = \cos x - 2\cos 2x$ which exists in $(0, \pi)$

So, $f(x)$ is differentiable on $(0, \pi)$.

Since the above conditions are satisfied, then there must exist atleast one point $c \in (0, \pi)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\cos c - 2\cos 2c = \frac{(\sin \pi - \sin 2\pi) - (\sin 0 - \sin 0)}{\pi - 0}$$

$$\Rightarrow \quad \cos c - 2(2\cos^2 c - 1) = 0 \Rightarrow \cos c - 4\cos^2 c + 2 = 0$$

$$\Rightarrow \quad 4\cos^2 c - \cos c - 2 = 0$$

$$\Rightarrow \quad \cos c = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \times 4 \times -2}}{2 \times 4}$$

$$\Rightarrow \quad \cos c = \frac{1 \pm \sqrt{1+32}}{8} = \frac{1 \pm \sqrt{33}}{8}$$

$$\Rightarrow \quad c = \cos^{-1}\left(\frac{1 \pm \sqrt{33}}{8}\right) \in (0, \pi).$$

Hence, Mean Value Theorem is verified.

Q76. $f(x) = \sqrt{25 - x^2}$ in [1, 5].

Sol. Given that: $f(x) = \sqrt{25 - x^2}$ in [1, 5]

(*i*) $f(x)$ is continuous if $25 - x^2 \geq 0 \Rightarrow -x^2 \geq -25$

$\Rightarrow x^2 \leq 25 \Rightarrow x \leq \pm 5 \Rightarrow -5 \leq x \leq 5$

So, $f(x)$ is continuous on [1, 5].

(*ii*) $f'(x) = \dfrac{1}{2\sqrt{25 - x^2}} \times (-2x) = \dfrac{-x}{\sqrt{25 - x^2}}$ which exists in (1, 5).

So, $f(x)$ is differentiable in [1, 5].

Since the above conditions are satisfied then there must exist atleast one point $c \in (1, 5)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\frac{-c}{\sqrt{25 - c^2}} = \frac{\sqrt{25 - 25} - \sqrt{25 - 1}}{5 - 1}$$

$$\Rightarrow \quad \frac{-c}{\sqrt{25 - c^2}} = \frac{0 - \sqrt{24}}{4}$$

$$\Rightarrow \quad \frac{c}{\sqrt{25 - c^2}} = \frac{2\sqrt{6}}{4} \quad \Rightarrow \quad \frac{c}{\sqrt{25 - c^2}} = \frac{\sqrt{6}}{2}$$

Squaring both sides

$$\frac{c^2}{25 - c^2} = \frac{6}{4} = \frac{3}{2}$$

$$\Rightarrow \quad 2c^2 = 75 - 3c^2 \Rightarrow 5c^2 = 75 \Rightarrow c^2 = 15$$

$$\therefore \quad c = \pm\sqrt{15} \in (1, 5)$$

Hence, Mean Value Theorem is verified.

Q77. Find a point on the curve $y = (x - 3)^2$, where the tangent is parallel to the chord joining the points (3, 0) and (4, 1).

Sol. Given that: $y = (x-3)^2$

Let $\quad f(x) = (x-3)^2$

(*i*) Being an algebraic polynomial, $f(x)$ is continuous at $x_1 = 3$ and $x_2 = 4$ i.e. in [3, 4].

(*ii*) $f'(x) = 2(x-3)$ which exists in (3, 4).

Hence, by mean value theorem, there must exist a point c on the curve at which the tangent is parallel to the chord joining the points (3, 0) and (4, 1).

$$\therefore \quad f'(c) = \frac{f(b)-f(a)}{b-a} \quad \text{where } b = 4 \text{ and } a = 3$$

$$\Rightarrow \quad 2(c-3) = \frac{(4-3)^2-(3-3)^2}{4-3}$$

$$\Rightarrow \quad 2c-6 = \frac{1-0}{1} = 1 \quad \Rightarrow 2c = 6+1 = 7$$

$$\therefore \quad c = \frac{7}{2}$$

If $x = \frac{7}{2} \quad \therefore y = \left(\frac{7}{2}-3\right)^2 = \frac{1}{4}$.

Hence, $\left(\frac{7}{2}, \frac{1}{4}\right)$ is the point on the curve at which the tangent is parallel to the chord joining the points (3, 0) and (4, 1).

Q78. Using Mean Value Theorem, prove that there is a point on the curve $y = 2x^2 - 5x + 3$ between the points A(1, 0) and B(2, 1), where tangent is parallel to the chord AB. Also, find that point.

Sol. Given that: $y = 2x^2 - 5x + 3$

Let $\quad f(x) = 2x^2 - 5x + 3$

(*i*) Being an algebraic polynomial, $f(x)$ is continuous in [1, 2].

(*ii*) $f'(x) = 4x - 5$ which exists in (1, 2).

As per the Mean Value Theorem, there must exist a point $c \in (1, 2)$ on the curve at which the tangent is parallel to the chord joining the points A(1, 0) and B(2, 1).

$$\text{So} \quad f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$4c-5 = \frac{(8-10+3)-(2-5+3)}{2-1}$$

$$\Rightarrow \quad 4c-5 = \frac{1-0}{1} = 1 \Rightarrow 4c = 1+5 \quad \Rightarrow \quad 4c = 6$$

$$\therefore \qquad c = \frac{6}{4} = \frac{3}{2}$$

$$\therefore \qquad y = 2\left(\frac{3}{2}\right)^2 - 5\left(\frac{3}{2}\right) + 3$$

$$= 2 \times \frac{9}{4} - \frac{15}{2} + 3 = \frac{9}{2} - \frac{15}{2} + 3 = \frac{9 - 15 + 6}{2} = 0$$

Hence, $\left(\frac{3}{2}, 0\right)$ is the point on the curve at which the tangent is parallel to the chord joining the points A(1, 0) and B(2, 1).

LONG ANSWER TYPE QUESTIONS

Q79. Find the values of p and q so that

$$f(x) = \begin{cases} x^2 + 3x + p, \text{ if } x \le 1 \\ qx + 2, \text{ if } x > 1 \end{cases} \text{ is differentiable at } x = 1.$$

Sol. Given that:

$$f(x) = \begin{cases} x^2 + 3x + p, \text{ if } x \le 1 \\ qx + 2, \text{ if } x > 1 \end{cases} \text{ at } x = 1.$$

$$\text{L.H.L. } f'(c) = \lim_{x \to 1^-} \frac{f(x) - f(c)}{x - c}$$

$$\Rightarrow \qquad f'(1) = \lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1}$$

$$= \lim_{x \to 1^-} \frac{(x^2 + 3x + p) - (1 + 3 + p)}{x - 1}$$

$$= \lim_{h \to 0} \frac{[(1-h)^2 + 3(1-h) + p] - [4 + p]}{1 - h - 1}$$

$$= \lim_{h \to 0} \frac{[1 + h^2 - 2h + 3 - 3h + p] - [4 + p]}{-h}$$

$$= \lim_{h \to 0} \frac{[h^2 - 5h + 4 + p] - [4 + p]}{-h}$$

$$= \lim_{h \to 0} \frac{h^2 - 5h + 4 + p - 4 - p}{-h}$$

$$= \lim_{h \to 0} \frac{h^2 - 5h}{-h} = \lim_{h \to 0} \frac{h[h - 5]}{-h} = 5$$

$$\text{R.H.L. } f'(1) = \lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1}$$

$$= \lim_{x\to 1^+} \frac{(qx+2)-(1+3+p)}{x-1}$$

$$= \lim_{h\to 0} \frac{[q(1+h)+2]-[4+p]}{1+h-1}$$

$$= \lim_{h\to 0} \frac{q+qh+2-4-p}{h} = \lim_{h\to 0} \frac{qh+q-2-p}{h}$$

For existing the limit

$q-2-p=0 \quad \Rightarrow \quad q-p=2$ ...(*i*)

$\Rightarrow \lim_{h\to 0} \frac{qh-0}{h} = q$

If L.H.L. $f'(1)$ = R.H.L. $f'(1)$ then $q = 5$.

Now putting the value of q in eqn. (*i*)

$$5-p=2 \quad \Rightarrow \quad p=3.$$

Hence, value of p is 3 and that of q is 5.

Q80. If $x^m.y^n = (x+y)^{m+n}$, prove that

(*i*) $\frac{dy}{dx} = \frac{y}{x}$ (*ii*) $\frac{d^2y}{dx^2} = 0$.

Sol. (*i*) Given that: $x^m.y^n = (x+y)^{m+n}$

Taking log on both sides

$$\log x^m.y^n = \log (x+y)^{m+n} \quad [\because \log xy = \log x + \log y]$$

$\Rightarrow \quad \log x^m + \log y^n = (m+n)\log(x+y)$

$\Rightarrow m\log x + n\log y = (m+n)\log(x+y)$

Differentiating both sides w.r.t. x

$$\Rightarrow \quad m.\frac{d}{dx}\log x + n.\frac{d}{dx}\log y = (m+n)\frac{d}{dx}\log(x+y)$$

$$\Rightarrow \quad m.\frac{1}{x} + n.\frac{1}{y}.\frac{dy}{dx} = (m+n).\frac{1}{x+y}\left(1+\frac{dy}{dx}\right)$$

$$\Rightarrow \quad \frac{m}{x} + \frac{n}{y}.\frac{dy}{dx} = \frac{m+n}{x+y}.\left(1+\frac{dy}{dx}\right)$$

$$\Rightarrow \quad \frac{m}{x} + \frac{n}{y}.\frac{dy}{dx} = \frac{m+n}{x+y} + \frac{m+n}{x+y}.\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{n}{y}.\frac{dy}{dx} - \frac{m+n}{x+y}.\frac{dy}{dx} = \frac{m+n}{x+y} - \frac{m}{x}$$

$$\Rightarrow \quad \left(\frac{n}{y}-\frac{m+n}{x+y}\right)\frac{dy}{dx} = \frac{m+n}{x+y}-\frac{m}{x}$$

$$\Rightarrow \left(\frac{nx+ny-my-ny}{y(x+y)}\right)\frac{dy}{dx} = \left(\frac{mx+nx-mx-my}{x(x+y)}\right)$$

$$\Rightarrow \quad \left(\frac{nx-my}{y(x+y)}\right)\frac{dy}{dx} = \left(\frac{nx-my}{x(x+y)}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{nx-my}{x(x+y)}\times\frac{y(x+y)}{nx-my}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x} \text{ Hence proved.}$$

(*ii*) Given that: $\frac{dy}{dx}=\frac{y}{x}$

Differentiating both sides w.r.t. x

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{y}{x}\right)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{x.\frac{dy}{dx} \quad y.1}{} = \frac{x.\frac{y}{x}-y}{x^2} \qquad \left[\because \frac{dy}{dx}=\frac{y}{x}\right]$$

$$= \frac{y-y}{x^2}=\frac{0}{x^2}=0$$

Hence, $\quad \frac{d^2y}{dx^2} = 0$. Hence, proved.

Q81. If $x = \sin t$ and $y = \sin pt$, prove that

$$(1-x^2)\frac{d^2y}{dx^2}-x.\frac{dy}{dx}+p^2y=0.$$

Sol. Given that: $x = \sin t$ and $y = \sin pt$

Differentiating both the parametric functions w.r.t. t

$$\frac{dx}{dt} = \cos t \quad \text{and} \quad \frac{dy}{dt} = \cos pt\,.\,p = p\cos pt$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}=\frac{p\cos pt}{\cos t}$$

$$\therefore \quad \frac{dy}{dx} = \frac{p\cos pt}{\cos t}$$

Again differentiating w.r.t. x,

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = p.\frac{d}{dx}\left(\frac{\cos pt}{\cos t}\right)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = p\left[\frac{\cos t.\frac{d}{dx}(\cos pt) - \cos pt.\frac{d}{dx}(\cos t)}{\cos^2 t}\right]$$

$$= p\left[\frac{\cos t(-\sin pt).p\frac{dt}{dx} - \cos pt(-\sin t).\frac{dt}{dx}}{\cos^2 t}\right]$$

$$= p\left[\frac{-p\cos t\sin pt + \cos pt\sin t}{\cos^2 t}\right]\frac{dt}{dx}$$

$$= p\left[\frac{-p\cos t\sin pt + \cos pt\sin t}{\cos^2 t}\right].\frac{1}{\cos t}$$

$$= p\left[\frac{-p\cos t\sin pt + \cos pt\sin t}{\cos^3 t}\right]$$

Now we have to prove that

$$(1-x^2).\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2y = 0$$

$$\text{L.H.S.} = (1-x^2)\left[p\left(\frac{-p\cos t\sin pt + \cos pt\sin t}{\cos^3 t}\right)\right] - x.p\frac{\cos pt}{\cos t} + p^2y$$

$$\Rightarrow (1-\sin^2 t)\left[p\left(\frac{-p\cos t\sin pt + \cos pt\sin t}{\cos^3 t}\right)\right] - \frac{p\sin t.\cos pt}{\cos t} + p^2.\sin pt$$

$$\Rightarrow \cos^2 t\left[\frac{-p^2\cos t\sin pt + p\cos pt\sin t}{\cos^3 t}\right] - \frac{p\sin t.\cos pt}{\cos t} + p^2.\sin pt$$

$$\Rightarrow \frac{-p^2\cos t\sin pt + p\cos pt\sin t}{\cos t} - \frac{p\sin t\cos pt}{\cos t} + p^2\sin pt$$

$$\Rightarrow \frac{-p^2\cos t\sin pt + p\cos pt\sin t - p\sin t\cos pt + p^2\sin pt\cos t}{\cos t}$$

$$\Rightarrow \frac{0}{\cos t} = 0 = \text{R.H.S.}$$

Hence, proved.

Q82. Find $\frac{dy}{dx}$, if $y = x^{\tan x} + \sqrt{\frac{x^2+1}{2}}$.

Sol. Given that: $y = x^{\tan x} + \sqrt{\frac{x^2+1}{2}}$

Let $u = x^{\tan x}$ and $v = \sqrt{\frac{x^2+1}{2}}$

$\therefore \quad y = u + v$

Differentiating both sides w.r.t. x

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad ...(i)$$

Now taking $u = x^{\tan x}$

Taking log on both sides $\log u = \log (x^{\tan x})$

$$\log u = \tan x \,.\, \log x$$

Differentiating both sides w.r.t. x

$$\frac{1}{u}.\frac{du}{dx} = \frac{d}{dx}(\tan x.\log x)$$

$$\Rightarrow \quad \frac{1}{u}.\frac{du}{dx} = \tan x.\frac{d}{dx}(\log x) + \log x.\frac{d}{dx}(\tan x)$$

$$\Rightarrow \quad \frac{1}{u}.\frac{du}{dx} = \tan x.\frac{1}{x} + \log x.\sec^2 x$$

$$\Rightarrow \quad \frac{du}{dx} = u\left[\frac{\tan x}{x} + \log x.\sec^2 x\right]$$

$$\therefore \quad \frac{du}{dx} = x^{\tan x}\left[\frac{\tan x}{x} + \log x \sec^2 x\right]$$

Taking $\quad v = \sqrt{\frac{x^2+1}{2}} \quad \Rightarrow v = \frac{1}{\sqrt{2}}\sqrt{x^2+1}$

Differentiating both sides w.r.t. x

$$\frac{dv}{dx} = \frac{1}{\sqrt{2}}.\frac{1}{2\sqrt{x^2+1}}.2x = \frac{x}{\sqrt{2}\sqrt{x^2+1}}$$

Putting the values of $\frac{du}{dx}$ and $\frac{dv}{dx}$ in eqn. (i)

$$\frac{dy}{dx} = x^{\tan x}\left[\log x \sec^2 x + \frac{\tan x}{x}\right] + \frac{x}{\sqrt{2}\sqrt{x^2+1}}$$

OBJECTIVE TYPE QUESTIONS

Choose the correct answers from the given four options in each of the Exercises 83 to 96.

Q83. If $f(x) = 2x$ and $g(x) = \frac{x^2}{2} + 1$, then which of the following can be a discontinuous function

(*a*) $f(x) + g(x)$ (*b*) $f(x) - g(x)$ (*c*) $f(x).g(x)$ (*d*) $\frac{g(x)}{f(x)}$

Sol. We know that the algebraic polynomials are continuous functions everywhere.

$\therefore f(x) + g(x)$ is continuous
$f(x) - g(x)$ is continuous
$f(x) \,.\, g(x)$ is continuous

[$\because$ Sum, difference and product of two continuous functions is also continuous]

$\frac{g(x)}{f(x)}$ is only continuous if $g(x) \neq 0$

$$\therefore \boxed{\frac{f(x)}{g(x)} = \frac{2x}{\frac{x^2}{2} + 1} = \frac{4x}{x^2 + 2}}$$

Here, $\frac{g(x)}{f(x)} = \frac{\frac{x^2}{2} + 1}{2x} = \frac{x^2 + 2}{4x}$ which is discontinuous at $x = 0$.

Hence, the correct option is (*d*).

Q84. The function $f(x) = \frac{4 - x^2}{4x - x^3}$ is

(*a*) discontinuous at only one point
(*b*) discontinuous at exactly two points
(*c*) discontinuous at exactly three points
(*d*) none of these

Sol. Given that: $f(x) = \frac{4 - x^2}{4x - x^3}$

For discontinuous function

$$4x - x^3 = 0$$
$$\Rightarrow \quad x(4 - x^2) = 0$$
$$\Rightarrow \quad x(2 - x)(2 + x) = 0$$
$$\Rightarrow \quad x = 0, x = -2, x = 2$$

Hence, the given function is discontinuous exactly at three points. Hence, the correct option is (*c*).

Q85. The set of points where the function f given by $f(x) = |2x - 1| \sin x$ is differentiable is (*a*) R (*b*) $R-\left\{\frac{1}{2}\right\}$ (*c*) $(0, \infty)$ (*d*) none of these

Sol. Given that: $f(x) = |2x-1| \sin x$

Clearly, $f(x)$ is not differentiable at $x = \frac{1}{2}$

$$\text{R.H.L.} = f'\left(\frac{1}{2}\right) = \lim_{h\to 0} \frac{f\left(\frac{1}{2}+h\right)-f\left(\frac{1}{2}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\left|2\left(\frac{1}{2}+h\right)-1\right| \sin\left(\frac{1}{2}+h\right)-0}{h}$$

$$= \lim_{h\to 0} \frac{|2h| \sin\left(\frac{1+2h}{2}\right)}{h} = 2\sin\left(\frac{1}{2}\right)$$

$$\text{Also L.H.L.} = f'\left(\frac{1}{2}\right) = \lim_{h\to 0} \frac{f\left(\frac{1}{2}-h\right)-f\left(\frac{1}{2}\right)}{-h}$$

$$= \lim_{h\to 0} \frac{\left|2\left(\frac{1}{2}-h\right)-1\right|\left[-\sin\left(\frac{1}{2}-h\right)\right]-0}{-h}$$

$$= \frac{|-2h|\left[-\sin\left(\frac{1}{2}-h\right)\right]}{-h} = -2\sin\left(\frac{1}{2}\right)$$

$$\therefore \text{R.H.L.} = f'\left(\frac{1}{2}\right) \neq \text{L.H.L. } f'\left(\frac{1}{2}\right)$$

So, the given function $f(x)$ is not differentiable at $x = \frac{1}{2}$.

$\therefore f(x)$ is differentiable in $R-\left\{\frac{1}{2}\right\}$.

Hence, the correct option is (*b*).

Q86. The function $f(x) = \cot x$ is discontinuous on the set

(*a*) $\{x = n\pi; n \in Z\}$ (*b*) $\{x = 2n\pi; n \in Z\}$

(*c*) $\left\{x = (2n+1)\frac{\pi}{2}; n \in Z\right\}$ (*d*) $\left\{x = \frac{n\pi}{2}; n \in Z\right\}$

Sol. Given that: $f(x) = \cot x$

$$\Rightarrow \quad f(x) = \frac{\cos x}{\sin x}$$

We know that $\sin x = 0$ if $f(x)$ is discontinuous.

$\therefore$ If $\quad \sin x = 0$

$\therefore \qquad x = n\pi, n \in n\pi.$

So, the given function $f(x)$ is discontinuous on the set $\{x = n\pi; n \in Z\}$.

Hence, the correct option is (*a*).

Q87. The function $f(x) = e^{|x|}$ is

(*a*) continuous everywhere but not differentiable at $x = 0$

(*b*) continuous and differentiable everywhere.

(*c*) Not continuous at $x = 0$ (*d*) None of these

Sol. Given that: $f(x) = e^{|x|}$

We know that modulus function is continuous but not differentiable in its domain.

Let $\quad g(x) = |x|$ and $t(x) = e^x$

$\therefore \quad f(x) = got(x) = g[t(x)] = e^{|x|}$

Since $g(x)$ and $t(x)$ both are continuous at $x = 0$ but $f(x)$ is not differentiable at $x = 0$.

Hence, the correct option is (*a*).

Q88. If $f(x) = x^2 \sin \frac{1}{x}$, where $x \neq 0$, then the value of the function f at $x = 0$, so that the function is continuous at $x = 0$, is

(*a*) 0 (*b*) -1 (*c*) 1 (*d*) none of these

Sol. Given that: $f(x) = x^2 \sin \frac{1}{x}$ where $x \neq 0$.

So, the value of the function f at $x = 0$, so that $f(x)$ is continuous is 0.

Hence, the correct option is (*a*).

Q89. If $f(x) = \begin{cases} mx + 1, \text{ if } x \leq \frac{\pi}{2} \\ \sin x + n, \text{ if } x > \frac{\pi}{2} \end{cases}$ is continuous at $x = \frac{\pi}{2}$, then

(*a*) $m = 1, n = 0$ (*b*) $m = \frac{n\pi}{2} + 1$ (*c*) $n = \frac{m\pi}{2}$ (*d*) $m = n = \frac{\pi}{2}$

Sol. Given that: $f(x) = \begin{cases} mx + 1, \text{ if } x \leq \frac{\pi}{2} \\ \sin x + n, \text{ if } x > \frac{\pi}{2} \end{cases}$ is continuous at $x = \frac{\pi}{2}$

$$\text{L.H.L.} = \lim_{x \to \frac{\pi}{2}^-} (mx + 1) = \lim_{h \to 0} \left[m\left(\frac{\pi}{2} - h\right) + 1 \right] = \frac{m\pi}{2} + 1$$

R.H.L. = $\lim_{x \to \frac{\pi}{2}^+} (\sin x + n) = \lim_{h \to 0}\left[\sin\left(\frac{\pi}{2} + h\right) + n\right]$

$$= \lim_{h \to 0} \cos h + n = 1 + n$$

When $f(x)$ is continuous at $x = \frac{\pi}{2}$

$\therefore$ L.H.L. = R.H.L.

$$\frac{m\pi}{2} + 1 = 1 + n \Rightarrow n = \frac{m\pi}{2}$$

Hence, the correct option is (*c*).

Q90. Let $f(x) = |\sin x|$. Then

(*a*) f is everywhere differentiable.

(*b*) f is everywhere continuous but not differentiable at $x = n\pi, n \in Z$.

(*c*) f is everywhere continuous but not differentiable at

$x = (2n + 1)\frac{\pi}{2}, n \in Z$.

(*d*) none of these

Sol. Given that: $f(x) = |\sin x|$

Let $g(x) = \sin x$ and $t(x) = |x|$

$\therefore \quad f(x) = tog(x) = t[g(x)] = t(\sin x) = |\sin x|$

where $g(x)$ and $t(x)$ both are continuous.

$\therefore$ $f(x) = got(x)$ is continuous but $t(x)$ is not differentiable at $x = 0$.

So, $f(x)$ is not continuous at $\sin x = 0 \Rightarrow x = n\pi, n \in Z$.

Hence, the correct option is (*b*).

Q91. If $y = \log\left(\frac{1 - x^2}{1 + x^2}\right)$, then $\frac{dy}{dx}$ is equal to

(*a*) $\frac{4x^3}{1 - x^4}$ (*b*) $\frac{-4x}{1 - x^4}$ (*c*) $\frac{1}{4 - x^4}$ (*d*) $\frac{-4x^3}{1 - x^4}$

Sol. Given that: $y = \log\left(\frac{1 - x^2}{1 + x^2}\right)$

$\Rightarrow y = \log(1 - x^2) - \log(1 + x^2)$ $\left[\because \log\frac{x}{y} = \log x - \log y\right]$

Differentiating both sides w.r.t. x

$$\frac{dy}{dx} = \frac{1}{1-x^2}.\frac{d}{dx}(1-x^2) - \frac{1}{1+x^2}.\frac{d}{dx}(1+x^2)$$

$$= \frac{-2x}{1-x^2} - \frac{2x}{1+x^2} = \frac{-2x-2x^3-2x+2x^3}{(1-x^2)(1+x^2)} = \frac{-4x}{1-x^4}$$

Hence, the correct option is (*b*).

Q92. If $y = \sqrt{\sin x + y}$, then $\frac{dy}{dx}$ is equal to

(*a*) $\frac{\cos x}{2y-1}$ (*b*) $\frac{\cos x}{1-2y}$ (*c*) $\frac{\sin x}{1-2y}$ (*d*) $\frac{\sin x}{2y-1}$

Sol. Given that: $y = \sqrt{\sin x + y}$

Differentiating both sides w.r.t. x

$$\frac{dy}{dx} = \frac{1}{2\sqrt{\sin x + y}}.\frac{d}{dx}(\sin x + y)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{\sin x + y}}.\left(\cos x + \frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2y}.\left[\cos x + \frac{dy}{dx}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos x}{2y} + \frac{1}{2y}.\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} - \frac{1}{2y}.\frac{dy}{dx} = \frac{\cos x}{2y}$$

$$\Rightarrow \left(1 - \frac{1}{2y}\right)\frac{dy}{dx} = \frac{\cos x}{2y} \Rightarrow \left(\frac{2y-1}{2y}\right)\frac{dy}{dx} = \frac{\cos x}{2y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos x}{2y} \times \frac{2y}{2y-1} \Rightarrow \frac{dy}{dx} = \frac{\cos x}{2y-1}$$

Hence, the correct option is (*a*).

Q93. The derivative of $\cos^{-1}(2x^2-1)$ w.r.t. $\cos^{-1} x$ is

(*a*) 2 (*b*) $\frac{-1}{2\sqrt{1-x^2}}$ (*c*) $\frac{2}{x}$ (*d*) $1-x^2$

Sol. Let $y = \cos^{-1}(2x^2-1)$ and $t = \cos^{-1} x$

Differentiating both the functions w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\cos^{-1}(2x^2-1) \text{ and } \frac{dt}{dx} = \frac{d}{dx}\cos^{-1} x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-1}{\sqrt{1-(2x^2-1)^2}}\cdot\frac{d}{dx}(2x^2-1) \text{ and } \frac{dt}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

$$= \frac{-1.4x}{\sqrt{1-(4x^4+1-4x^2)}} \text{ and } \frac{dt}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

$$= \frac{-4x}{\sqrt{1-4x^4-1+4x^2}} = \frac{-4x}{\sqrt{4x^2-4x^4}} = \frac{-4x}{2x\sqrt{1-x^2}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-2}{\sqrt{1-x^2}}$$

$$\text{Now } \frac{dy}{dt} = \frac{dy/dx}{dt/dx} = \frac{\frac{-2}{\sqrt{1-x^2}}}{\frac{-1}{\sqrt{1-x^2}}} = 2$$

Hence, the correct option is (*a*).

Q94. If $x = t^2$ and $y = t^3$, then $\dfrac{d^2y}{dx^2}$ is

(*a*) $\dfrac{3}{2}$ (*b*) $\dfrac{3}{4t}$ (*c*) $\dfrac{3}{2t}$ (*d*) $\dfrac{2}{3t}$

Sol. Given that $x = t^2$ and $y = t^3$

Differentiating both the parametric functions w.r.t. t

$$\frac{dx}{dt} = 2t \quad \text{and} \quad \frac{dy}{dt} = 3t^2$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2}{2t} = \frac{3}{2}t \Rightarrow \frac{dy}{dx} = \frac{3}{2}t$$

Now differentiating again w.r.t. x

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{3}{2}\cdot\frac{dt}{dx} \Rightarrow \frac{d^2y}{dx^2} = \frac{3}{2}\cdot\frac{1}{2t} = \frac{3}{4t}$$

Hence, the correct option is (*b*).

Q95. The value of 'c' in Rolle's Theorem for the function $f(x) = x^3 - 3x$ in the interval $[0, \sqrt{3}]$ is

(*a*) 1 (*b*) -1 (*c*) $\dfrac{3}{2}$ (*d*) $\dfrac{1}{3}$

Sol. Given that: $f(x) = x^3 - 3x$ in $[0, \sqrt{3}]$

We know that if $f(x) = x^3 - 3x$ satisfies the conditions of Rolle's

Theorem in $[0, \sqrt{3}]$, then

$$f'(c) = 0$$

$\Rightarrow \quad 3c^2 - 3 = 0 \Rightarrow 3c^2 = 3 \quad \Rightarrow c^2 = 1$

$\therefore \quad c = \pm 1 \quad \Rightarrow \quad 1 \in (0, \sqrt{3})$

Hence, the correct option is (*a*).

Q96. For the function $f(x) = x + \frac{1}{x}, x \in [1, 3]$, the value of '*c*' for mean value theorem is

(*a*) 1 (*b*) $\sqrt{3}$ (*c*) 2 (*d*) none of these

Sol. Given that: $f(x) = x + \frac{1}{x}, x \in [1, 3]$

We know that if $f(x) = x + \frac{1}{x}, x \in [1, 3]$ satisfies all the conditions of mean value theorem then

$$f'(c) = \frac{f(b) - f(a)}{b - a} \text{ where } a = 1 \text{ and } b = 3$$

$$\Rightarrow \quad 1 - \frac{1}{c^2} = \frac{\left(3 + \frac{1}{3}\right) - \left(1 + \frac{1}{1}\right)}{3 - 1}$$

$$\Rightarrow \quad 1 - \frac{1}{c^2} = \frac{\frac{10}{3} - 2}{2} \quad \Rightarrow 1 - \frac{1}{c^2} = \frac{4}{6} = \frac{2}{3} \quad \Rightarrow -\frac{1}{c^2} = \frac{2}{3} - 1$$

$$\Rightarrow \quad -\frac{1}{c^2} = -\frac{1}{3} \quad \Rightarrow \frac{1}{c^2} = \frac{1}{3} \quad \Rightarrow c = \pm\sqrt{3}.$$

Here $c = \sqrt{3} \in (1, 3)$.

Hence, the correct option is (*b*).

Fill in the blanks in each of the Exercises 97 to 101

Q97. An example of a function which is continuous everywhere but fails to be differentiable exactly at two points is ....

Sol. $|x| + |x - 1|$ is the function which is continuous everywhere but fails to be differentiable at $x = 0$ and $x = 1$.

We can have more such examples.

Q98. Derivative of x^2 w.r.t. x^3 is ....

Sol. Let $y = x^2$ and $t = x^3$

Differentiating both the parametric functions w.r.t. x

$$\frac{dy}{dx} = 2x \text{ and } \frac{dt}{dx} = 3x^2$$

$$\therefore \quad \frac{dy}{dt} = \frac{dy/dx}{dt/dx} = \frac{2x}{3x^2} = \frac{2}{3x}$$

So, the derivative of x^2 w.r.t. x^3 is $\frac{2}{3x}$

Q99. If $f(x) = |\cos x|$, then $f'\left(\frac{\pi}{4}\right) =$ ....

Sol. Given that: $f(x) = |\cos x|$

$$\Rightarrow \quad f(x) = \cos x \text{ if } x \in \left(0, \frac{\pi}{2}\right)$$

Differentiating both sides w.r.t. x, we get $f'(x) = -\sin x$

at $x = \frac{\pi}{4}$, $f'\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$

Q100. If $f(x) = |\cos x - \sin x|$, then $f'\left(\frac{\pi}{3}\right) =$ ....

Sol. Given that: $f(x) = |\cos x - \sin x|$

We know that $\sin x > \cos x$ if $x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

$\Rightarrow \cos x - \sin x < 0$

$$\therefore \quad f(x) = -(\cos x - \sin x)$$

$$f'(x) = -(-\sin x - \cos x) \quad \Rightarrow f'(x) = (\sin x + \cos x)$$

$$\therefore \quad f'\left(\frac{\pi}{3}\right) = \sin\frac{\pi}{3} + \cos\frac{\pi}{3} = \frac{\sqrt{3}}{2} + \frac{1}{2} = \frac{\sqrt{3}+1}{2}$$

Q101. For the curve $\sqrt{x} + \sqrt{y} = 1$, $\frac{dy}{dx}$ at $\left(\frac{1}{4}, \frac{1}{4}\right)$ is ....

Sol. Given that: $\sqrt{x} + \sqrt{y} = 1$

Differentiating both sides w.r.t. x

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}.\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{1}{\sqrt{y}}\frac{dy}{dx} = \frac{-1}{\sqrt{x}} \Rightarrow \frac{dy}{dx} = \frac{-\sqrt{y}}{\sqrt{x}}$$

$$\therefore \quad \frac{dy}{dx} \text{ at } \left(\frac{1}{4}, \frac{1}{4}\right) = -\frac{\sqrt{\frac{1}{4}}}{\sqrt{\frac{1}{4}}} = -1$$

State True or False for the statements in each of the Exercises 102 to 106.

Q102. Rolle's Theorem is applicable for the function $f(x) = |x - 1|$ in $[0, 2]$.

Sol. False. Given that $f(x) = |x - 1|$ in $[0, 2]$
We know that modulus function is not differentiable. So, it is false.

Q103. If f is continuous on its domain D, then $|f|$ is also continuous on D.

Sol. True. We know that modulus function is continuous function on its domain. So, it is true.

Q104. The composition of two continuous functions is a continuous function.

Sol. True. We know that the sum and difference of two or more functions is always continuous. So, it is true.

Q105. Trigonometric and inverse trigonometric functions are differentiable in their respective domain.

Sol. True.

Q106. If $f.g$ is continuous at $x = a$, then f and g are separately continuous at $x = a$.

Sol. False. Let us take an example: $f(x) = \sin x$ and $g(x) = \cot x$

$\therefore f(x).g(x) = \sin x.\cot x = \sin x.\dfrac{\cos x}{\sin x} = \cos x$ which is continuous at $x = 0$ but $\cot x$ is not continuous at $x = 0$.

❑❑❑

6 Application of Derivatives

6.3 EXERCISE

SHORT ANSWER TYPE QUESTIONS

Q1. A spherical ball of salt is dissolving in water in such a manner that the rate of decrease of the volume at any instant is proportional to the surface. Prove that the radius is decreasing at a constant rate.

Sol. Ball of salt is spherical

$\therefore$ Volume of ball, $V = \frac{4}{3}\pi r^3$, where r = radius of the ball

As per the question, $\frac{dV}{dt} \propto S$, where S = surface area of the ball

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) \propto 4\pi r^2 \qquad [\because \quad S = 4\pi r^2]$$

$$\Rightarrow \quad \frac{4}{3}\pi \cdot 3r^2 \cdot \frac{dr}{dt} \propto 4\pi r^2$$

$$\Rightarrow \quad 4\pi r^2 \cdot \frac{dr}{dt} = K \cdot 4\pi r^2 \quad (K = \text{Constant of proportionality})$$

$$\Rightarrow \quad \frac{dr}{dt} = K \cdot \frac{4\pi r^2}{4\pi r^2}$$

$$\therefore \quad \frac{dr}{dt} = K \cdot 1 = K$$

Hence, the radius of the ball is decreasing at constant rate.

Q2. If the area of a circle increases at a uniform rate, then prove that perimeter varies inversely as the radius.

Sol. We know that:

Area of circle, $A = \pi r^2$, where r = radius of the circle.

and perimeter $= 2\pi r$

As per the question,

$$\frac{dA}{dt} = K, \text{ where } K = \text{constant}$$

$$\Rightarrow \quad \frac{d}{dt}(\pi r^2) = K \quad \Rightarrow \quad \pi \cdot 2r \cdot \frac{dr}{dt} = K$$

$$\therefore \quad \frac{dr}{dt} = \frac{K}{2\pi r} \qquad \ldots(1)$$

Now Perimeter $\quad c = 2\pi r$

Differentiating both sides w.r.t., t, we get

$$\Rightarrow \quad \frac{dc}{dt} = \frac{d}{dt}(2\pi r) \quad \Rightarrow \quad \frac{dc}{dt} = 2\pi \cdot \frac{dr}{dt}$$

$$\Rightarrow \quad \frac{dc}{dt} = 2\pi \cdot \frac{K}{2\pi r} = \frac{K}{r} \qquad \text{[From (1)]}$$

$$\Rightarrow \quad \frac{dc}{dt} \propto \frac{1}{r}$$

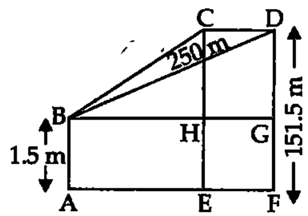
Hence, the perimeter of the circle varies inversely as the radius of the circle.

Q3. A kite is moving horizontally at a height of 151.5 metres. If the speed of the kite is 10 m/s, how fast is the string being let out; when the kite is 250 m away from the boy who is flying the kite? The height of the boy is 1.5 m.

Sol. Given that height of the kite (h) = 151.5 m

Speed of the kite(V) = 10 m/s

Let FD be the height of the kite and AB be the height of the boy.



Let AF = x m

$\therefore$ BG = AF = x m

and $\frac{dx}{dt} = 10$ m/s

From the figure, we get that

$$GD = DF - GF \Rightarrow DF - AB$$
$$= (151.5 - 1.5) \text{ m} = 150 \text{ m} \qquad [\because \; AB = GF]$$

Now in ΔBGD,

$$BG^2 + GD^2 = BD^2 \qquad \text{(By Pythagoras Theorem)}$$

$$\Rightarrow \; x^2 + (150)^2 = (250)^2$$

$$\Rightarrow \; x^2 + 22500 = 62500 \quad \Rightarrow x^2 = 62500 - 22500$$

$$\Rightarrow \; x^2 = 40000 \quad \Rightarrow x = 200 \text{ m}$$

Let initially the length of the string be y m

$\therefore$ In ΔBGD

$$BG^2 + GD^2 = BD^2 \quad \Rightarrow x^2 + (150)^2 = y^2$$

Differentiating both sides w.r.t., t, we get

$$\Rightarrow \quad 2x \cdot \frac{dx}{dt} + 0 = 2y \cdot \frac{dy}{dt} \qquad \left[\because \frac{dx}{dt} = 10 \text{ m/s}\right]$$

$$\Rightarrow \quad 2 \times 200 \times 10 = 2 \times 250 \times \frac{dy}{dt}$$

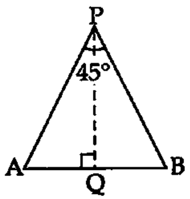
$$\therefore \quad \frac{dy}{dt} = \frac{2 \times 200 \times 10}{2 \times 250} = 8 \text{ m/s}$$

Hence, the rate of change of the length of the string is 8 m/s.

Q4. Two men A and B start with velocities V at the same time from the junction of two roads inclined at 45° to each other. If they travel by different roads, find the rate at which they are being separated.

Sol. Let P be any point at which the two roads are inclined at an angle of 45°.

Two men A and B are moving along the roads PA and PB respectively with the same speed 'V'.



Let A and B be their final positions such that

$AB = y$

$\angle APB = 45°$ and they move with the same speed.

$\therefore$ ΔAPB is an isosceles triangle. Draw $PQ \perp AB$

$AB = y \quad \therefore \quad AQ = \dfrac{y}{2}$ and $PA = PB = x$ (let)

$$\angle APQ = \angle BPQ = \frac{45}{2} = 22\frac{1}{2}°$$

[$\because$ In an isosceles Δ, the altitude drawn from the vertex, bisects the base]

Now in right ΔAPQ,

$$\sin 22\frac{1}{2}° = \frac{AQ}{AP}$$

$$\Rightarrow \quad \sin 22\frac{1}{2}° = \frac{\frac{y}{2}}{x} = \frac{y}{2x} \quad \Rightarrow y = 2x \cdot \sin 22\frac{1}{2}°$$

Differentiating both sides w.r.t, t, we get

$$\frac{dy}{dt} = 2 \cdot \frac{dx}{dt} \cdot \sin 22\frac{1}{2}°$$

$$= 2 \cdot V \cdot \frac{\sqrt{2-\sqrt{2}}}{2} \qquad \left[\because \sin 22\frac{1}{2}° = \frac{\sqrt{2-\sqrt{2}}}{2}\right]$$

$$= \sqrt{2-\sqrt{2}}\ V \text{ m/s}$$

Hence, the rate of their separation is $\sqrt{2-\sqrt{2}}$ V unit/s.

Q5. Find an angle θ, $0 < \theta < \dfrac{\pi}{2}$, which increases twice as fast as its sine.

Sol. As per the given condition,

$$\frac{d\theta}{dt} = 2\frac{d}{dt}(\sin\theta)$$

$$\Rightarrow \quad \frac{d\theta}{dt} = 2\cos\theta \cdot \frac{d\theta}{dt} \quad \Rightarrow 1 = 2\cos\theta$$

$$\therefore \quad \cos\theta = \frac{1}{2} \quad \Rightarrow \quad \cos\theta = \cos\frac{\pi}{3} \quad \Rightarrow \theta = \frac{\pi}{3}$$

Hence, the required angle is $\dfrac{\pi}{3}$.

Q6. Find the approximate value of $(1.999)^5$.

Sol. $(1.999)^5 = (2 - 0.001)^5$

Let $x = 2$ and $\Delta x = -0.001$

Let $y = x^5$

Differentiating both sides w.r.t, x, we get

$$\frac{dy}{dx} = 5x^4 = 5(2)^4 = 80$$

Now $\Delta y = \left(\frac{dy}{dx}\right) \cdot \Delta x = 80 \cdot (-0.001) = -0.080$

$\therefore$ $(1.999)^5 = y + \Delta y$

$$= x^5 - 0.080 = (2)^5 - 0.080 = 32 - 0.080 = 31.92$$

Hence, approximate value of $(1.999)^5$ is 31.92.

Q7. Find the approximate volume of metal in a hollow spherical shell whose internal and external radii are 3 cm and 3.0005 cm respectively.

Sol. Internal radius $r = 3$ cm

and external radius $R = r + \Delta r = 3.0005$ cm

$\therefore$ $\Delta r = 3.0005 - 3 = 0.0005$ cm

Let $y = r^3 \Rightarrow y + \Delta y = (r + \Delta r)^3 = R^3 = (3.0005)^3$ ...(*i*)

Differentiating both sides w.r.t., r, we get

$$\frac{dy}{dr} = 3r^2$$

$\therefore$ $\Delta y = \frac{dy}{dr} \times \Delta r = 3r^2 \times 0.0005$

$$= 3 \times (3)^2 \times 0.0005 = 27 \times 0.0005 = 0.0135$$

$\therefore$ $(3.0005)^3 = y + \Delta y$ [From eq. (*i*)]

$$= (3)^3 + 0.0135 = 27 + 0.0135 = 27.0135$$

Volume of the shell $= \frac{4}{3}\pi[R^3 - r^3]$

$$= \frac{4}{3}\pi[27.0135 - 27] = \frac{4}{3}\pi \times 0.0135$$

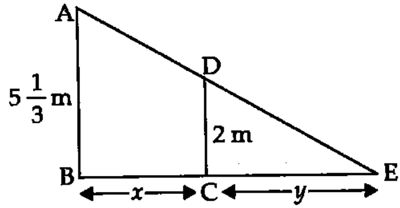
$$= 4\pi \times 0.005 = 4 \times 3.14 \times 0.0045 = 0.018\,\pi \text{ cm}^3$$

Hence, the approximate volume of the metal in the shell is 0.018π cm^3.

Q8. A man, 2m tall, walks at the rate of $1\frac{2}{3}$ m/s towards a street light which is $5\frac{1}{3}$ m above the ground. At what rate is the tip of his shadow moving? At what rate is the length of the shadow changing when he is $3\frac{1}{3}$ m from the base of the light?

Sol. Let AB is the height of street light post and CD is the height of the man such that

$AB = 5\frac{1}{3} = \frac{16}{3}$ m and $CD = 2$ m



Let $BC = x$ length (the distance of the man from the lamp post) and $CE = y$ is the length of the shadow of the man at any instant.

From the figure, we see that

$\Delta ABE \sim \Delta DCE$ [by AAA Similarity]

$\therefore$ Taking ratio of their corresponding sides, we get

$$\frac{AB}{CD} = \frac{BE}{CE} \Rightarrow \frac{AB}{CD} = \frac{BC+CE}{CE}$$

$$\Rightarrow \quad \frac{16/3}{2} = \frac{x+y}{y} \Rightarrow \frac{8}{3} = \frac{x+y}{y}$$

$$\Rightarrow \quad 8y = 3x + 3y \Rightarrow 8y - 3y = 3x \Rightarrow 5y = 3x$$

Differentiating both sides w.r.t, t, we get

$$5\frac{dy}{dt} = 3\cdot\frac{dx}{dt}$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{3}{5}\cdot\frac{dx}{dt} \Rightarrow \frac{dy}{dt} = \frac{3}{5}\cdot\left(-1\frac{2}{3}\right) = \frac{3}{5}\cdot\left(\frac{-5}{3}\right)$$

[$\because$ man is moving in opposite direction]

$= -1$ m/s

Hence, the length of shadow is decreasing at the rate of 1 m/s.

Now let $u = x + y$

(u = distance of the tip of shadow from the light post)

Differentiating both sides w.r.t. t, we get

$$\frac{du}{dt} = \frac{dx}{dt} + \frac{dy}{dt}$$

$$= \left(-1\frac{2}{3} - 1\right) = -\left(\frac{5}{3} + 1\right) = -\frac{8}{3} = -2\frac{2}{3} \text{ m/s}$$

Hence, the tip of the shadow is moving at the rate of $2\frac{2}{3}$ m/s towards the light post and the length of shadow decreasing at the rate of 1 m/s.

Q9. A swimming pool is to be drained for cleaning. If L represents the number of litres of water in the pool t seconds after the pool has been plugged off to drain and $L = 200(10 - t)^2$. How fast is the water running out at the end of 5 seconds? What is the average rate at which the water flows out during the first 5 seconds?

Sol. Given that $L = 200(10 - t)^2$

where L represents the number of litres of water in the pool.

Differentiating both sides w.r.t, t, we get

$$\frac{dL}{dt} = 200 \times 2(10 - t)(-1) = -400(10 - t)$$

But the rate at which the water is running out

$$= -\frac{dL}{dt} = 400(10 - t) \qquad ...(1)$$

Rate at which the water is running after 5 seconds

$$= 400 \times (10 - 5) = 2000 \text{ L/s (final rate)}$$

For initial rate put $t = 0$

$$= 400(10 - 0) = 4000 \text{ L/s}$$

The average rate at which the water is running out

$$= \frac{\text{Initial rate} + \text{Final rate}}{2} = \frac{4000 + 2000}{2} = \frac{6000}{2} = 3000 \text{ L/s}$$

Hence, the required rate = 3000 L/s.

Q10. The volume of a cube increases at a constant rate. Prove that the increase in its surface area varies inversely as the length of the side.

Sol. Let x be the length of the cube

$\therefore$ Volume of the cube $V = x^3$ ...(1)

Given that $\frac{dV}{dt} = K$

Differentiating Eq. (1) w.r.t. t, we get

$$\frac{dV}{dt} = 3x^2 \cdot \frac{dx}{dt} = K \text{ (constant)}$$

$$\therefore \quad \frac{dx}{dt} = \frac{K}{3x^2}$$

Now surface area of the cube, $S = 6x^2$

Differentiating both sides w.r.t. t, we get

$$\frac{ds}{dt} = 6 \cdot 2 \cdot x \cdot \frac{dx}{dt} = 12x \cdot \frac{K}{3x^2}$$

$$\Rightarrow \quad \frac{ds}{dt} = \frac{4K}{x} \quad \Rightarrow \quad \frac{ds}{dt} \propto \frac{1}{x} \qquad (4K = \text{constant})$$

Hence, the surface area of the cube varies inversely as the length of the side.

Q11. x and y are the sides of two squares such that $y = x - x^2$. Find the rate of change of the area of second square with respect to the area of first square.

Sol. Let area of the first square $A_1 = x^2$

and area of the second square $A_2 = y^2$

Now $A_1 = x^2$ and $A_2 = y^2 = (x - x^2)^2$

Differentiating both A_1 and A_2 w.r.t. t, we get

$$\frac{dA_1}{dt} = 2x \cdot \frac{dx}{dt} \text{ and } \frac{dA_2}{dt} = 2(x - x^2)(1 - 2x) \cdot \frac{dx}{dt}$$

$$\therefore \quad \frac{dA_2}{dA_1} = \frac{\frac{dA_2}{dt}}{\frac{dA_1}{dt}} = \frac{2(x - x^2)(1 - 2x) \cdot \frac{dx}{dt}}{2x \cdot \frac{dx}{dt}}$$

$$= \frac{x(1 - x)(1 - 2x)}{x} = (1 - x)(1 - 2x)$$

$$= 1 - 2x - x + 2x^2 = 2x^2 - 3x + 1$$

Hence, the rate of change of area of the second square with respect to first is $2x^2 - 3x + 1$.

Q12. Find the condition that the curves $2x = y^2$ and $2xy = k$ intersect orthogonally.

Sol. The two circles intersect orthogonally if the angle between the tangents drawn to the two circles at the point of their intersection is 90°.

Equation of the two circles are given as

$$2x = y^2 \quad \text{...(i)}$$

$$\text{and} \quad 2xy = k \quad \text{...(ii)}$$

Differentiating eq. (*i*) and (*ii*) w.r.t. x, we get

$$2.1 = 2y \cdot \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{y} \Rightarrow m_1 = \frac{1}{y}$$

(m_1 = slope of the tangent)

$$\Rightarrow \quad 2xy = k$$

$$\Rightarrow \quad 2\left[x \cdot \frac{dy}{dx} + y \cdot 1\right] = 0$$

$$\therefore \quad \frac{dy}{dx} = -\frac{y}{x} \Rightarrow m_2 = -\frac{y}{x}$$

[m_2 = slope of the other tangent]

If the two tangents are perpendicular to each other,

then $\quad m_1 \times m_2 = -1$

$$\Rightarrow \quad \frac{1}{y} \times \left(-\frac{y}{x}\right) = -1 \Rightarrow \frac{1}{x} = 1 \Rightarrow x = 1$$

Now solving $\quad 2x = y^2 \quad$ [From (*i*)]

and $\quad 2xy = k \quad$ [From (*ii*)]

From eq. (*ii*) $\quad y = \dfrac{k}{2x}$

Putting the value of y in eq. (*i*)

$$2x = \left(\frac{k}{2x}\right)^2 \Rightarrow 2x = \frac{k^2}{4x^2}$$

$\Rightarrow \quad 8x^3 = k^2 \Rightarrow 8(1)^3 = k^2 \Rightarrow 8 = k^2$

Hence, the required condition is $k^2 = 8$.

Q13. Prove that the curves $xy = 4$ and $x^2 + y^2 = 8$ touch each other.

Sol. Given circles are $\quad xy = 4 \quad$ ...(*i*)

and $\quad x^2 + y^2 = 8 \quad$ ...(*ii*)

Differentiating eq. (*i*) w.r.t., x

$$x \cdot \frac{dy}{dx} + y \cdot 1 = 0$$

$\Rightarrow \quad \dfrac{dy}{dx} = -\dfrac{y}{x} \Rightarrow m_1 = -\dfrac{y}{x} \quad$ ...(*iii*)

where, m_1 is the slope of the tangent to the curve.

Differentiating eq. (*ii*) w.r.t. x

$$2x + 2y \cdot \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y} \Rightarrow m_2 = -\frac{x}{y}$$

where, m_2 is the slope of the tangent to the circle.

To find the point of contact of the two circles

$$m_1 = m_2 \Rightarrow -\frac{y}{x} = -\frac{x}{y} \Rightarrow x^2 = y^2$$

Putting the value of y^2 in eq. (*ii*)

$$x^2 + x^2 = 8 \Rightarrow 2x^2 = 8 \Rightarrow x^2 = 4$$

$\therefore \quad x = \pm 2$

$\because \quad x^2 = y^2 \Rightarrow y = \pm 2$

$\therefore$ The point of contact of the two circles are (2, 2) and (–2, 2).

Q14. Find the coordinates of the point on the curve $\sqrt{x} + \sqrt{y} = 4$ at which tangent is equally inclined to the axes.

Sol. Equation of curve is given by $\sqrt{x} + \sqrt{y} = 4$

Let (x_1, y_1) be the required point on the curve

$\therefore \quad \sqrt{x_1} + \sqrt{y_1} = 4$

Differentiating both sides w.r.t. x_1, we get

$$\frac{d}{dx_1}\sqrt{x_1} + \frac{d}{dx_1}\sqrt{y_1} = \frac{d}{dx_1}(4)$$

$$\Rightarrow \quad \frac{1}{2\sqrt{x_1}} + \frac{1}{2\sqrt{y_1}} \cdot \frac{dy_1}{dx_1} = 0$$

$$\Rightarrow \quad \frac{1}{\sqrt{x_1}} + \frac{1}{\sqrt{y_1}} \cdot \frac{dy_1}{dx_1} = 0 \Rightarrow \frac{dy_1}{dx_1} = -\frac{\sqrt{y_1}}{\sqrt{x_1}} \qquad \text{...}(i)$$

Since the tangent to the given curve at (x_1, y_1) is equally inclined to the axes.

$\therefore$ Slope of the tangent $\frac{dy_1}{dx_1} = \pm\tan\frac{\pi}{4} = \pm 1$

So, from eq. (*i*) we get

$$-\frac{\sqrt{y_1}}{\sqrt{x_1}} = \pm 1$$

Squaring both sides, we get

$$\frac{y_1}{x_1} = 1 \quad \Rightarrow \quad y_1 = x_1$$

Putting the value of y_1 in the given equation of the curve.

$$\sqrt{x_1} + \sqrt{y_1} = 4$$

$$\Rightarrow \quad \sqrt{x_1} + \sqrt{x_1} = 4 \Rightarrow 2\sqrt{x_1} = 4 \Rightarrow \sqrt{x_1} = 2 \Rightarrow x_1 = 4$$

Since $\quad y_1 = x_1$

$\therefore \quad y_1 = 4$

Hence, the required point is (4, 4).

Q15. Find the angle of intersection of the curves $y = 4 - x^2$ and $y = x^2$.

Sol. We know that the angle of intersection of two curves is equal to the angle between the tangents drawn to the curves at their point of intersection.

The given curves are $\quad y = 4 - x^2$ ... (*i*) and $\quad y = x^2 \qquad$ ...(*ii*)

Differentiating eq. (*i*) and (*ii*) with respect to *x*, we have

$$\frac{dy}{dx} = -2x \quad \Rightarrow \quad m_1 = -2x$$

m_1 is the slope of the tangent to the curve (*i*).

and $\quad \frac{dy}{dx} = 2x \quad \Rightarrow \quad m_2 = 2x$

m_2 is the slope of the tangent to the curve (*ii*).

So, $\quad m_1 = -2x \quad$ and $\quad m_2 = 2x$

Now solving eq. (*i*) and (*ii*) we get

$$\Rightarrow \quad 4 - x^2 = x^2 \Rightarrow 2x^2 = 4 \Rightarrow x^2 = 2 \Rightarrow x = \pm\sqrt{2}$$

So, $\quad m_1 = -2x = -2\sqrt{2}$ and $m_2 = 2x = 2\sqrt{2}$

Let θ be the angle of intersection of two curves

$\therefore \quad \tan\theta = \left|\dfrac{m_2 - m_1}{1 + m_1 m_2}\right|$

$$= \left|\frac{2\sqrt{2} + 2\sqrt{2}}{1 - (2\sqrt{2})(2\sqrt{2})}\right| = \left|\frac{4\sqrt{2}}{1 - 8}\right| = \left|\frac{4\sqrt{2}}{-7}\right| = \frac{4\sqrt{2}}{7}$$

$\therefore \quad \theta = \tan^{-1}\left(\dfrac{4\sqrt{2}}{7}\right)$

Hence, the required angle is $\tan^{-1}\left(\dfrac{4\sqrt{2}}{7}\right)$.

Q16. Prove that the curves $y^2 = 4x$ and $x^2 + y^2 - 6x + 1 = 0$ touch each other at the point (1, 2).

Sol. Given that the equation of the two curves are $y^2 = 4x$ ...(*i*)

and $x^2 + y^2 - 6x + 1 = 0$ ...(*ii*)

Differentiating (*i*) w.r.t. x, we get $2y\dfrac{dy}{dx} = 4 \Rightarrow \dfrac{dy}{dx} = \dfrac{2}{y}$

Slope of the tangent at (1, 2), $m_1 = \dfrac{2}{2} = 1$

Differentiating (*ii*) w.r.t. $x \Rightarrow 2x + 2y \cdot \dfrac{dy}{dx} - 6 = 0$

$$\Rightarrow \quad 2y \cdot \frac{dy}{dx} = 6 - 2x \Rightarrow \frac{dy}{dx} = \frac{6 - 2x}{2y}$$

$\therefore$ Slope of the tangent at the same point (1, 2)

$$\Rightarrow \quad m_2 = \frac{6 - 2 \times 1}{2 \times 2} = \frac{4}{4} = 1$$

We see that $m_1 = m_2 = 1$ at the point (1, 2).

Hence, the given circles touch each other at the same point (1, 2).

Q17. Find the equation of the normal lines to the curve $3x^2 - y^2 = 8$ which are parallel to the line $x + 3y = 4$.

Sol. We have equation of the curve $3x^2 - y^2 = 8$

Differentiating both sides w.r.t. x, we get

$$\Rightarrow \quad 6x - 2y \cdot \frac{dy}{dx} = 0 \Rightarrow -2y\frac{dy}{dx} = -6x \Rightarrow \frac{dy}{dx} = \frac{3x}{y}$$

Slope of the tangent to the given curve $= \dfrac{3x}{y}$

$\therefore$ Slope of the normal to the curve $= -\dfrac{1}{\dfrac{3x}{y}} = -\dfrac{y}{3x}$.

Now differentiating both sides the given line $x + 3y = 4$

$$\Rightarrow \quad 1+3\cdot\frac{dy}{dx}=0 \quad \Rightarrow \quad \frac{dy}{dx}=-\frac{1}{3}$$

Since the normal to the curve is parallel to the given line $x+3y=4$.

$$\therefore \quad -\frac{y}{3x}=-\frac{1}{3} \quad \Rightarrow \quad y=x$$

Putting the value of y in $3x^2-y^2=8$, we get

$$3x^2-x^2=8 \Rightarrow 2x^2=8 \Rightarrow x^2=4 \Rightarrow x=\pm 2$$

$$\therefore \quad y=\pm 2$$

$\therefore$ The points on the curve are $(2, 2)$ and $(-2, -2)$.

Now equation of the normal to the curve at $(2, 2)$ is

$$y-2=-\frac{1}{3}(x-2)$$

$$\Rightarrow \quad 3y-6=-x+2 \Rightarrow x+3y=8$$

at $(-2,-2)$ $\quad y+2=-\frac{1}{3}(x+2)$

$$\Rightarrow \quad 3y+6=-x-2 \Rightarrow x+3y=-8$$

Hence, the required equations are $x+3y=8$ and $x+3y=-8$ or $x+3y=\pm 8$.

Q18. At what points on the curve $x^2+y^2-2x-4y+1=0$, the tangents are parallel to the y-axis?

Sol. Given that the equation of the curve is

$$x^2+y^2-2x-4y+1=0 \qquad \text{...}(i)$$

Differentiating both sides w.r.t. x, we have

$$2x+2y\cdot\frac{dy}{dx}-2-4\cdot\frac{dy}{dx}=0$$

$$\Rightarrow \quad (2y-4)\frac{dy}{dx}=2-2x \Rightarrow \frac{dy}{dx}=\frac{2-2x}{2y-4} \qquad \text{...}(ii)$$

Since the tangent to the curve is parallel to the y-axis.

$$\therefore \quad \text{Slope } \frac{dy}{dx}=\tan\frac{\pi}{2}=\infty=\frac{1}{0}$$

So, from eq. (*ii*) we get

$$\frac{2-2x}{2y-4}=\frac{1}{0} \Rightarrow 2y-4=0 \Rightarrow y=2$$

Now putting the value of y in eq. (*i*), we get

$$\Rightarrow \quad x^2+(2)^2-2x-8+1=0$$

$$\Rightarrow \quad x^2-2x+4-8+1=0$$

$$\Rightarrow \quad x^2-2x-3=0 \Rightarrow x^2-3x+x-3=0$$

$$\Rightarrow \quad x(x-3)+1(x-3)=0 \Rightarrow (x-3)(x+1)=0$$

$$\Rightarrow \quad x=-1 \text{ or } 3$$

Hence, the required points are $(-1, 2)$ and $(3, 2)$.

Q19. Show that the line $\frac{x}{a}+\frac{y}{b}=1$, touches the curve $y=b\cdot e^{-x/a}$ at the point where the curve intersects the axis of y.

Sol. Given that $y=b\cdot e^{-x/a}$, the equation of curve

and $\frac{x}{a}+\frac{y}{b}=1$, the equation of line.

Let the coordinates of the point where the curve intersects the y-axis be $(0, y_1)$

Now differentiating $y=b\cdot e^{-x/a}$ both sides w.r.t. x, we get

$$\frac{dy}{dx}=b\cdot e^{-x/a}\left(-\frac{1}{a}\right)=-\frac{b}{a}\cdot e^{-x/a}$$

So, the slope of the tangent, $m_1=-\frac{b}{a}e^{-x/a}$.

Differentiating $\frac{x}{a}+\frac{y}{b}=1$ both sides w.r.t. x, we get

$$\frac{1}{a}+\frac{1}{b}\cdot\frac{dy}{dx}=0$$

So, the slope of the line, $m_2=\frac{-b}{a}$.

If the line touches the curve, then $m_1=m_2$

$$\Rightarrow \quad \frac{-b}{a}\cdot e^{-x/a}=\frac{-b}{a} \quad\Rightarrow\quad e^{-x/a}=1$$

$$\Rightarrow \quad \frac{-x}{a}\log e=\log 1 \qquad \text{(Taking log on both sides)}$$

$$\Rightarrow \quad \frac{-x}{a}=0 \quad\Rightarrow\quad x=0$$

Putting $x=0$ in equation $y=b\cdot e^{-x/a}$

$$\Rightarrow \quad y=b\cdot e^0=b$$

Hence, the given equation of curve intersect at $(0, b)$ i.e. on y-axis.

Q20. Show that $f(x)=2x+\cot^{-1}x+\log(\sqrt{1+x^2}-x)$ is increasing in R.

Sol. Given that $f(x)=2x+\cot^{-1}x+\log(\sqrt{1+x^2}-x)$

Differentiating both sides w.r.t. x, we get

$$f'(x)=2-\frac{1}{1+x^2}+\frac{1}{\sqrt{1+x^2}-x}\times\frac{d}{dx}\left(\sqrt{1+x^2}-x\right)$$

$$=2-\frac{1}{1+x^2}+\frac{\left(\frac{1}{2\sqrt{1+x^2}}\times(2x-1)\right)}{\sqrt{1+x^2}-x}$$

$$= 2 - \frac{1}{1+x^2} + \frac{x - \sqrt{1+x^2}}{\sqrt{1+x^2}\left(\sqrt{1+x^2} - x\right)}$$

$$= 2 - \frac{1}{1+x^2} - \frac{\left(\sqrt{1+x^2} - x\right)}{\sqrt{1+x^2}\left(\sqrt{1+x^2} - x\right)}$$

$$= 2 - \frac{1}{1+x^2} - \frac{1}{\sqrt{1+x^2}}$$

For increasing function, $f'(x) \geq 0$

$$\therefore \quad 2 - \frac{1}{1+x^2} - \frac{1}{\sqrt{1+x^2}} \geq 0$$

$$\Rightarrow \frac{2(1+x^2) - 1 + \sqrt{1+x^2}}{(1+x^2)} \geq 0 \quad \Rightarrow 2 + 2x^2 - 1 + \sqrt{1+x^2} \geq 0$$

$$\Rightarrow \quad 2x^2 + 1 + \sqrt{1+x^2} \geq 0 \quad \Rightarrow 2x^2 + 1 \geq -\sqrt{1+x^2}$$

Squaring both sides, we get $4x^4 + 1 + 4x^2 \geq 1 + x^2$

$\Rightarrow \quad 4x^4 + 4x^2 - x^2 \geq 0 \quad \Rightarrow 4x^4 + 3x^2 \geq 0 \quad \Rightarrow x^2(4x^2 + 3) \geq 0$

which is true for any value of $x \in R$.

Hence, the given function is an increasing function over R.

Q21. Show that for $a \geq 1$, $f(x) = \sqrt{3} \sin x - \cos x - 2ax + b$ is decreasing in R.

Sol. Given that: $f(x) = \sqrt{3} \sin x - \cos x - 2ax + b, a \geq 1$

Differentiating both sides w.r.t. x, we get

$$f'(x) = \sqrt{3} \cos x + \sin x - 2a$$

For decreasing function, $f'(x) < 0$

$$\therefore \quad \sqrt{3} \cos x + \sin x - 2a < 0$$

$$\Rightarrow \quad 2\left(\frac{\sqrt{3}}{2} \cos x + \frac{1}{2} \sin x\right) - 2a < 0$$

$$\Rightarrow \quad \frac{\sqrt{3}}{2} \cos x + \frac{1}{2} \sin x - a < 0$$

$$\Rightarrow \quad \left(\cos\frac{\pi}{6} \cos x + \sin\frac{\pi}{6} \sin x\right) - a < 0$$

$$\Rightarrow \quad \cos\left(x - \frac{\pi}{6}\right) - a < 0$$

Since $\cos x \in [-1, 1]$ and $a \geq 1$

$$\therefore \quad f'(x) < 0$$

Hence, the given function is decreasing in R.

Q22. Show that $f(x) = \tan^{-1}(\sin x + \cos x)$ is an increasing function in $\left(0, \frac{\pi}{4}\right)$.

Sol. Given that: $f(x) = \tan^{-1}(\sin x + \cos x)$ in $\left(0, \frac{\pi}{4}\right)$

Differentiating both sides w.r.t. x, we get

$$f'(x) = \frac{1}{1+(\sin x+\cos x)^2} \cdot \frac{d}{dx}(\sin x + \cos x)$$

$$\Rightarrow \quad f'(x) = \frac{1 \times (\cos x - \sin x)}{1+(\sin x + \cos x)^2}$$

$$\Rightarrow \quad f'(x) = \frac{\cos x - \sin x}{1+\sin^2 x + \cos^2 x + 2\sin x \cos x}$$

$$\Rightarrow \quad f'(x) = \frac{\cos x - \sin x}{1+1+2\sin x \cos x} \quad \Rightarrow f'(x) = \frac{\cos x - \sin x}{2+2\sin x \cos x}$$

For an increasing function $f'(x) \geq 0$

$$\therefore \quad \frac{\cos x - \sin x}{2+2\sin x \cos x} \geq 0$$

$$\Rightarrow \quad \cos x - \sin x \geq 0 \quad \left[\because \ (2+\sin 2x) \geq 0 \text{ in } \left(0, \frac{\pi}{4}\right)\right]$$

$$\Rightarrow \quad \cos x \geq \sin x, \text{ which is true for } \left(0, \frac{\pi}{4}\right)$$

Hence, the given function $f(x)$ is an increasing function in $\left(0, \frac{\pi}{4}\right)$.

Q23. At what point, the slope of the curve $y = -x^3 + 3x^2 + 9x - 27$ is maximum ? Also find the maximum slope.

Sol. Given that: $y = -x^3 + 3x^2 + 9x - 27$

Differentiating both sides w.r.t. x, we get $\frac{dy}{dx} = -3x^2 + 6x + 9$

Let slope of the cuve $\frac{dy}{dx} = Z$

$$\therefore \quad z = -3x^2 + 6x + 9$$

Differentiating both sides w.r.t. x, we get $\frac{dz}{dx} = -6x + 6$

For local maxima and local minima, $\frac{dz}{dx} = 0$

$$\therefore \quad -6x + 6 = 0 \quad \Rightarrow \quad x = 1$$

$$\Rightarrow \quad \frac{d^2z}{dx^2} = -6 < 0 \quad \text{Maxima}$$

Put $x = 1$ in equation of the curve $y = (-1)^3 + 3(1)^2 + 9(1) - 27$

$$= -1 + 3 + 9 - 27 = -16$$

Maximum slope $= -3(1)^2 + 6(1) + 9 = 12$

Hence, $(1, -16)$ is the point at which the slope of the given curve is maximum and maximum slope = 12.

Q24. Prove that $f(x) = \sin x + \sqrt{3}\cos x$ has maximum value at $x = \frac{\pi}{6}$.

Sol. We have: $f(x) = \sin x + \sqrt{3}\cos x = 2\left(\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x\right)$

$$= 2\left(\cos\frac{\pi}{3}\sin x + \sin\frac{\pi}{3}\cos x\right) = 2\sin\left(x + \frac{\pi}{3}\right)$$

$$f'(x) = 2\cos\left(x + \frac{\pi}{3}\right);\ f''(x) = -2\sin\left(x + \frac{\pi}{3}\right)$$

$$f''(x)_{x=\frac{\pi}{6}} = -2\sin\left(\frac{\pi}{6} + \frac{\pi}{3}\right)$$

$$= -2\sin\frac{\pi}{2} = -2.1 = -2 < 0 \text{ (Maxima)}$$

$$= -2 \times \frac{\sqrt{3}}{2} = -\sqrt{3} < 0 \text{ (Maxima)}$$

Maximum value of the function at $x = \frac{\pi}{6}$ is

$$\sin\frac{\pi}{6} + \sqrt{3}\cos\frac{\pi}{6} = \frac{1}{2} + \sqrt{3}\cdot\frac{\sqrt{3}}{2} = 2$$

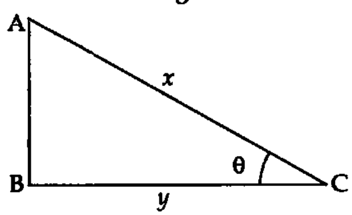
Hence, the given function has maximum value at $x = \frac{\pi}{6}$ and the maximum value is 2.

LONG ANSWER TYPE QUESTIONS

Q25. If the sum of the lengths of the hypotenuse and a side of a right angled triangle is given, show that the area of the triangle is maximum when the angle between them is $\frac{\pi}{3}$.

Sol. Let ΔABC be the right angled triangle in which $\angle B = 90°$

Let $AC = x$, $BC = y$



$$\therefore \quad AB = \sqrt{x^2 - y^2}$$

$$\angle ACB = \theta$$

Let $\quad Z = x + y \quad$ (given)

Now area of ΔABC, $A = \frac{1}{2} \times AB \times BC$

$\Rightarrow\ A = \frac{1}{2} y \cdot \sqrt{x^2 - y^2} \quad \Rightarrow A = \frac{1}{2} y \cdot \sqrt{(Z-y)^2 - y^2}$

Squaring both sides, we get

$$A^2 = \frac{1}{4} y^2 \left[(Z-y)^2 - y^2\right] \Rightarrow A^2 = \frac{1}{4} y^2 [Z^2 + y^2 - 2Zy - y^2]$$

$$\Rightarrow\ P = \frac{1}{4} y^2 [Z^2 - 2Zy] \Rightarrow P = \frac{1}{4}[y^2Z^2 - 2Zy^3] \qquad [A^2 = P]$$

Differentiating both sides w.r.t. y we get

$$\frac{dP}{dy} = \frac{1}{4}[2yZ^2 - 6Zy^2] \qquad ...(i)$$

For local maxima and local minima, $\frac{dP}{dy} = 0$

$\therefore\ \frac{1}{4}(2yZ^2 - 6Zy^2) = 0$

$\Rightarrow\ \frac{2yZ}{4}(Z - 3y) = 0 \quad \Rightarrow \quad yZ(Z-3y) = 0$

$\Rightarrow\ yZ \neq 0 \qquad (\because\ y \neq 0 \text{ and } Z \neq 0)$

$\therefore\ Z - 3y = 0$

$\Rightarrow\ y = \frac{Z}{3} \quad \Rightarrow y = \frac{x+y}{3} \qquad (\because\ Z = x + y)$

$\Rightarrow\ 3y = x + y \quad \Rightarrow 3y - y = x \quad \Rightarrow 2y = x$

$\Rightarrow\ \frac{y}{x} = \frac{1}{2} \quad \Rightarrow \cos\theta = \frac{1}{2}$

$\therefore\ \theta = \frac{\pi}{3}$

Differentiating eq. (*i*) w.r.t. y, we have $\frac{d^2P}{dy^2} = \frac{1}{4}[2Z^2 - 12Zy]$

$$\frac{d^2P}{dy^2} \text{ at } y = \frac{Z}{3} = \frac{1}{4}\left[2Z^2 - 12Z \cdot \frac{Z}{3}\right]$$

$$= \frac{1}{4}[2Z^2 - 4Z^2] = \frac{-Z^2}{2} < 0 \text{ Maxima}$$

Hence, the area of the given triangle is maximum when the angle between its hypotenuse and a side is $\frac{\pi}{3}$.

Q26. Find the points of local maxima, local minima and the points of inflection of the function $f(x) = x^5 - 5x^4 + 5x^3 - 1$. Also find the corresponding local maximum and local minimum values.

Sol. We have $\quad f(x) = x^5 - 5x^4 + 5x^3 - 1$

$\Rightarrow \quad f'(x) = 5x^4 - 20x^3 + 15x^2$

For local maxima and local minima, $f'(x) = 0$

$\Rightarrow \quad 5x^4 - 20x^3 + 15x^2 = 0 \quad \Rightarrow 5x^2(x^2 - 4x + 3) = 0$

$\Rightarrow \quad 5x^2(x^2 - 3x - x + 3) = 0 \quad \Rightarrow x^2(x - 3)(x - 1) = 0$

$\therefore \quad x = 0, x = 1$ and $x = 3$

Now $\qquad f''(x) = 20x^3 - 60x^2 + 30x$

$\Rightarrow \qquad f''(x)_{\text{at } x=0} = 20(0)^3 - 60(0)^2 + 30(0) = 0$ which is neither maxima nor minima.

$\therefore \quad f(x)$ has the point of inflection at $x = 0$

$$f''(x)_{\text{at } x=1} = 20(1)^3 - 60(1)^2 + 30(1)$$
$$= 20 - 60 + 30 = -10 < 0 \text{ Maxima}$$
$$f''(x)_{\text{at } x=3} = 20(3)^3 - 60(3)^2 + 30(3)$$
$$= 540 - 540 + 90 = 90 > 0 \text{ Minima}$$

The maximum value of the function at $x = 1$

$$f(x) = (1)^5 - 5(1)^4 + 5(1)^3 - 1$$
$$= 1 - 5 + 5 - 1 = 0$$

The minimum value at $x = 3$ is

$$f(x) = (3)^5 - 5(3)^4 + 5(3)^3 - 1$$
$$= 243 - 405 + 135 - 1 = 378 - 406 = -28$$

Hence, the function has its maxima at $x = 1$ and the maximum value = 0 and it has minimum value at x = 3 and its minimum value is – 28.

$x = 0$ is the point of inflection.

Q27. A telephone company in a town has 500 subscribers on its list and collects fixed charges of ₹ 300 per subscriber per year. The company proposes to increase the annual subscription and it is believed that for every increase of ₹ 1.00, one subscriber will discontinue the service. Find what increase will bring maximum profit?

Sol. Let us consider that the company increases the annual subscription by ₹ x.

So, x is the number of subscribers who discontinue the services.

$\therefore \quad$ Total revenue, $\quad R(x) = (500 - x)(300 + x)$

$$= 150000 + 500x - 300x - x^2$$
$$= -x^2 + 200x + 150000$$

Differentiating both sides w.r.t. x, we get $\quad R'(x) = -2x + 200$

For local maxima and local minima, $R'(x) = 0$

$$-2x + 200 = 0 \quad \Rightarrow \quad x = 100$$
$$R''(x) = -2 < 0 \text{ Maxima}$$

So, $R(x)$ is maximum at $x = 100$

Hence, in order to get maximum profit, the company should increase its annual subscription by ₹ 100.

Q28. If the straight line $x \cos \alpha + y \sin \alpha = p$ touches the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then prove that $a^2 \cos^2 \alpha + b^2 \sin^2 \alpha = p^2$.

Sol. The given curve is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ...(*i*)

and the straight line $x \cos \alpha + y \sin \alpha = p$ ...(*ii*)

Differentiating eq. (*i*) w.r.t. x, we get

$$\frac{1}{a^2} \cdot 2x + \frac{1}{b^2} \cdot 2y \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{x}{a^2} + \frac{y}{b^2}\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{b^2}{a^2} \cdot \frac{x}{y}$$

So the slope of the curve $= \frac{-b^2}{a^2} \cdot \frac{x}{y}$

Now differentiating eq. (*ii*) w.r.t. x, we have

$$\cos \alpha + \sin \alpha \cdot \frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx} = \frac{-\cos \alpha}{\sin \alpha} = -\cot \alpha$$

So, the slope of the straight line $= -\cot \alpha$

If the line is the tangent to the curve, then

$$\frac{-b^2}{a^2} \cdot \frac{x}{y} = -\cot \alpha \quad \Rightarrow \quad \frac{x}{y} = \frac{a^2}{b^2} \cdot \cot \alpha \quad \Rightarrow \quad x = \frac{a^2}{b^2} \cot \alpha \cdot y$$

Now from eq. (*ii*) we have $x \cos \alpha + y \sin \alpha = p$

$$\Rightarrow \quad \frac{a^2}{b^2} \cdot \cot \alpha \cdot y \cdot \cos \alpha + y \sin \alpha = p$$

$$\Rightarrow \quad a^2 \cot \alpha \cdot \cos \alpha\, y + b^2 \sin \alpha\, y = b^2 p$$

$$\Rightarrow \quad a^2 \frac{\cos \alpha}{\sin \alpha} \cdot \cos \alpha\, y + b^2 \sin \alpha\, y = b^2 p$$

$$\Rightarrow \quad a^2 \cos^2 \alpha\, y + b^2 \sin^2 \alpha\, y = b^2 \sin \alpha\, p$$

$$\Rightarrow \quad a^2 \cos^2 \alpha + b^2 \sin^2 \alpha = \frac{b^2}{y} \cdot \sin \alpha \cdot p$$

$$\Rightarrow \quad a^2 \cos^2 \alpha + b^2 \sin^2 \alpha = p \cdot p \qquad \left[\because \frac{b^2}{y} \sin \alpha = p\right]$$

Hence, $\quad a^2 \cos^2 \alpha + b^2 \sin^2 \alpha = p^2$

Alternate method:

We know that $y = mx + c$ will touch the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ if } c^2 = a^2m^2 + b^2$$

Here equation of straight line is $x \cos \alpha + y \sin \alpha = p$ and that of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$x \cos \alpha + y \sin \alpha = p$$

$$\Rightarrow \quad y \sin \alpha = -x \cos \alpha + p$$

$$\Rightarrow \quad y = -x \frac{\cos \alpha}{\sin \alpha} + \frac{p}{\sin \alpha} \quad \Rightarrow y = -x \cot \alpha + \frac{p}{\sin \alpha}$$

Comparing with $y = mx + c$, we get

$$m = -\cot \alpha \quad \text{and} \quad c = \frac{p}{\sin \alpha}$$

So, according to the condition, we get $c^2 = a^2m^2 + b^2$

$$\frac{p^2}{\sin^2 \alpha} = a^2(-\cot \alpha)^2 + b^2$$

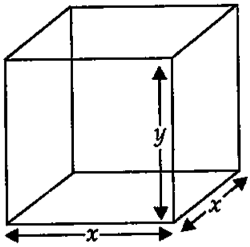
$$\Rightarrow \quad \frac{p^2}{\sin^2 \alpha} = \frac{a^2 \cos^2 \alpha}{\sin^2 \alpha} + b^2 \quad \Rightarrow p^2 = a^2 \cos^2 \alpha + b^2 \sin^2 \alpha$$

Hence, $a^2 \cos^2 \alpha + b^2 \sin^2 \alpha = p^2$ Hence proved.

Q29. An open box with square base is to be made of a given quantity of card board of area c^2. Show that the maximum volume of the box is $\frac{c^3}{6\sqrt{3}}$ cubic units.

Sol. Let x be the length of the side of the square base of the cubical open box and y be its height.

$\therefore$ Surface area of the open box



$$c^2 = x^2 + 4xy \quad \Rightarrow y = \frac{c^2 - x^2}{4x} \quad \text{...}(i)$$

Now volume of the box, $V = x \times x \times y$

$$\Rightarrow \quad V = x^2y$$

$$\Rightarrow \quad V = x^2\left(\frac{c^2 - x^2}{4x}\right)$$

$$\Rightarrow \quad V = \frac{1}{4}(c^2x - x^3)$$

Differentiating both sides w.r.t. x, we get

$$\frac{dV}{dx} = \frac{1}{4}(c^2 - 3x^2) \qquad ...(ii)$$

For local maxima and local minima, $\frac{dV}{dx} = 0$

$$\therefore \quad \frac{1}{4}(c^2 - 3x^2) = 0 \Rightarrow c^2 - 3x^2 = 0$$

$$\Rightarrow \qquad x^2 = \frac{c^2}{3}$$

$$\therefore \qquad x = \sqrt{\frac{c^2}{3}} = \frac{c}{\sqrt{3}}$$

Now again differentiating eq. (*ii*) w.r.t. x, we get

$$\frac{d^2V}{dx^2} = \frac{1}{4}(-6x) = \frac{-3}{2}\cdot\frac{c}{\sqrt{3}} < 0 \quad \text{(maxima)}$$

Volume of the cubical box (V) = $x^2 y$

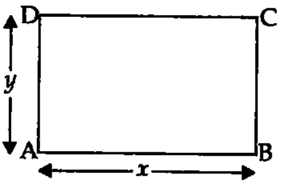
$$= x^2\left(\frac{c^2 - x^2}{4x}\right) = \frac{c}{\sqrt{3}}\left[\frac{c^2 - \frac{c^2}{3}}{4}\right] = \frac{c}{\sqrt{3}} \times \frac{2c^2}{3 \times 4} = \frac{c^3}{6\sqrt{3}}$$

Hence, the maximum volume of the open box is

$\frac{c^3}{6\sqrt{3}}$ cubic units.

Q30. Find the dimensions of the rectangle of perimeter 36 cm which will sweep out a volume as large as possible, when revolved about one of its sides. Also find the maximum volume.

Sol. Let x and y be the length and breadth of a given rectangle ABCD as per question, the rectangle be revolved about side AD which will make a cylinder with radius x and height y.



$$\therefore \quad \text{Volume of the cylinder} \quad V = \pi r^2 h$$

$$\Rightarrow \qquad V = \pi x^2 y \qquad ...(i)$$

Now perimeter of rectangle $P = 2(x + y) \Rightarrow 36 = 2(x + y)$

$$\Rightarrow \qquad x + y = 18 \Rightarrow y = 18 - x \qquad ...(ii)$$

Putting the value of y in eq. (*i*) we get

$$V = \pi x^2(18 - x)$$

$$\Rightarrow \qquad V = \pi(18x^2 - x^3)$$

Differentiating both sides w.r.t. x, we get

$$\frac{dV}{dx} = \pi(36x - 3x^2) \qquad ...(iii)$$

For local maxima and local minima $\frac{dV}{dx} = 0$

$\therefore \quad \pi(36x - 3x^2) = 0 \quad \Rightarrow 36x - 3x^2 = 0$

$\Rightarrow \quad 3x(12 - x) = 0$

$\Rightarrow \quad x \neq 0 \quad \therefore \quad 12 - x = 0 \Rightarrow x = 12$

From eq. (*ii*) $y = 18 - 12 = 6$

Differentiating eq. (*iii*) w.r.t. x, we get $\frac{d^2V}{dx^2} = \pi(36 - 6x)$

at $x = 12$ $\quad \frac{d^2V}{dx^2} = \pi(36 - 6 \times 12)$

$= \pi(36 - 72) = -36\pi < 0$ maxima

Now volume of the cylinder so formed $= \pi x^2 y$

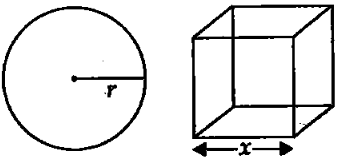
$= \pi \times (12)^2 \times 6 = \pi \times 144 \times 6 = 864\pi \text{ cm}^3$

Hence, the required dimensions are 12 cm and 6 cm and the maximum volume is 864π cm^3.

Q31. If the sum of the surface areas of cube and a sphere is constant, what is the ratio of an edge of the cube to the diameter of the sphere, when the sum of their volumes is minimum?

Sol. Let x be the edge of the cube and r be the radius of the sphere.

Surface area of cube $= 6x^2$



and surface area of the sphere $= 4\pi r^2$

$\therefore \quad 6x^2 + 4\pi r^2 = K(\text{constant}) \quad \Rightarrow \quad r = \sqrt{\frac{K - 6x^2}{4\pi}}$ ...(*i*)

Volume of the cube $= x^3$ and the volume of sphere $= \frac{4}{3}\pi r^3$

$\therefore$ Sum of their volumes (V) = Volume of cube + Volume of sphere

$$\Rightarrow \quad V = x^3 + \frac{4}{3}\pi r^3$$

$$\Rightarrow \quad V = x^3 + \frac{4}{3}\pi \times \left(\frac{K - 6x^2}{4\pi}\right)^{3/2}$$

Differentiating both sides w.r.t. x, we get

$$\frac{dV}{dx} = 3x^2 + \frac{4\pi}{3} \times \frac{3}{2}(K - 6x^2)^{1/2}(-12x) \times \frac{1}{(4\pi)^{3/2}}$$

$$= 3x^2 + \frac{2\pi}{(4\pi)^{3/2}} \times (-12x)(K - 6x^2)^{1/2}$$

$$= 3x^2 + \frac{1}{4\pi^{1/2}} \times (-12x)(K - 6x^2)^{1/2}$$

$$\therefore \quad \frac{dV}{dx} = 3x^2 - \frac{3x}{\sqrt{\pi}}(K - 6x^2)^{1/2} \qquad \text{...}(ii)$$

For local maxima and local minima, $\frac{dV}{dx} = 0$

$$\therefore \quad 3x^2 - \frac{3x}{\sqrt{\pi}}(K - 6x^2)^{1/2} = 0$$

$$\Rightarrow \quad 3x\left[x - \frac{(K - 6x^2)^{1/2}}{\sqrt{\pi}}\right] = 0$$

$$x \neq 0 \quad \therefore \quad x - \frac{(K - 6x^2)^{1/2}}{\sqrt{\pi}} = 0$$

$$\Rightarrow \quad x = \frac{(K - 6x^2)^{1/2}}{\sqrt{\pi}}$$

Squaring both sides, we get

$$x^2 = \frac{K - 6x^2}{\pi} \quad \Rightarrow \pi x^2 = K - 6x^2$$

$$\Rightarrow \quad \pi x^2 + 6x^2 = K \quad \Rightarrow x^2(\pi + 6) = K \quad \Rightarrow x^2 = \frac{K}{\pi + 6}$$

$$\therefore \quad x = \sqrt{\frac{K}{\pi + 6}}$$

Now putting the value of K in eq. (i), we get

$$6x^2 + 4\pi r^2 = x^2(\pi + 6)$$

$$\Rightarrow \quad 6x^2 + 4\pi r^2 = \pi x^2 + 6x^2 \quad \Rightarrow 4\pi r^2 = \pi x^2 \quad \Rightarrow 4r^2 = x^2$$

$$\therefore \quad 2r = x$$

$$\therefore \quad x : 2r = 1 : 1$$

Now differentiating eq. (ii) w.r.t x, we have

$$\frac{d^2V}{dx^2} = 6x - \frac{3}{\sqrt{\pi}} \frac{d}{dx}[x(K - 6x^2)^{1/2}]$$

$$= 6x - \frac{3}{\sqrt{\pi}}\left[x \cdot \frac{1}{2\sqrt{K - 6x^2}} \times (-12x) + (K - 6x^2)^{1/2} \cdot 1\right]$$

$$= 6x - \frac{3}{\sqrt{\pi}}\left[\frac{-6x^2}{\sqrt{K - 6x^2}} + \sqrt{K - 6x^2}\right]$$

$$= 6x - \frac{3}{\sqrt{\pi}}\left[\frac{-6x^2 + K - 6x^2}{\sqrt{K - 6x^2}}\right] = 6x + \frac{3}{\sqrt{\pi}}\left[\frac{12x^2 - K}{\sqrt{K - 6x^2}}\right]$$

Put $\quad x = \sqrt{\frac{K}{\pi + 6}} = 6\sqrt{\frac{K}{\pi + 6}} + \frac{3}{\sqrt{\pi}}\left[\frac{\frac{12K}{\pi + 6} - K}{\sqrt{K - \frac{6K}{\pi + 6}}}\right]$

$$= 6\sqrt{\frac{K}{\pi + 6}} + \frac{3}{\sqrt{\pi}}\left[\frac{12K - \pi K - 6K}{\sqrt{\frac{\pi K + 6K - 6K}{\pi + 6}}}\right]$$

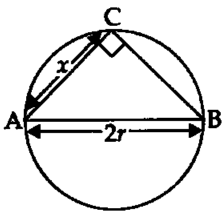
$$= 6\sqrt{\frac{K}{\pi + 6}} + \frac{3}{\sqrt{\pi}}\left[\frac{6K - \pi K}{\sqrt{\frac{\pi K}{\pi + 6}}}\right]$$

$$= 6\sqrt{\frac{K}{\pi + 6}} + \frac{3}{\pi\sqrt{K}}[(6K - \pi K)\sqrt{\pi + 6}] > 0$$

So it is minima.

Hence, the required ratio is $1:1$ when the combined volume is minimum.

Q32. AB is a diameter of a circle and C is any point on the circle. Show that the area of ΔABC is maximum, when it is isosceles.



Sol. Let AB be the diameter and C be any point on the circle with radius r.

$\angle ACB = 90°$ [angle in the semi circle is 90°]

Let $\quad AC = x$

$\therefore \quad BC = \sqrt{AB^2 - AC^2}$

$\Rightarrow \quad BC = \sqrt{(2r)^2 - x^2} \quad \Rightarrow BC = \sqrt{4r^2 - x^2} \qquad \ldots(i)$

Now area of ΔABC, $A = \frac{1}{2} \times AC \times BC$

$\Rightarrow \qquad A = \frac{1}{2}x \cdot \sqrt{4r^2 - x^2}$

Squaring both sides, we get

$$A^2 = \frac{1}{4}x^2(4r^2 - x^2)$$

Let $A^2 = Z$

$\therefore \qquad Z = \frac{1}{4}x^2(4r^2 - x^2) \quad \Rightarrow Z = \frac{1}{4}(4x^2r^2 - x^4)$

Differentiating both sides w.r.t. x, we get

$$\frac{dZ}{dx} = \frac{1}{4}[8xr^2 - 4x^3] \quad ...(ii)$$

For local maxima and local minima $\frac{dZ}{dx} = 0$

$\therefore \quad \frac{1}{4}[8xr^2 - 4x^3] = 0 \Rightarrow x[2r^2 - x^2] = 0$

$x \neq 0 \quad \therefore \quad 2r^2 - x^2 = 0$

$\Rightarrow \quad x^2 = 2r^2 \Rightarrow x = \sqrt{2}r = AC$

Now from eq. (*i*) we have

$$BC = \sqrt{4r^2 - 2r^2} \Rightarrow BC = \sqrt{2r^2} \Rightarrow BC = \sqrt{2}r$$

So $\quad AC = BC$

Hence, ΔABC is an isosceles triangle.

Differentiating eq. (*ii*) w.r.t. x, we get $\frac{d^2Z}{dx^2} = \frac{1}{4}[8r^2 - 12x^2]$

Put $x = \sqrt{2}r$

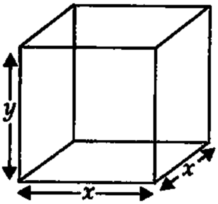
$$\therefore \quad \frac{d^2Z}{dx^2} = \frac{1}{4}[8r^2 - 12 \times 2r^2] = \frac{1}{4}[8r^2 - 24r^2]$$

$$= \frac{1}{4} \times (-16r^2) = -4r^2 < 0 \quad \text{maxima}$$

Hence, the area of ΔABC is maximum when it is an isosceles triangle.

Q33. A metal box with a square base and vertical sides is to contain 1024 cm³. The material for the top and botttom costs ₹ 5/cm² and the material for the sides costs ₹ 2.50/cm². Find the least cost of the box.

Sol. Let x be the side of the square base and y be the length of the vertical sides.



Area of the base and bottom $= 2x^2$ cm²

$\therefore$ Cost of the material required $=$ ₹ $5 \times 2x^2$

$=$ ₹ $10x^2$

Area of the 4 sides $= 4xy$ cm²

$\therefore$ Cost of the material for the four sides

$=$ ₹ $2.50 \times 4xy =$ ₹ $10xy$

Total cost $\quad C = 10x^2 + 10xy \quad ...(i)$

New volume of the box $= x \times x \times y$

$\Rightarrow \quad 1024 = x^2y$

$$\therefore \quad y = \frac{1024}{x^2} \quad ...(ii)$$

Putting the value of y in eq. (i) we get

$$C = 10x^2 + 10x \times \frac{1024}{x^2} \quad \Rightarrow C = 10x^2 + \frac{10240}{x}$$

Differentiating both sides w.r.t. x, we get

$$\frac{dC}{dx} = 20x - \frac{10240}{x^2} \qquad ...(iii)$$

For local maxima and local minima $\frac{dC}{dx} = 0$

$$20 - \frac{10240}{x^2} = 0$$

$$\Rightarrow \quad 20x^3 - 10240 = 0 \quad \Rightarrow x^3 = 512 \quad \Rightarrow \quad x = 8 \text{ cm}$$

Now from eq. (ii)

$$y = \frac{10240}{(8)^2} = \frac{10240}{64} = 16 \text{ cm}$$

$\therefore$ Cost of material used $C = 10x^2 + 10xy$

$$= 10 \times 8 \times 8 + 10 \times 8 \times 16 = 640 + 1280 = 1920$$

Now differentiating eq. (iii) we get

$$\frac{d^2C}{dx^2} = 20 + \frac{20480}{x^3}$$

Put $x = 8$

$$= 20 + \frac{20480}{(8)^3} = 20 + \frac{20480}{512} = 20 + 40 = 60 > 0 \text{ minima}$$

Hence, the required cost is ₹ 1920 which is the minimum.

Q34. The sum of the surface areas of a rectangular parallelopiped with sides x, $2x$ and $\frac{x}{3}$ and a sphere is given to be constant. Prove that the sum of their volumes is minimum, if x is equal to three times the radius of the sphere. Also find the minimum value of the sum of their volumes.

Sol. Let 'r' be the radius of the sphere.

$\therefore$ Surface area of the sphere $= 4\pi r^2$

Volume of the sphere $= \frac{4}{3}\pi r^3$

The sides of the parallelopiped are x, $2x$ and $\frac{x}{3}$

$$\therefore \quad \text{Its surface area} = 2\left[x \times 2x + 2x \times \frac{x}{3} + x \times \frac{x}{3}\right]$$

$$= 2\left[2x^2 + \frac{2x^2}{3} + \frac{x^2}{3}\right] = 2[2x^2 + x^2]$$

$$= 2[3x^2] = 6x^2$$

Volume of the parallelopiped $= x \times 2x \times \frac{x}{3} = \frac{2}{3}x^3$

As per the conditions of the question,

Surface area of the parallelopiped

+ Surface area of the sphere = constant

$\Rightarrow \quad 6x^2 + 4\pi r^2 = \text{K (constant)} \quad \Rightarrow 4\pi r^2 = \text{K} - 6x^2$

$$\therefore \quad r^2 = \frac{\text{K} - 6x^2}{4\pi} \qquad ...(i)$$

Now let V = Volume of parallelopiped + Volume of the sphere

$$\Rightarrow \quad \text{V} = \frac{2}{3}x^3 + \frac{4}{3}\pi r^3$$

$$\Rightarrow \quad \text{V} = \frac{2}{3}x^3 + \frac{4}{3}\pi\left[\frac{\text{K} - 6x^2}{4\pi}\right]^{3/2} \qquad \text{[from eq. (}i\text{)]}$$

$$\Rightarrow \quad \text{V} = \frac{2}{3}x^3 + \frac{4}{3}\pi \times \frac{1}{(4)^{3/2}\pi^{3/2}}[\text{K} - 6x^2]^{3/2}$$

$$\Rightarrow \quad \text{V} = \frac{2}{3}x^3 + \frac{4}{3}\pi \times \frac{1}{8 \times \pi^{3/2}}[\text{K} - 6x^2]^{3/2}$$

$$\Rightarrow \quad = \frac{2}{3}x^3 + \frac{1}{6\sqrt{\pi}}[\text{K} - 6x^2]^{3/2}$$

Differentiating both sides w.r.t. x, we have

$$\frac{d\text{V}}{dx} = \frac{2}{3}\cdot 3x^2 + \frac{1}{6\sqrt{\pi}}\left[\frac{3}{2}(K - 6x^2)^{1/2}(-12x)\right]$$

$$= 2x^2 + \frac{1}{6\sqrt{\pi}} \times \frac{3}{2} \times (-12x)(\text{K} - 6x^2)^{1/2}$$

$$= 2x^2 - \frac{3x}{\sqrt{\pi}}[\text{K} - 6x^2)^{1/2}$$

For local maxima and local minima, we have $\frac{d\text{V}}{dx} = 0$

$$\therefore \quad 2x^2 - \frac{3x}{\sqrt{\pi}}(\text{K} - 6x^2)^{1/2} = 0$$

$$\Rightarrow \quad 2\sqrt{\pi}x^2 - 3x(\text{K} - 6x^2)^{1/2} = 0$$

$$\Rightarrow \quad x[2\sqrt{\pi}x - 3(\text{K} - 6x^2)^{1/2}] = 0$$

Here $x \neq 0$ and $2\sqrt{\pi}x - 3(\text{K} - 6x^2)^{1/2} = 0$

$$\Rightarrow \quad 2\sqrt{\pi}x = 3(\text{K} - 6x^2)^{1/2}$$

Squaring both sides, we get

$$4\pi x^2 = 9(\text{K} - 6x^2) \Rightarrow 4\pi x^2 = 9\text{K} - 54x^2$$

$$\Rightarrow \quad 4\pi x^2 + 54x^2 = 9K$$

$$\Rightarrow \quad K = \frac{4\pi x^2 + 54x^2}{9} \qquad ...(ii)$$

$$\Rightarrow \quad 2x^2(2\pi + 27) = 9K$$

$$\therefore \quad x^2 = \frac{9K}{2(2\pi + 27)} = 3\sqrt{\frac{K}{4\pi + 54}}$$

Now from eq. (*i*) we have

$$r^2 = \frac{K - 6x^2}{4\pi}$$

$$\Rightarrow \quad r^2 = \frac{\dfrac{4\pi x^2 + 54x^2}{9} - 6x^2}{4\pi}$$

$$\Rightarrow \quad r^2 = \frac{4\pi x^2 + 54x^2 - 54x^2}{9 \times 4\pi} = \frac{4\pi x^2}{9 \times 4\pi}$$

$$\Rightarrow \quad r^2 = \frac{x^2}{9} \quad \Rightarrow \quad r = \frac{x}{3} \quad \therefore\ x = 3r$$

Now we have $\quad \dfrac{dV}{dx} = 2x^2 - \dfrac{3x}{\sqrt{\pi}}(K - 6x^2)^{1/2}$

Differentiating both sides w.r.t. x, we get

$$\frac{d^2V}{dx^2} = 4x - \frac{3}{\sqrt{\pi}}\left[x \cdot \frac{d}{dx}(K - 6x^2)^{1/2} + (K - 6x^2)^{1/2} \cdot \frac{d}{dx} \cdot x\right]$$

$$= 4x - \frac{3}{\sqrt{\pi}}\left[x \cdot \frac{1 \times (-12x)}{2\sqrt{K - 6x^2}} + (K - 6x^2)^{1/2} \cdot 1\right]$$

$$= 4x - \frac{3}{\sqrt{\pi}}\left[\frac{-6x^2}{(K - 6x^2)^{1/2}} + (K - 6x^2)^{1/2}\right]$$

$$= 4x - \frac{3}{\sqrt{\pi}}\left[\frac{-6x^2 + K - 6x^2}{(K - 6x^2)^{1/2}}\right] = 4x - \frac{3}{\sqrt{\pi}}\left[\frac{K - 12x^2}{(K - 6x^2)^{1/2}}\right]$$

Put $\quad x = 3 \cdot \sqrt{\dfrac{K}{4\pi + 54}}$

$$\frac{d^2V}{dx^2} = 4 \cdot 3\sqrt{\frac{K}{4\pi + 54}} - \frac{3}{\sqrt{\pi}}\left[\frac{K - 12 \cdot \dfrac{9K}{4\pi + 54}}{\sqrt{(K - 6 \cdot \dfrac{9K}{4\pi + 54})}}\right]$$

$$= 12\sqrt{\frac{K}{4\pi+54}} - \frac{3}{\sqrt{\pi}} \frac{\frac{4K\pi + 54K - 108K}{4\pi+54}}{\sqrt{\frac{4K\pi+54K-54K}{4\pi+54}}}$$

$$= 12\sqrt{\frac{K}{4\pi+54}} - \frac{3}{\sqrt{\pi}} \left[\frac{\frac{4K\pi - 54K}{4\pi+54}}{\sqrt{\frac{4K\pi}{4\pi+54}}}\right]$$

$$= 12\sqrt{\frac{K}{4\pi+54}} - \frac{3}{\sqrt{\pi}} \left[\frac{4K\pi - 54K}{\sqrt{4K\pi}\cdot\sqrt{4\pi+54}}\right]$$

$$= 12\sqrt{\frac{K}{4\pi+54}} - \frac{6K}{\sqrt{\pi}} \left(\frac{2\pi - 27}{\sqrt{4K\pi}\cdot\sqrt{4\pi+54}}\right)$$

$$= 12\sqrt{\frac{K}{4\pi+54}} + \frac{6K}{\sqrt{\pi}} \left[\frac{27 - 2\pi}{\sqrt{4k\pi}\cdot\sqrt{4\pi+54}}\right] > 0$$

$[\because \quad 27 - 2\pi > 0]$

$\therefore \quad \frac{d^2V}{dx^2} > 0$ so, it is minima.

Hence, the sum of volume is minimum for $x = 3\sqrt{\frac{K}{4\pi+54}}$

$\therefore$ Minimum volume,

$$V \text{ at } \left(x = 3\sqrt{\frac{K}{4\pi+54}}\right) = \frac{2}{3}x^3 + \frac{4}{3}\pi r^3 = \frac{2}{3}x^3 + \frac{4}{3}\pi\cdot\left(\frac{x}{3}\right)^3$$

$$= \frac{2}{3}x^3 + \frac{4}{3}\pi\cdot\frac{x^3}{27} = \frac{2}{3}x^3 + \frac{4}{81}\pi x^3$$

$$= \frac{2}{3}x^3\left(1 + \frac{2\pi}{27}\right)$$

Hence, the required minimum volume is $\frac{2}{3}x^3\left(1+\frac{2\pi}{27}\right)$ and $x = 3r$.

OBJECTIVE TYPE QUESTIONS

Choose the correct answer from the given four options in each of the following questions 35 to 59:

Q35. The sides of an equilateral triangle are increasing at the rate of 2 cm/sec. The rate at which the area increases, when side is 10 cm is:

(a) $10 \text{ cm}^2/\text{s}$ (b) $\sqrt{3} \text{ cm}^2/\text{s}$

(c) $10\sqrt{3} \text{ cm}^2/\text{s}$ (d) $\frac{10}{3} \text{ cm}^2/\text{s}$

Sol. Let the length of each side of the given equilateral triangle be x cm.

$$\therefore \quad \frac{dx}{dt} = 2 \text{ cm/sec}$$

Area of equilateral triangle $A = \frac{\sqrt{3}}{4}x^2$

$$\therefore \quad \frac{dA}{dt} = \frac{\sqrt{3}}{4} \cdot 2x \cdot \frac{dx}{dt} = \frac{\sqrt{3}}{2} \times 10 \times 2 = 10\sqrt{3} \text{ cm}^2/\text{sec}$$

Hence, the rate of increasing of area $= 10\sqrt{3} \text{ cm}^2/\text{sec}$.

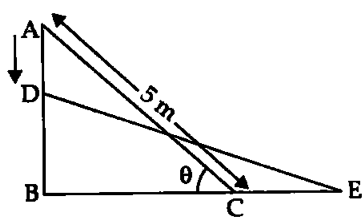
Hence, the correct option is (*c*).

Q36. A ladder, 5 m long, standing on a horizontal floor, leans against a vertical wall. If the top of the ladder slides downwards at the rate of 10 cm/sec, then the rate at which the angle between the floor and the ladder is decreasing when lower end of ladder is 2 metres from the wall is:

(*a*) $\frac{1}{10}$ radian/sec (*b*) $\frac{1}{20}$ radian/sec

(*c*) 20 radian/sec

(*d*) 10 radian/sec



Sol. Length of ladder = 5 m

Let AB = y m and BC = x m

$\therefore$ In right ΔABC,

$$AB^2 + BC^2 = AC^2$$

$$\Rightarrow \quad x^2 + y^2 = (5)^2 \Rightarrow x^2 + y^2 = 25$$

Differentiating both sides w.r.t x, we have

$$2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 0$$

$$\Rightarrow \quad x\frac{dx}{dt} + y \cdot \frac{dy}{dt} = 0$$

$$\Rightarrow \quad 2 \cdot \frac{dx}{dt} + y \times (-0.1) = 0 \quad [\because x = 2m]$$

$$\Rightarrow \quad 2 \cdot \frac{dx}{dt} + \sqrt{25 - x^2} \times (-0.1) = 0$$

$$\Rightarrow \quad 2 \cdot \frac{dx}{dt} + \sqrt{25 - 4} \times (-0.1) = 0$$

$$\Rightarrow \quad 2 \cdot \frac{dx}{dt} - \frac{\sqrt{21}}{10} = 0 \Rightarrow \frac{dx}{dt} = \frac{\sqrt{21}}{20}$$

Now $\cos\theta = \frac{BC}{AC}$ (θ is in radian)

$\Rightarrow$ $\cos\theta = \frac{x}{5}$

Differentiating both sides w.r.t. t, we get

$$\frac{d}{dt}\cos\theta = \frac{1}{5}\cdot\frac{dx}{dt} \Rightarrow -\sin\theta\cdot\frac{d\theta}{dt} = \frac{1}{5}\cdot\frac{\sqrt{21}}{20}$$

$$\Rightarrow \quad \frac{d\theta}{dt} = \frac{\sqrt{21}}{100}\times\left(-\frac{1}{\sin\theta}\right) = \frac{\sqrt{21}}{100}\times-\left(\frac{1}{\frac{AB}{AC}}\right)$$

$$= -\frac{\sqrt{21}}{100}\times\frac{AC}{AB} = -\frac{\sqrt{21}}{100}\times\frac{5}{\sqrt{21}} = -\frac{1}{20}\text{ radian/sec}$$

[(–) sign shows the decrease of change of angle]

Hence, the required rate $= \frac{1}{20}$ radian/sec

Hence, the correct option is (*b*).

Q37. The curve $y = x^{1/5}$ has at (0, 0)

(*a*) a vertical tangent (parallel to y-axis)
(*b*) a horizontal tangent (parallel to x-axis)
(*c*) an oblique tangent
(*d*) no tangent

Sol. Equation of curve is $y = x^{1/5}$

Differentiating w.r.t. x, we get $\frac{dy}{dx} = \frac{1}{5}x^{-4/5}$

(at $x = 0$) $\quad \frac{dy}{dx} = \frac{1}{5}(0)^{-4/5} = \frac{1}{5}\times\frac{1}{0} = \infty$

$$\frac{dy}{dx} = \infty$$

$\therefore$ The tangent is parallel to y-axis.

Hence, the correct option is (*a*).

Q38. The equation of normal to the curve $3x^2 - y^2 = 8$ which is parallel to the line $x + 3y = 8$ is

(*a*) $3x - y = 8$ (*b*) $3x + y + 8 = 0$
(*c*) $x + 3y \pm 8 = 0$ (*d*) $x + 3y = 0$

Sol. Given equation of the curve is $3x^2 - y^2 = 8$ ...(*i*)

Differentiating both sides w.r.t. x, we get

$$6x - 2y\cdot\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{3x}{y}$$

$\frac{3x}{y}$ is the slope of the tangent

$\therefore$ Slope of the normal $= \frac{-1}{dy/dx} = \frac{-y}{3x}$

Now $x + 3y = 8$ is parallel to the normal

Differentiating both sides w.r.t. x, we have

$$1 + 3\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{1}{3}$$

$$\therefore \quad \frac{-y}{3x} = -\frac{1}{3} \quad \Rightarrow \quad y = x$$

Putting $y = x$ in eq. (*i*) we get

$$3x^2 - x^2 = 8 \quad \Rightarrow \quad 2x^2 = 8 \quad \Rightarrow \quad x^2 = 4$$

$$\therefore \quad x = \pm 2 \text{ and } y = \pm 2$$

So the points are (2, 2) and (–2, –2).

Equation of normal to the given curve at (2, 2) is

$$y - 2 = -\frac{1}{3}(x - 2)$$

$$\Rightarrow \quad 3y - 6 = -x + 2 \quad \Rightarrow x + 3y - 8 = 0$$

Equation of normal at (–2, –2) is

$$y + 2 = -\frac{1}{3}(x + 2)$$

$$\Rightarrow \quad 3y + 6 = -x - 2 \quad \Rightarrow x + 3y + 8 = 0$$

$\therefore$ The equations of the normals to the curve are

$$x + 3y \pm 8 = 0$$

Hence, the correct option is (*c*).

Q39. If the curve $ay + x^2 = 7$ and $x^3 = y$, cut orthogonally at (1, 1), then the value of 'a' is:

(*a*) 1 (*b*) 0 (*c*) –6 (*d*) 6

Sol. Equation of the given curves are $ay + x^2 = 7$ ...(*i*)

and $x^3 = y$ ...(*ii*)

Differentiating eq. (*i*) w.r.t. x, we have

$$a\frac{dy}{dx} + 2x = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{2x}{a}$$

$$\therefore \quad m_1 = -\frac{2x}{a} \qquad \left(m_1 = \frac{dy}{dx}\right)$$

Now differentiating eq. (*ii*) w.r.t. x, we get

$$3x^2 = \frac{dy}{dx} \quad \Rightarrow \quad m_2 = 3x^2 \qquad \left(m_2 = \frac{dy}{dx}\right)$$

The two curves are said to be orthogonal if the angle between the tangents at the point of intersection is 90°.

$\therefore \quad m_1 \times m_2 = -1$

$\Rightarrow \quad \dfrac{-2x}{a} \times 3x^2 = -1 \quad \Rightarrow \quad \dfrac{-6x^3}{a} = -1 \quad \Rightarrow 6x^3 = a$

(1, 1) is the point of intersection of two curves.

$\therefore \quad 6(1)^3 = a$

So $\quad a = 6$

Hence, the correct option is (*d*).

Q40. If $y = x^4 - 10$ and if x changes from 2 to 1.99, what is the change in y?

(*a*) 0.32 (*b*) 0.032 (*c*) 5.68 (*d*) 5.968

Sol. Given that $y = x^4 - 10$

$$\frac{dy}{dx} = 4x^3$$

$$\Delta x = 2.00 - 1.99 = 0.01$$

$$\therefore \quad \Delta y = \frac{dy}{dx} \cdot \Delta x = 4x^3 \times \Delta x$$

$$= 4 \times (2)^3 \times 0.01 = 32 \times 0.01 = 0.32$$

Hence, the correct option is (*a*).

Q41. The equation of tangent to the curve $y(1 + x^2) = 2 - x$, where it crosses x-axis is:

(*a*) $x + 5y = 2$ (*b*) $x - 5y = 2$

(*c*) $5x - y = 2$ (*d*) $5x + y = 2$

Sol. Given that $y(1 + x^2) = 2 - x$ ...(*i*)

If it cuts x-axis, then y-coordinate is 0.

$\therefore \quad 0(1 + x^2) = 2 - x \quad \Rightarrow \quad x = 2$

Put $x = 2$ in equation (*i*)

$$y(1 + 4) = 2 - 2 \quad \Rightarrow y(5) = 0 \quad \Rightarrow \quad y = 0$$

Point of contact = (2, 0)

Differentiating eq. (*i*) w.r.t. x, we have

$$y \times 2x + (1 + x^2)\frac{dy}{dx} = -1$$

$$\Rightarrow \quad 2xy + (1 + x^2)\frac{dy}{dx} = -1 \quad \Rightarrow (1 + x^2)\frac{dy}{dx} = -1 - 2xy$$

$$\therefore \quad \frac{dy}{dx} = \frac{-(1 + 2xy)}{(1 + x^2)} \quad \Rightarrow \frac{dy}{dx}_{(2,0)} = \frac{-1}{(1 + 4)} = \frac{-1}{5}$$

Equation of tangent is $y - 0 = -\dfrac{1}{5}(x - 2)$

$\Rightarrow \quad 5y = -x + 2 \quad \Rightarrow x + 5y = 2$

Hence, the correct option is (a).

Q42. The points at which the tangents to the curve $y = x^3 - 12x + 18$ are parallel to x-axis are:

(a) $(2, -2), (-2, -34)$ (b) $(2, 34), (-2, 0)$
(c) $(0, 34), (-2, 0)$ (d) $(2, 2), (-2, 34)$

Sol. Given that $y = x^3 - 12x + 18$

Differentiating both sides w.r.t. x, we have

$$\Rightarrow \quad \frac{dy}{dx} = 3x^2 - 12$$

Since the tangents are parallel to x-axis, then $\frac{dy}{dx} = 0$

$\therefore \quad 3x^2 - 12 = 0 \quad \Rightarrow \quad x = \pm 2$

$\therefore \quad y_{x=2} = (2)^3 - 12(2) + 18 = 8 - 24 + 18 = 2$

$y_{x=-2} = (-2)^3 - 12(-2) + 18 = -8 + 24 + 18 = 34$

$\therefore$ Points are $(2, 2)$ and $(-2, 34)$

Hence, the correct option is (d).

Q43. The tangent to the curve $y = e^{2x}$ at the point $(0, 1)$ meets x-axis at:

(a) $(0, 1)$ (b) $\left(-\frac{1}{2}, 0\right)$ (c) $(2, 0)$ (d) $(0, 2)$

Sol. Equation of the curve is $y = e^{2x}$

Slope of the tangent $\frac{dy}{dx} = 2e^{2x} \Rightarrow \left.\frac{dy}{dx}\right|_{(0,1)} = 2 \cdot e^0 = 2$

$\therefore$ Equation of tangent to the curve at $(0, 1)$ is

$y - 1 = 2(x - 0)$

$\Rightarrow \quad y - 1 = 2x \quad \Rightarrow y - 2x = 1$

Since the tangent meets x-axis where $y = 0$

$\therefore \quad 0 - 2x = 1 \quad \Rightarrow \quad x = \frac{-1}{2}$

So the point is $\left(-\frac{1}{2}, 0\right)$

Hence, the correct option is (b).

Q44. The slope of tangent to the curve $x = t^2 + 3t - 8$ and $y = 2t^2 - 2t - 5$ at the point $(2, -1)$ is:

(a) $\frac{22}{7}$ (b) $\frac{6}{7}$ (c) $-\frac{6}{7}$ (d) -6

Sol. The given curve is $x = t^2 + 3t - 8$ and $y = 2t^2 - 2t - 5$

$$\frac{dx}{dt} = 2t + 3 \quad \text{and} \quad \frac{dy}{dt} = 4t - 2$$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4t - 2}{2t + 3}$$

Now $(2, -1)$ lies on the curve

$$\therefore \quad 2 = t^2 + 3t - 8 \Rightarrow t^2 + 3t - 10 = 0$$

$$\Rightarrow t^2 + 5t - 2t - 10 = 0$$

$$\Rightarrow t(t+5) - 2(t+5) = 0$$

$$\Rightarrow (t+5)(t-2) = 0$$

$\therefore t = 2, t = -5$ and $-1 = 2t^2 - 2t - 5$

$$\Rightarrow 2t^2 - 2t - 4 = 0$$

$$\Rightarrow t^2 - t - 2 = 0 \Rightarrow t^2 - 2t + t - 2 = 0$$

$$\Rightarrow t(t-2) + 1(t-2) = 0 \Rightarrow (t+1)(t-2) = 0$$

$$\Rightarrow t = -1 \text{ and } t = 2$$

So t = 2 is common value

$$\therefore \quad \text{Slope } \frac{dy}{dx_{x=2}} = \frac{4 \times 2 - 2}{2 \times 2 + 3} = \frac{6}{7}$$

Hence, the correct option is (b).

Q45. The two curves $x^3 - 3xy^2 + 2 = 0$ and $3x^2y - y^3 - 2 = 0$ intersect at an angle of:

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{6}$

Sol. The given curves are $x^3 - 3xy^2 + 2 = 0$ ...(i)

and $3x^2y - y^3 - 2 = 0$ ...(ii)

Differentiating eq. (i) w.r.t. x, we get

$$3x^2 - 3\left(x \cdot 2y\frac{dy}{dx} + y^2 \cdot 1\right) = 0$$

$$\Rightarrow x^2 - 2xy\frac{dy}{dx} - y^2 = 0 \Rightarrow 2xy\frac{dy}{dx} = x^2 - y^2$$

$$\therefore \quad \frac{dy}{dx} = \frac{x^2 - y^2}{2xy}$$

So slope of the curve $\quad m_1 = \frac{x^2 - y^2}{2xy}$

Differentiating eq. (ii) w.r.t. x, we get

$$3\left[x^2\frac{dy}{dx} + y \cdot 2x\right] - 3y^2 \cdot \frac{dy}{dx} = 0$$

$$x^2\frac{dy}{dx} + 2xy - y^2\frac{dy}{dx} = 0 \Rightarrow (x^2 - y^2)\frac{dy}{dx} = -2xy$$

$$\therefore \quad \frac{dy}{dx} = \frac{-2xy}{x^2 - y^2}$$

So the slope of the curve $m_2 = \dfrac{-2xy}{x^2 - y^2}$

Now $$m_1 \times m_2 = \frac{x^2 - y^2}{2xy} \times \frac{-2xy}{x^2 - y^2} = -1$$

So the angle between the curves is $\dfrac{\pi}{2}$.

Hence, the correct option is (*c*).

Q46. The interval on which the function $f(x) = 2x^3 + 9x^2 + 12x - 1$ is decreasing is:

(*a*) $[-1, \infty)$ (*b*) $[-2, -1]$

(*c*) $(-\infty, -2]$ (*d*) $[-1, 1]$

Sol. The given function is $f(x) = 2x^3 + 9x^2 + 12x - 1$

$$f'(x) = 6x^2 + 18x + 12$$

For increasing and decreasing $f'(x) = 0$

$\therefore$ $6x^2 + 18x + 12 = 0$

$\Rightarrow$ $x^2 + 3x + 2 = 0 \Rightarrow x^2 + 2x + x + 2 = 0$

$\Rightarrow$ $x(x+2) + 1(x+2) = 0 \Rightarrow (x+2)(x+1) = 0$

$\Rightarrow$ $x = -2, x = -1$

The possible intervals are $(-\infty, -2), (-2, -1), (-1, \infty)$

Now $f'(x) = (x+2)(x+1)$

$\Rightarrow$ $f'(x)_{(-\infty, -2)} = (-)(-) = (+)$ increasing

$\Rightarrow$ $f'(x)_{(-2, -1)} = (+)(-) = (-)$ decreasing

$\Rightarrow$ $f'(x)_{(-1, \infty)} = (+)(+) = (+)$ increasing

Hence, the correct option is (*b*).

Q47. Let the $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = 2x + \cos x$, then f:

(*a*) has a minimum at $x = \pi$ (*b*) has a maximum at $x = 0$

(*c*) is a decreasing function (*d*) is an increasing function

Sol. Given that $f(x) = 2x + \cos x$

$$f'(x) = 2 - \sin x$$

Since $f'(x) > 0 \ \forall \ x$

So $f(x)$ is an increasing function.

Hence, the correct option is (*d*).

Q48. $y = x(x-3)^2$ decreases for the values of x given by:

(*a*) $1 < x < 3$ (*b*) $x < 0$ (*c*) $x > 0$ (*d*) $0 < x < \dfrac{3}{2}$

Sol. Here $y = x(x-3)^2$

$$\frac{dy}{dx} = x \cdot 2(x-3) + (x-3)^2 \cdot 1 \Rightarrow \frac{dy}{dx} = 2x(x-3) + (x-3)^2$$

For increasing and decreasing $\frac{dy}{dx} = 0$

$\therefore \quad 2x(x-3) + (x-3)^2 = 0 \quad \Rightarrow (x-3)(2x+x-3) = 0$

$\Rightarrow \quad (x-3)(3x-3) = 0 \quad \Rightarrow 3(x-3)(x-1) = 0$

$\therefore \quad x = 1, 3$

$\therefore$ Possible intervals are $(-\infty, 1), (1, 3), (3, \infty)$

$$\frac{dy}{dx} = (x-3)(x-1)$$

For $(-\infty, 1) = (-)(-) = (+)$ increasing

For $(1, 3) = (-)(+) = (-)$ decreasing

For $(3, \infty) = (+)(+) = (+)$ increasing

So the function decreases in $(1, 3)$ or $1 < x < 3$

Hence, the correct option is (*a*).

Q49. The function $f(x) = 4\sin^3 x - 6\sin^2 x + 12\sin x + 100$ is strictly

(*a*) increasing in $\left(\pi, \frac{3\pi}{2}\right)$ (*b*) decreasing in $\left(\frac{\pi}{2}, \pi\right)$

(*c*) decreasing in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (*d*) decreasing in $\left[0, \frac{\pi}{2}\right]$

Sol. Here,

$$f(x) = 4\sin^3 x - 6\sin^2 x + 12\sin x + 100$$

$$f'(x) = 12\sin^2 x \cdot \cos x - 12\sin x\cos x + 12\cos x$$

$$= 12\cos x[\sin^2 x - \sin x + 1]$$

$$= 12\cos x[\sin^2 x + (1 - \sin x)]$$

$\because \quad 1 - \sin x \geq 0$ and $\sin^2 x \geq 0$

$\therefore \quad \sin^2 x + 1 - \sin x \geq 0$ (when $\cos x > 0$)

Hence, $f'(x) > 0$, when $\cos x > 0$ i.e., $x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

So, $f(x)$ is increasing where $x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ and $f'(x) < 0$

when $\cos x < 0$ i.e. $x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

Hence, $f(x)$ is decreasing when $x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

As $\left(\frac{\pi}{2}, \pi\right) \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

So $f(x)$ is decreasing in $\left(\frac{\pi}{2}, \pi\right)$

Hence, the correct option is (*b*).

Q50. Which of the following functions is decreasing in $\left(0, \frac{\pi}{2}\right)$?

(*a*) $\sin 2x$ (*b*) $\tan x$ (*c*) $\cos x$ (*d*) $\cos 3x$

Sol. Here, Let $f(x) = \cos x$; So, $f'(x) = -\sin x$

$$f'(x) < 0 \text{ in } \left(0, \frac{\pi}{2}\right)$$

So $f(x) = \cos x$ is decreasing in $\left(0, \frac{\pi}{2}\right)$

Hence, the correct option is (*c*).

Q51. The function $f(x) = \tan x - x$

(*a*) always increases (*b*) always decreases

(*c*) never increases

(*d*) sometimes increases and sometimes decreases.

Sol. Here, $f(x) = \tan x - x$ So, $f'(x) = \sec^2 x - 1$

$$f'(x) > 0 \ \forall \ x \in R$$

So $f(x)$ is always increasing

Hence, the correct option is (*a*).

Q52. If x is real, the minimum value of $x^2 - 8x + 17$ is

(*a*) -1 (*b*) 0 (*c*) 1 (*d*) 2

Sol. Let $f(x) = x^2 - 8x + 17$

$$f'(x) = 2x - 8$$

For local maxima and local minima, $f'(x) = 0$

$\therefore$ $2x - 8 = 0 \Rightarrow x = 4$

So, $x = 4$ is the point of local maxima and local minima.

$$f''(x) = 2 > 0 \text{ minima at } x = 4$$

$$\therefore \quad f(x)_{x=4} = (4)^2 - 8(4) + 17$$
$$= 16 - 32 + 17 = 33 - 32 = 1$$

So the minimum value of the function is 1

Hence, the correct option is (*c*).

Q53. The smallest value of the polynomial $x^3 - 18x^2 + 96x$ in $[0, 9]$ is:

(*a*) 126 (*b*) 0 (*c*) 135 (*d*) 160

Sol. Let $f(x) = x^3 - 18x^2 + 96x$; So, $f'(x) = 3x^2 - 36x + 96$

For local maxima and local minima $f'(x) = 0$

$$\therefore \quad 3x^2 - 36x + 96 = 0$$
$$\Rightarrow \quad x^2 - 12x + 32 = 0 \quad \Rightarrow x^2 - 8x - 4x + 32 = 0$$
$$\Rightarrow x(x - 8) - 4(x - 8) = 0 \quad \Rightarrow (x - 8)(x - 4) = 0$$
$$\therefore \quad x = 8, 4 \in [0, 9]$$

So, $x = 4, 8$ are the points of local maxima and local minima.

Now we will calculate the absolute maxima or absolute minima at $x = 0, 4, 8, 9$

$$\therefore \quad f(x) = x^3 - 18x^2 + 96x$$
$$f(x)_{x=0} = 0 - 0 + 0 = 0$$

$$f(x)_{x=4} = (4)^3 - 18(4)^2 + 96(4)$$
$$= 64 - 288 + 384 = 448 - 288 = 160$$
$$f(x)_{x=8} = (8)^3 - 18(8)^2 + 96(8)$$
$$= 512 - 1152 + 768 = 1280 - 1152 = 128$$
$$f(x)_{x=9} = (9)^3 - 18(9)^2 + 96(9)$$
$$= 729 - 1458 + 864 = 1593 - 1458 = 135$$

So, the absolute minimum value of f is 0 at $x = 0$

Hence, the correct option is (*b*).

Q54. The function $f(x) = 2x^3 - 3x^2 - 12x + 4$, has

(*a*) two points of local maximum
(*b*) two points of local minimum
(*c*) one maxima and one minima
(*d*) no maxima or minima

Sol. We have $\quad f(x) = 2x^3 - 3x^2 - 12x + 4$

$$f'(x) = 6x^2 - 6x - 12$$

For local maxima and local minima $f'(x) = 0$

$\therefore \quad 6x^2 - 6x - 12 = 0$

$\Rightarrow \quad x^2 - x - 2 = 0 \quad \Rightarrow x^2 - 2x + x - 2 = 0$

$\Rightarrow \quad x(x-2) + 1(x-2) = 0 \quad \Rightarrow (x+1)(x-2) = 0$

$\Rightarrow \quad x = -1, 2$ are the points of local maxima and local minima

Now $\quad f''(x) = 12x - 6$

$$f''(x)_{x=-1} = 12(-1) - 6 = -12 - 6 = -18 < 0, \text{ maxima}$$
$$f''(x)_{x=2} = 12(2) - 6 = 24 - 6 = 18 > 0 \text{ minima}$$

So, the function is maximum at $x = -1$ and minimum at $x = 2$

Hence, the correct option is (*c*).

Q55. The maximum value of $\sin x \cos x$ is

(*a*) $\frac{1}{4}$ (*b*) $\frac{1}{2}$ (*c*) $\sqrt{2}$ (*d*) $2\sqrt{2}$

Sol. We have $\quad f(x) = \sin x \cos x$

$\Rightarrow \quad f(x) = \frac{1}{2} \cdot 2 \sin x \cos x = \frac{1}{2} \sin 2x$

$$f'(x) = \frac{1}{2} \cdot 2 \cos 2x$$

$\Rightarrow \quad f'(x) = \cos 2x$

Now for local maxima and local minima $f'(x) = 0$

$\therefore \quad \cos 2x = 0$

$$2x = (2n+1)\frac{\pi}{2}, \quad n \in I$$

$\Rightarrow \quad x = (2n+1)\frac{\pi}{4}$

$$\therefore \qquad x = \frac{\pi}{4}, \frac{3\pi}{4} \ldots$$

$$f''(x) = -2 \sin 2x$$

$$f''(x)_{x=\frac{\pi}{4}} = -2 \sin 2 \cdot \frac{\pi}{4} = -2 \sin \frac{\pi}{2} = -2 < 0 \text{ maxima}$$

$$f''(x)_{x=\frac{3\pi}{4}} = -2 \sin 2 \cdot \frac{3\pi}{4} = -2 \sin \frac{3\pi}{2} = 2 > 0 \text{ minima}$$

So $f(x)$ is maximum at $x = \frac{\pi}{4}$

$\therefore$ Maximum value of $f(x) = \sin \frac{\pi}{4} \cdot \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2}$

Hence, the correct option is (*b*).

Q56. At $x = \frac{5\pi}{6}$, $f(x) = 2 \sin 3x + 3 \cos 3x$ is:

(*a*) maximum (*b*) minimum

(*c*) zero (*d*) neither maximum nor minimum.

Sol. We have $f(x) = 2 \sin 3x + 3 \cos 3x$

$$f'(x) = 2 \cos 3x \cdot 3 - 3 \sin 3x \cdot 3 = 6 \cos 3x - 9 \sin 3x$$

$$f''(x) = -6 \sin 3x \cdot 3 - 9 \cos 3x \cdot 3$$

$$= -18 \sin 3x - 27 \cos 3x$$

$$f''\left(\frac{5\pi}{6}\right) = -18 \sin 3\left(\frac{5\pi}{6}\right) - 27 \cos 3\left(\frac{5\pi}{6}\right)$$

$$= -18 \sin \left(\frac{5\pi}{2}\right) - 27 \cos \left(\frac{5\pi}{2}\right)$$

$$= -18 \sin \left(2\pi + \frac{\pi}{2}\right) - 27 \cos \left(2\pi + \frac{\pi}{2}\right)$$

$$= -18 \sin \frac{\pi}{2} - 27 \cos \frac{\pi}{2} = -18 \cdot 1 - 27 \cdot 0$$

$$= -18 < 0 \text{ maxima}$$

Maximum value of $f(x)$ at $x = \frac{5\pi}{6}$

$$f\left(\frac{5\pi}{6}\right) = 2 \sin 3\left(\frac{5\pi}{6}\right) + 3 \cos 3\left(\frac{5\pi}{6}\right) = 2 \sin \frac{5\pi}{2} + 3 \cos \frac{5\pi}{2}$$

$$= 2 \sin\left(2\pi + \frac{\pi}{2}\right) + 3 \cos\left(2\pi + \frac{\pi}{2}\right) = 2 \sin \frac{\pi}{2} + 3 \cos \frac{\pi}{2} = 2$$

Hence, the correct option is (a).

Q57. Maximum slope of the curve $y = -x^3 + 3x^2 + 9x - 27$ is:

(*a*) 0 (*b*) 12 (*c*) 16 (*d*) 32

Sol. Given that $y = -x^3 + 3x^2 + 9x - 27$

$$\frac{dy}{dx} = -3x^2 + 6x + 9$$

$\therefore$ Slope of the given curve,

$$m = -3x^2 + 6x + 9 \qquad \left(\frac{dy}{dx} = m\right)$$

$$\frac{dm}{dx} = -6x + 6$$

For local maxima and local minima, $\frac{dm}{dx} = 0$

$\therefore$ $$-6x + 6 = 0 \Rightarrow x = 1$$

Now $$\frac{d^2m}{dx^2} = -6 < 0 \quad \text{maxima}$$

$\therefore$ Maximum value of the slope at $x = 1$ is

$$m_{x=1} = -3(1)^2 + 6(1) + 9 = -3 + 6 + 9 = 12$$

Hence, the correct option is (*b*).

Q58. $f(x) = x^x$ has a stationary point at

(*a*) $x = e$ (*b*) $x = \frac{1}{e}$ (*c*) $x = 1$ (*d*) $x = \sqrt{e}$

Sol. We have $$f(x) = x^x$$

Taking log of both sides, we have

$$\log f(x) = x \log x$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{f(x)} \cdot f'(x) = x \cdot \frac{1}{x} + \log x \cdot 1$$

$$\Rightarrow \quad f'(x) = f(x)\,[1 + \log x] = x^x\,[1 + \log x]$$

To find stationary point, $f'(x) = 0$

$\therefore$ $$x^x[1 + \log x] = 0$$

$x^x \neq 0$ $\therefore$ $$1 + \log x = 0$$

$$\Rightarrow \quad \log x = -1 \Rightarrow x = e^{-1} \Rightarrow x = \frac{1}{e}$$

Hence, the correct option is (*b*).

Q59. The maximum value of $\left(\frac{1}{x}\right)^x$ is:

(*a*) e (*b*) e^e (*c*) $e^{1/e}$ (*d*) $\left(\frac{1}{e}\right)^{1/e}$

Sol. Let $$f(x) = \left(\frac{1}{x}\right)^x$$

Taking log on both sides, we get

$$\log\,[f(x)] = x \log \frac{1}{x}$$

$$\Rightarrow \quad \log\,[f(x)] = x \log x^{-1} \Rightarrow \log\,[f(x)] = -\,[x \log x]$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{f(x)} \cdot f'(x) = -\left[x \cdot \frac{1}{x} + \log x \cdot 1\right] = -f(x)\,[1 + \log x]$$

$$\Rightarrow \qquad f'(x) = -\left(\frac{1}{x}\right)^x [1 + \log x]$$

For local maxima and local minima $f'(x) = 0$

$$-\left(\frac{1}{x}\right)^x [1 + \log x] = 0 \quad \Rightarrow \quad \left(\frac{1}{x}\right)^x [1 + \log x] = 0$$

$\left(\frac{1}{x}\right)^x \neq 0$

$$\therefore \qquad 1 + \log x = 0 \Rightarrow \log x = -1 \Rightarrow x = e^{-1}$$

So, $x = \frac{1}{e}$ is the stationary point.

Now $\quad f'(x) = -\left(\frac{1}{x}\right)^x [1 + \log x]$

$$f''(x) = -\left[\left(\frac{1}{x}\right)^x \left(\frac{1}{x}\right) + (1 + \log x) \cdot \frac{d}{dx}(x)^x\right]$$

$$f''(x) = -\left[(e)^{1/e}(e) + \left(1 + \log \frac{1}{e}\right) \frac{d}{dx}\left(\frac{1}{e}\right)^{1/e}\right]$$

$$x = \frac{1}{e} = -e^{\frac{1}{e}+1} < 0 \text{ maxima}$$

$\therefore$ Maximum value of the function at $x = \frac{1}{e}$ is

$$f\left(\frac{1}{e}\right) = \left(\frac{1}{1/e}\right)^{1/e} = e^{1/e}$$

Hence, the correct option is (*c*).

Fill in the blanks in each of the following exercises 60 to 64.

Q60. The curves $y = 4x^2 + 2x - 8$ and $y = x^3 - x + 13$ touch each other at the point __________ .

Sol. We have

$$y = 4x^2 + 2x - 8 \qquad ...(i)$$

and $$y = x^3 - x + 13 \qquad ...(ii)$$

Differentiating eq. (*i*) w.r.t. x, we have

$$\frac{dy}{dx} = 8x + 2 \quad \Rightarrow \quad m_1 = 8x + 2$$

[m is the slope of curve (*i*)]

Differentiating eq. (*ii*) w.r.t. x, we get

$$\frac{dy}{dx} = 3x^2 - 1 \quad \Rightarrow \quad m_2 = 3x^2 - 1$$

[m_2 is the slope of curve (*ii*)]

If the two curves touch each other, then $m_1 = m_2$

$\therefore \quad 8x + 2 = 3x^2 - 1$

$\Rightarrow \quad 3x^2 - 8x - 3 = 0 \quad \Rightarrow 3x^2 - 9x + x - 3 = 0$

$\Rightarrow \quad 3x(x-3) + 1(x-3) = 0 \quad \Rightarrow (x-3)(3x+1) = 0$

$\therefore \quad x = 3, \ \frac{-1}{3}$

Putting $x = 3$ in eq. (*i*), we get

$$y = 4(3)^2 + 2(3) - 8 = 36 + 6 - 8 = 34$$

So, the required point is (3, 34)

Now for $x = -\frac{1}{3}$

$$y = 4\left(\frac{-1}{3}\right)^2 + 2\left(\frac{-1}{3}\right) - 8 = 4 \times \frac{1}{9} - \frac{2}{3} - 8$$

$$= \frac{4}{9} - \frac{2}{3} - 8 = \frac{4 - 6 - 72}{9} = \frac{-74}{9}$$

$\therefore$ Other required point is $\left(-\frac{1}{3}, \frac{-74}{9}\right)$.

Hence, the required points are (3, 34) and $\left(-\frac{1}{3}, \frac{-74}{9}\right)$.

Q61. The equation of normal to the curve $y = \tan x$ at (0, 0) is __________.

Sol. We have $y = \tan x$. So, $\frac{dy}{dx} = \sec^2 x$

$\therefore$ Slope of the normal $= \frac{-1}{\sec^2 x} = -\cos^2 x$

at the point (0, 0) the slope $= -\cos^2(0) = -1$

So the equation of normal at (0, 0) is $y - 0 = -1(x - 0)$

$\Rightarrow \quad y = -x \quad \Rightarrow y + x = 0$

Hence, the required equation is $y + x = 0$.

Q62. The values of a for which the function $f(x) = \sin x - ax + b$ increases on **R** are __________.

Sol. We have $\quad f(x) = \sin x - ax + b \quad \Rightarrow f'(x) = \cos x - a$

For increasing the function $f'(x) > 0$

$\therefore \quad \cos x - a > 0$

Since $\quad \cos x \in [-1, 1]$

$\therefore \quad a < -1 \;\Rightarrow\; a \in (-\infty, -1)$

Hence, the value of a is $(-\infty, -1)$.

Q63. The function $f(x) = \dfrac{2x^2 - 1}{x^4}$, $x > 0$, decreases in the interval ________.

Sol. We have $f(x) = \dfrac{2x^2 - 1}{x^4}$

$$f'(x) = \frac{x^4(4x) - (2x^2 - 1)\cdot 4x^3}{x^8}$$

$$\Rightarrow f'(x) = \frac{4x^5 - (2x^2 - 1)\cdot 4x^3}{x^8} = \frac{4x^3[x^2 - 2x^2 + 1]}{x^8} = \frac{4(-x^2 + 1)}{x^5}$$

For decreasing the function $f'(x) < 0$

$$\therefore \quad \frac{4(-x^2 + 1)}{x^5} < 0 \;\Rightarrow\; -x^2 + 1 < 0 \;\Rightarrow\; x^2 > 1$$

$$\therefore \quad x > \pm 1 \;\Rightarrow\; x \in (1, \infty)$$

Hence, the required interval is $(1, \infty)$.

Q64. The least value of the function $f(x) = ax + \dfrac{b}{x}$ (where $a > 0$, $b > 0, x > 0$) is ________.

Sol. Here, $\quad f(x) = ax + \dfrac{b}{x} \;\Rightarrow\; f'(x) = a - \dfrac{b}{x^2}$

For maximum and minimum value $f'(x) = 0$

$$\therefore \quad a - \frac{b}{x^2} = 0 \;\Rightarrow\; x^2 = \frac{b}{a} \;\Rightarrow\; x = \pm\sqrt{\frac{b}{a}}$$

Now $\quad f''(x) = \dfrac{2b}{x^3}$

$$f''(x)\Big|_{x=\sqrt{\frac{b}{a}}} = \frac{2b}{\left(\frac{b}{a}\right)^{3/2}} = 2\,\frac{a^{3/2}}{b^{1/2}} > 0 \qquad (\because \; a, b > 0)$$

Hence, minima

So the least value of the function at $x = \sqrt{\dfrac{b}{a}}$ is

$$f\left(\sqrt{\frac{b}{a}}\right) = a\cdot\sqrt{\frac{b}{a}} + \frac{b}{\sqrt{\frac{b}{a}}} = \sqrt{ab} + \sqrt{ab} = 2\sqrt{ab}$$

Hence, least value is $2\sqrt{ab}$.

❑❑❑

7 Integrals

7.3 EXERCISE

SHORT ANSWER TYPE QUESTIONS

Verify the following:

Q1. $\int \frac{2x-1}{2x+3} dx = x - \log|(2x+3)^2| + C$

Sol. L.H.S. $= \int \frac{2x-1}{2x+3} dx$

$\Rightarrow \int \left(1 - \frac{4}{2x+3}\right) dx$ [Dividing the numerator by the denominator]

$\Rightarrow \int 1.dx - 4\int \frac{1}{2x+3} dx \Rightarrow \int 1.dx - \frac{4}{2}\int \frac{1}{x+\frac{3}{2}} dx$

$\Rightarrow \int 1.dx - 2\int \frac{1}{x+\frac{3}{2}} dx \Rightarrow x - 2\log\left|x+\frac{3}{2}\right| + C$

$\Rightarrow x - 2\log\left|\frac{2x+3}{2}\right| + C \Rightarrow x - \log\left|\left(\frac{2x+3}{2}\right)^2\right| + C$

[$\because n \log m = \log m^n$]

$\Rightarrow x - \log|(2x+3)^2| - \log 2^2 + C$

$\Rightarrow x - \log|(2x+3)^2| + C_1 \Rightarrow$ R.H.S. [where $C_1 = C - \log 2^2$]

L.H.S. = R.H.S.

Hence proved.

Q2. $\int \frac{2x+3}{x^2+3x} dx = \log|x^2+3x| + C$

Sol. L.H.S. $= \int \frac{2x+3}{x^2+3x} dx$

Put $x^2 + 3x = t$

$\therefore$ $(2x+3)\, dx = dt$

$\Rightarrow$ $\int \frac{dt}{t} = \log|t| \Rightarrow \log|x^2+3x| + C =$ R.H.S.

L.H.S. = R.H.S.

Hence verified.

Evaluate the following:

Q3. $\int \frac{(x^2+2)}{x+1}\,dx$

Sol. Let $I = \int \frac{x^2+2}{x+1}\,dx$

$$\begin{array}{r|l} x+1 & x^2+2 \; (x-1 \\ & \underline{x^2+x} \\ & (-)\;(-) \\ & -x+2 \\ & \underline{-x-1} \\ & (+)\;(+) \\ & 3 \end{array}$$

$\therefore \quad I = \int \left[(x-1) + \frac{3}{x+1}\right] dx$

$= \int (x-1)\,dx + 3\int \frac{1}{x+1}\,dx$

$= \frac{x^2}{2} - x + 3\log|x+1| + C$

Hence, the required solution is $\frac{x^2}{2} - x + 3\log|x+1| + C$.

Q4. $\int \frac{e^{6\log x} - e^{5\log x}}{e^{4\log x} - e^{3\log x}}\,dx$

Sol. Let $I = \int \frac{e^{6\log x} - e^{5\log x}}{e^{4\log x} - e^{3\log x}}\,dx = \int \frac{e^{\log x^6} - e^{\log x^5}}{e^{\log x^4} - e^{\log x^3}}\,dx$

$= \int \frac{x^6 - x^5}{x^4 - x^3}\,dx = \int \frac{x^2(x^4 - x^3)}{x^4 - x^3}\,dx = \int x^2 dx = \frac{1}{3}x^3 + C$

Hence, the required solution is $\frac{1}{3}x^3 + C$.

Q5. $\int \frac{(1+\cos x)}{x+\sin x}\,dx$

Sol. Let $I = \int \frac{1+\cos x}{x+\sin x}\,dx$

Put $x + \sin x = t \Rightarrow (1+\cos x)\,dx = dt$

$\therefore \quad I = \int \frac{dt}{t} = \log|t| = \log|x+\sin x| + C$

Hence, the required solution is $\log|x+\sin x| + C$.

Q6. $\int \frac{dx}{1+\cos x}$

Sol. Let $I = \int \frac{dx}{1+\cos x} = \int \frac{dx}{2\cos^2 x/2}$ $\left[\because 1+\cos x = 2\cos^2\frac{x}{2}\right]$

$= \frac{1}{2}\int \sec^2\frac{x}{2}\,dx = \frac{1}{2}.2\tan\frac{x}{2} + C = \tan\frac{x}{2} + C$

Hence, the required solution is $\tan\frac{x}{2} + C$.

Q7. $\int \tan^2 x . \sec^4 x \, dx$

Sol. Let $I = \int \tan^2 x . \sec^4 x \, dx$

$$= \int \tan^2 x \sec^2 x . \sec^2 x \, dx = \int \tan^2 x (1 + \tan^2 x) . \sec^2 x \, dx$$

Put $\tan x = t$, $\therefore \sec^2 x \, dx = dt$

$$\therefore \quad I = \int t^2(1+t^2)\, dt = \int (t^2 + t^4)\, dt = \int t^2 \, dt + \int t^4 \, dt$$

$$= \frac{1}{3}t^3 + \frac{1}{5}t^5 = \frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x + C$$

Hence, the required solution is $\frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x + C$.

Q8. $\int \frac{\sin x + \cos x}{\sqrt{1 + \sin 2x}} dx$

Sol. Let $I = \int \frac{\sin x + \cos x}{\sqrt{1 + 2 \sin x \cos x}} dx$

$$= \int \frac{(\sin x + \cos x)}{\sqrt{\sin^2 x + \cos^2 x + 2 \sin x \cos x}} dx$$

$$= \int \frac{\sin x + \cos x}{\sqrt{(\sin x + \cos x)^2}} dx = \int \frac{\sin x + \cos x}{\sin x + \cos x} dx$$

$$= \int 1 \, dx$$

$$= x + C$$

Hence, the required solution is $x + C$.

Q9. $\int \sqrt{1 + \sin x} \, dx$

Sol. Let $I = \int \sqrt{1 + \sin x} \, dx$

$$= \int \sqrt{\left(\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} + 2 \sin \frac{x}{2} \cos \frac{x}{2}\right)} dx$$

$$= \int \sqrt{\left(\sin \frac{x}{2} + \cos \frac{x}{2}\right)^2} dx = \int \left(\sin \frac{x}{2} + \cos \frac{x}{2}\right) dx$$

$$= \int \sin \frac{x}{2} dx + \int \cos \frac{x}{2} dx = -2 \cos \frac{x}{2} + 2 \sin \frac{x}{2} + C$$

$$= 2\left(\sin \frac{x}{2} - \cos \frac{x}{2}\right) + C$$

Hence, the required solution is $2\left(\sin \frac{x}{2} - \cos \frac{x}{2}\right) + C$.

Q10. $\int \frac{x}{\sqrt{x} + 1} dx$ (Hint: Put $\sqrt{x} = z$)

Sol. $I = \int \frac{x}{\sqrt{x}+1} dx$

Put $\sqrt{x} = t \Rightarrow x = t^2 \quad \therefore dx = 2t \,.\, dt$

$$\therefore I = \int \frac{t^2 . 2t . dt}{t+1} = 2\int \frac{t^3}{t+1} dt = 2\int \frac{t^3+1-1}{t+1} dt$$

$$= 2\int \frac{t^3+1}{t+1} dt - 2\int \frac{1}{t+1} dt$$

$$= 2\int \frac{(t+1)(t^2-t+1)}{t+1} dt - 2\int \frac{1}{t+1} dt$$

$$= 2\int (t^2-t+1)\, dt - 2\int \frac{1}{t+1} dt$$

$$= 2\left[\frac{t^3}{3} - \frac{t^2}{2} + t\right] - 2\log|t+1|$$

$$= 2\left[\frac{x^{3/2}}{3} - \frac{x}{2} + \sqrt{x}\right] - 2\log\left|\sqrt{x}+1\right| + C$$

$$= 2\left[\frac{x\sqrt{x}}{3} - \frac{x}{2} + \sqrt{x} - \log\left|\sqrt{x}+1\right|\right] + C$$

Hence, $I = 2\left[\frac{x\sqrt{x}}{3} - \frac{x}{2} + \sqrt{x} - \log\left|\sqrt{x}+1\right|\right] + C$

Q11. $\int \sqrt{\frac{a+x}{a-x}}\, dx$

Sol. Let $I = \int \sqrt{\frac{a+x}{a-x}}\, dx$

$$= \int \sqrt{\frac{a+x}{a-x} \times \frac{a+x}{a+x}}\, dx = \int \frac{a+x}{\sqrt{(a-x)(a+x)}} dx$$

$$= \int \frac{a+x}{\sqrt{a^2-x^2}} dx = \int \frac{a}{\sqrt{a^2-x^2}} dx + \int \frac{x}{\sqrt{a^2-x^2}} dx$$

Let $I = I_1 + I_2$

Now $I_1 = \int \frac{a}{\sqrt{a^2-x^2}} dx = a . \sin^{-1}\frac{x}{a} + C_1$

and $I_2 = \int \frac{x}{\sqrt{a^2-x^2}} dx$

Put $a^2 - x^2 = t \Rightarrow -2x\, dx = dt$

$$x\, dx = \frac{dt}{-2}$$

$\therefore \quad I_2 = -\frac{1}{2}\int \frac{dt}{\sqrt{t}} = -\frac{1}{2} \times 2\sqrt{t} = -\sqrt{a^2 - x^2} + C_2$

Since $\quad I = I_1 + I_2$

$= a \sin^{-1}\frac{x}{a} + C_1 - \sqrt{a^2 - x^2} + C_2$

$\therefore \quad I = a \sin^{-1}\frac{x}{a} - \sqrt{a^2 - x^2} + (C_1 + C_2)$

Hence, $\quad I = a \sin^{-1}\frac{x}{a} - \sqrt{a^2 - x^2} + C \qquad [C = C_1 + C_2]$

Alternate method:

$$I = \int \sqrt{\frac{a + x}{a - x}}\, dx$$

Put $\quad x = a \cos 2\theta$

$\therefore \quad dx = a\,(-2 \sin 2\theta)\, d\theta = -2a \sin 2\theta\, d\theta$

$$\therefore \quad I = \int \sqrt{\frac{a + a\cos 2\theta}{a - a\cos 2\theta}}.(-2a \sin 2\theta)\, d\theta$$

$$= \int \sqrt{\frac{1 + \cos 2\theta}{1 - \cos 2\theta}}.(-2a \sin 2\theta)\, d\theta$$

$$= -2a\int \sqrt{\frac{2\cos^2\theta}{2\sin^2\theta}}.\sin 2\theta\, d\theta$$

$$= -2a\int \sqrt{\frac{\cos^2\theta}{\sin^2\theta}}.2\sin\theta\cos\theta\, d\theta$$

$$= -2a\int \frac{\cos\theta}{\sin\theta}.2\sin\theta\cos\theta\, d\theta$$

$$= -4a\int \cos\theta\cos\theta\, d\theta = -4a\int \cos^2\theta\, d\theta$$

$$= -4a\int \frac{1 + \cos 2\theta}{2}\, d\theta = -2a\int (1 + \cos 2\theta)\, d\theta$$

$$= -2a\left[\int 1\, d\theta + \int \cos 2\theta\, d\theta\right] = -2a\left[\theta + \frac{1}{2}\sin 2\theta\right]$$

Now $\quad x = a \cos 2\theta$

$$\frac{x}{a} = \cos 2\theta \Rightarrow 2\theta = \cos^{-1}\frac{x}{a} \Rightarrow \theta = \frac{1}{2}\cos^{-1}\frac{x}{a}$$

$$\sin 2\theta = \sqrt{1 - \cos^2 2\theta} = \sqrt{1 - \frac{x^2}{a^2}} = \frac{\sqrt{a^2 - x^2}}{a}$$

$$\therefore \quad I = -2a\left[\frac{1}{2}\cos^{-1}\frac{x}{a} + \frac{1}{2}\frac{\sqrt{a^2 - x^2}}{a}\right] + C_1$$

$$= -a\cos^{-1}\frac{x}{a} - \sqrt{a^2 - x^2} + C_1$$

$$= -a\left[\frac{\pi}{2} - \sin^{-1}\frac{x}{a}\right] - \sqrt{a^2 - x^2} + C_1$$

$$= \frac{-\pi a}{2} + a\sin^{-1}\frac{x}{a} - \sqrt{a^2 - x^2} + C_1$$

$$= a\sin^{-1}\frac{x}{a} - \sqrt{a^2 - x^2} + \left(C_1 - \frac{\pi a}{2}\right)$$

$$= a\sin^{-1}\frac{x}{a} - \sqrt{a^2 - x^2} + C \qquad \left[C = \left(C_1 - \frac{\pi a}{2}\right)\right]$$

Hence, $\mathrm{I} = a\sin^{-1}\frac{x}{a} - \sqrt{a^2 - x^2} + C$

Q12. $\displaystyle\int \frac{x^{1/2}}{1 + x^{3/4}}\,dx$ $\qquad$ (Hint: Put $x = z^4$)

Sol. Let $\quad \mathrm{I} = \displaystyle\int \frac{x^{1/2}}{1 + x^{3/4}}\,dx$

Put $\quad x = t^4 \Rightarrow dx = 4t^3\,dt$

$$t^3 + 1)\ t^5\ (t^2$$
$$t^5 + t^2$$
$$(-)\ \ (-)$$
$$-t^2$$

$$= \int \frac{t^2 . 4t^3}{1 + t^3}\,dt = 4\int \frac{t^5}{1 + t^3}\,dt$$

$$= 4\int\left(t^2 - \frac{t^2}{t^3 + 1}\right)dt = 4\int t^2\,dt - 4\int \frac{t^2}{t^3 + 1}.dt$$

$$\mathrm{I} = \mathrm{I}_1 - \mathrm{I}_2$$

Now $\quad \mathrm{I}_1 = 4\displaystyle\int t^2\,dt = 4.\frac{t^3}{3} + C_1 = \frac{4}{3}x^{3/4} + C_1$

$$\mathrm{I}_2 = 4\int \frac{t^2}{t^3 + 1}\,dt$$

Put $t^3 + 1 = z \Rightarrow 3t^2\,dt = dz$

$$t^2\,dt = \frac{1}{3}dz$$

$$\therefore \quad \mathrm{I}_2 = \frac{4}{3}\int \frac{dz}{z} = \frac{4}{3}\log|z| + C_2 = \frac{4}{3}\log\left|t^3 + 1\right| + C_2$$

$$= \frac{4}{3}\log\left|(x)^{3/4} + 1\right| + C_2$$

$$\therefore \quad \mathrm{I} = \mathrm{I}_1 - \mathrm{I}_2$$

$$= \frac{4}{3}x^{3/4} + C_1 - \frac{4}{3}\log\left|(x)^{3/4} + 1\right| - C_2$$

$$= \frac{4}{3}\left[x^{3/4} - \log\left|(x)^{3/4} + 1\right|\right] + C_1 - C_2$$

Hence, $I = \frac{4}{3}\left[x^{3/4} - \log\left|(x)^{3/4} + 1\right|\right] + C \qquad [\because C = C_1 - C_2]$

Q13. $\int \frac{\sqrt{1+x^2}}{x^4}\,dx$

Sol. Let $I = \int \frac{\sqrt{1+x^2}}{x^4}\,dx = \int \sqrt{\frac{1+x^2}{x^2}}\cdot\frac{1}{x^3}\,dx = \int \sqrt{\frac{1}{x^2}+1}\cdot\frac{1}{x^3}\,dx$

Put $\frac{1}{x^2} + 1 = t^2$

$\frac{-2}{x^3}\,dx = 2t\,dt \Rightarrow \frac{dx}{x^3} = -t\,dt$

$\therefore \quad I = \int t(\ t\,dt) = -\int t^2\,dt = -\frac{1}{3}t^3 + C$

Hence, $I = -\frac{1}{3}\left(\frac{1}{x^2}+1\right)^{3/2} + C$

Q14. $\int \frac{dx}{\sqrt{16-9x^2}}$

Sol. Let $I = \int \frac{dx}{\sqrt{16-9x^2}}$

$$= \frac{1}{3}\int \frac{dx}{\sqrt{\frac{16}{9}-x^2}} = \frac{1}{3}\int \frac{dx}{\sqrt{\left(\frac{4}{3}\right)^2 - x^2}}$$

$$= \frac{1}{3}\sin^{-1}\frac{x}{4/3} + C \qquad \left[\because \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C\right]$$

$$= \frac{1}{3}\sin^{-1}\frac{3x}{4} + C$$

Hence, $I = \frac{1}{3}\sin^{-1}\frac{3x}{4} + C.$

Q15. $\int \frac{dt}{\sqrt{3t-2t^2}}$

Sol. Let $I = \int \frac{dt}{\sqrt{3t-2t^2}} = \int \frac{dt}{\sqrt{-2\left(t^2 - \frac{3}{2}t\right)}}$

$$= \frac{1}{\sqrt{2}}\int \frac{dt}{\sqrt{-\left(t^2 - \frac{3}{2}t + \frac{9}{16} - \frac{9}{16}\right)}} \qquad \text{[Making perfect square]}$$

$$= \frac{1}{\sqrt{2}}\int \frac{dt}{\sqrt{-\left[\left(t-\frac{3}{4}\right)^2 - \frac{9}{16}\right]}} = \frac{1}{\sqrt{2}}\int \frac{dt}{\sqrt{\frac{9}{16} - \left(t-\frac{3}{4}\right)^2}}$$

$$= \frac{1}{\sqrt{2}}\int \frac{dt}{\sqrt{\left(\frac{3}{4}\right)^2 - \left(t-\frac{3}{4}\right)^2}} = \frac{1}{\sqrt{2}}.\sin^{-1}\frac{t-\frac{3}{4}}{\frac{3}{4}} + C$$

$$= \frac{1}{\sqrt{2}}\sin^{-1}\frac{4t-3}{3} + C$$

Hence, $\mathrm{I} = \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{4t-3}{3}\right) + C.$

Q16. $\int \frac{3x-1}{\sqrt{x^2+9}}\,dx$

Sol. Let $\mathrm{I} = \int \frac{3x-1}{\sqrt{x^2+9}}\,dx = \int \frac{3x}{\sqrt{x^2+9}}\,dx - \int \frac{1}{\sqrt{x^2+9}}\,dx$

$\mathrm{I} = \mathrm{I}_1 - \mathrm{I}_2$

Now $\mathrm{I}_1 = \int \frac{3x}{\sqrt{x^2+9}}\,dx$

Put $x^2+9 = t \Rightarrow 2x\,dx = dt$

$x\,dx = -dt$

$$\therefore \mathrm{I}_1 = \frac{3}{2}\int \frac{dt}{\sqrt{t}} = \frac{3}{2}.2\sqrt{t} + C_1 = 3\sqrt{x^2+9} + C_1$$

$$\mathrm{I}_2 = \int \frac{1}{\sqrt{x^2+9}}\,dx = \int \frac{1}{\sqrt{x^2+(3)^2}}\,dx = \log\left|x + \sqrt{x^2+(3)^2}\right| + C_2$$

$$\left[\because \int \frac{1}{\sqrt{x^2+a^2}}\,dx = \log\left|x+\sqrt{x^2+a^2}\right| + C\right]$$

$$= \log\left|x + \sqrt{x^2+9}\right| + C_2$$

$\therefore \mathrm{I} = \mathrm{I}_1 - \mathrm{I}_2$

$$= 3\sqrt{x^2+9} + C_1 - \log\left|x + \sqrt{x^2+9}\right| - C_2$$

$$= 3\sqrt{x^2+9} - \log\left|x + \sqrt{x^2+9}\right| + (C_1 - C_2)$$

Hence, $\mathrm{I} = 3\sqrt{x^2+9} - \log\left|x + \sqrt{x^2+9}\right| + C$

Q17. $\int \sqrt{5-2x+x^2}\,dx$

Sol. Let $I = \int \sqrt{5-2x+x^2}\,dx = \int \sqrt{x^2-2x+5}\,dx$

$$= \int \sqrt{x^2-2x+1-1+5}\,dx \qquad \text{(Making perfect square)}$$

$$= \int \sqrt{(x-1)^2+4}\,dx = \int \sqrt{(x-1)^2+(2)^2}\,dx$$

$$= \frac{x-1}{2}\sqrt{(x-1)^2+(2)^2} + \frac{4}{2}\log\left|(x-1)+\sqrt{(x-1)^2+(2)^2}\right| + C$$

$$\left[\because \int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\left\{\log\left|x+\sqrt{x^2+a^2}\right|\right\} + C\right]$$

$$= \frac{x-1}{2}\sqrt{x^2+1-2x+4} + 2\log\left|(x-1)+\sqrt{x^2+1-2x+4}\right| + C$$

$$= \frac{x-1}{2}\sqrt{x^2-2x+5} + 2\log\left|(x-1)+\sqrt{x^2-2x+5}\right| + C$$

Hence,

$$I = \frac{x-1}{2}\sqrt{x^2-2x+5} + 2\log\left|(x-1)+\sqrt{x^2-2x+5}\right| + C$$

Q18. $\int \frac{x}{x^4-1}\,dx$

Sol. Let $\qquad I = \int \frac{x}{x^4-1}\,dx$

Put $\qquad x^2 = t \Rightarrow 2x\,dx = dt \Rightarrow x\,dx = \frac{dt}{2}$

$$\frac{1}{2}\int \frac{dt}{t^2-1} = \frac{1}{2}\int \frac{dt}{t^2-(1)^2} = \frac{1}{2}\cdot\frac{1}{2\cdot 1}\log\left|\frac{t-1}{t+1}\right| + C$$

$$\left[\because \int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C\right]$$

$$= \frac{1}{4}\log\left|\frac{x^2-1}{x^2+1}\right| + C$$

Hence, $I = \frac{1}{4}\log\left|\frac{x^2-1}{x^2+1}\right| + C.$

Q19. $\int \frac{x^2}{1-x^4}\,dx$ (Put $x^2 = t$)

Sol. Let $\qquad I = \int \frac{x^2}{1-x^4}\,dx = \int \frac{x^2}{(1-x^2)(1+x^2)}\,dx$

Put $x^2 = t$ for the purpose of partial fractions.

We get $\dfrac{t}{(1-t)(1+t)}$

Resolving into partial fractions we put

$$\frac{t}{(1-t)(1+t)} = \frac{A}{1-t} + \frac{B}{1+t}$$

[where A and B are arbitrary constants]

$$\Rightarrow \quad \frac{t}{(1-t)(1+t)} = \frac{A(1+t)+B(1-t)}{(1-t)(1+t)}$$

$$\Rightarrow \quad t = A + At + B - Bt$$

Comparing the like terms, we get $\quad A - B = 1$ and $A + B = 0$

Solving the above equations, we have $A = \dfrac{1}{2}$ and $B = -\dfrac{1}{2}$

$$\therefore \quad I = \int \frac{1/2}{1-x^2}\,dx + \int \frac{-1/2}{1+x^2}\,dx \qquad \text{(Putting } t = x^2\text{)}$$

$$= \frac{1}{2}\cdot\frac{1}{2.1}\log\left|\frac{1+x}{1-x}\right| - \frac{1}{2}\tan^{-1}x + C$$

$$= \frac{1}{4}\log\left|\frac{1+x}{1-x}\right| - \frac{1}{2}\tan^{-1}x + C$$

Hence, $\quad I = \dfrac{1}{4}\log\left|\dfrac{1+x}{1-x}\right| - \dfrac{1}{2}\tan^{-1}x + C.$

Q20. $\int \sqrt{2ax - x^2}\,dx$

Sol. Let $\quad I = \int \sqrt{2ax - x^2}\,dx$

$$= \int \sqrt{-(x^2 - 2ax)}\,dx = \int \sqrt{-(x^2 - 2ax + a^2 - a^2)}\,dx$$

$$= \int \sqrt{-\left[(x-a)^2 - a^2\right]}\,dx = \int \sqrt{a^2 - (x-a)^2}\,dx$$

$$= \frac{x-a}{2}\sqrt{a^2-(x-a)^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right) + C$$

$$\left[\because \int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C\right]$$

$$= \frac{x-a}{2}\sqrt{a^2-(x^2-2ax+a^2)} + \frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right) + C$$

$$= \frac{x-a}{2}\sqrt{2ax-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right) + C$$

Hence, $I = \frac{x-a}{2}\sqrt{2ax-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right) + C.$

Q21. $\int \frac{\sin^{-1} x}{(1-x^2)^{3/2}} dx$

Sol. Let $I = \int \frac{\sin^{-1} x}{(1-x^2)^{3/2}} dx$

Put $x = \sin\theta \Rightarrow dx = \cos\theta\, d\theta$

$$I = \int \frac{\sin^{-1}(\sin\theta)}{(1-\sin^2\theta)^{3/2}} . \cos\theta\, d\theta$$

$$= \int \frac{\theta.\cos\theta\, d\theta}{(\cos^2\theta)^{3/2}} = \int \frac{\theta.\cos\theta}{\cos^3\theta} d\theta$$

$$= \int \frac{\theta}{\cos^2\theta} d\theta = \int \underset{\mathrm{I}}{\theta} \underset{\mathrm{II}}{\sec^2\theta}\, d\theta$$

$$= \theta.\int \sec^2\theta\, d\theta - \int \left(D(\theta).\int \sec^2\theta\, d\theta\right) d\theta$$

$$\left[\because \int \underset{\mathrm{I}}{u}.\underset{\mathrm{II}}{v}\, dx = u.\int v\, dx - \int \left(D(u)\int v\, dv\right) dv + C\right]$$

$$= \theta.\tan\theta - \int 1.\tan\theta\, d\theta$$

$$= \theta.\tan\theta - \log\sec\theta + C$$

$$= \sin^{-1} x.\frac{x}{\sqrt{1-x^2}} - \log\left|\sqrt{1-x^2}\right| + C$$

$$\left[\begin{array}{l}\text{when } x = \sin\theta \\ \therefore \tan\theta = \frac{x}{\sqrt{1-x^2}} \text{ and } \sec\theta = \sqrt{1-x^2}\end{array}\right]$$

Hence, $I = \frac{x\sin^{-1} x}{\sqrt{1-x^2}} - \log\left|\sqrt{1-x^2}\right| + C$

Q22. $\int \frac{(\cos 5x + \cos 4x)}{1-2\cos 3x} dx$

Sol. Let $I = \int \frac{\cos 5x + \cos 4x}{1-2\cos 3x} dx = \int \frac{2\cos\frac{5x+4x}{2}.\cos\frac{5x-4x}{2}}{1-2\left(2\cos^2\frac{3x}{2}-1\right)} dx$

$$= \int \frac{2\cos\frac{9x}{2}.\cos\frac{x}{2}}{1-4\cos^2\frac{3x}{2}+2} dx = \int \frac{2\cos\frac{9x}{2}.\cos\frac{x}{2}}{3-4\cos^2\frac{3x}{2}} dx$$

$$= -\int \frac{2\cos\frac{9x}{2}.\cos\frac{x}{2}}{4\cos^2\frac{3x}{2} - 3}\,dx = -\int \frac{2\cos\frac{9x}{2}.\cos\frac{x}{2}.\cos\frac{3x}{2}}{4\cos^3\frac{3x}{2} - 3\cos\frac{3x}{2}}\,dx$$

$\left[\text{Multiplying and dividing by } \cos\frac{3x}{2}\right]$

$$= -\int \frac{2\cos\frac{9x}{2}.\cos\frac{x}{2}.\cos\frac{3x}{2}}{\cos 3.\frac{3x}{2}}\,dx$$

$[\because \cos 3x = 4\cos^3 x - 3\cos x]$

$$= -\int \frac{2\cos\frac{9x}{2}.\cos\frac{x}{2}.\cos\frac{3x}{2}}{\cos\frac{9x}{2}}\,dx = -\int 2\cos\frac{3x}{2}.\cos\frac{x}{2}\,dx$$

$$= -\int\left[\cos\left(\frac{3x}{2} + \frac{x}{2}\right) + \cos\left(\frac{3x}{2} - \frac{x}{2}\right)\right]dx$$

$$= -\int (\cos 2x + \cos x)\,dx$$

$[\because 2\cos A\cos B = \cos(A+B) + \cos(A-B)]$

$$= -\int \cos 2x\,dx - \int \cos x\,dx = -\frac{1}{2}\sin 2x - \sin x + C$$

Hence, $I = -\left[\frac{1}{2}\sin 2x + \sin x\right] + C.$

Q23. $\int \frac{\sin^6 x + \cos^6 x}{\sin^2 x.\cos^2 x}\,dx$

Sol. Let $I = \int \frac{\sin^6 x + \cos^6 x}{\sin^2 x.\cos^2 x}\,dx = \int \frac{(\sin^2 x)^3 + (\cos^2 x)^3}{\sin^2 x.\cos^2 x}\,dx$

$$= \int \frac{(\sin^2 x + \cos^2 x)^3 - 3\sin^2 x\cos^2 x(\sin^2 x + \cos^2 x)}{\sin^2 x.\cos^2 x}\,dx$$

$[\because a^3 + b^3 = (a+b)^3 - 3ab\,(a+b)]$

$$= \int \frac{(1)^3 - 3\sin^2 x\cos^2 x.(1)}{\sin^2 x\cos^2 x}\,dx$$

$$= \int \frac{1 - 3\sin^2 x\cos^2 x}{\sin^2 x\cos^2 x}\,dx$$

$$= \int\left(\frac{1}{\sin^2 x\cos^2 x} - \frac{3\sin^2 x\cos^2 x}{\sin^2 x\cos^2 x}\right)dx$$

$$= \int\left(\frac{1}{\sin^2 x\cos^2 x} - 3\right)dx = \int\left(\frac{\sin^2 x + \cos^2 x}{\sin^2 x\cos^2 x} - 3\right)dx$$

$$= \int\left[\left(\frac{1}{\cos^2 x} + \frac{1}{\sin^2 x}\right) - 3\right]dx$$

$$= \int(\sec^2 x + \operatorname{cosec}^2 x - 3)\,dx$$

$$= \int\sec^2 x\,dx + \int\operatorname{cosec}^2 x\,dx - 3\int 1\,dx$$

$$= \tan x - \cot x - 3x + C$$

Hence, $I = \tan x - \cot x - 3x + C.$

Q24. $\displaystyle\int\frac{\sqrt{x}}{\sqrt{a^3 - x^3}}dx$

Sol. Let $\displaystyle I = \int\frac{\sqrt{x}}{\sqrt{a^3 - x^3}}dx = \int\frac{x^{1/2}}{\sqrt{(a^{3/2})^2 - (x^{3/2})^2}}dx$

Put $\displaystyle x^{3/2} = t \Rightarrow \frac{3}{2}x^{1/2}\,dx = dt \Rightarrow x^{1/2}\,dx = \frac{2}{3}dt$

$$\therefore \quad I = \frac{2}{3}\int\frac{dt}{\sqrt{(a^{3/2})^2 - (t)^2}}$$

$$= \frac{2}{3}\sin^{-1}\frac{t}{a^{3/2}} + C = \frac{2}{3}\sin^{-1}\left(\frac{x^{3/2}}{a^{3/2}}\right) + C$$

Hence, $\displaystyle I = \frac{2}{3}\sin^{-1}\left(\frac{x}{a}\right)^{3/2} + C.$

Q25. $\displaystyle\int\frac{\cos x - \cos 2x}{1 - \cos x}dx$

Sol. Let $\displaystyle I = \int\frac{\cos x - \cos 2x}{1 - \cos x}dx$

$$= \int\frac{2\sin\frac{x + 2x}{2}.\sin\left(\frac{2x - x}{2}\right)}{2\sin^2 x/2}dx$$

$$\left[\because \cos C - \cos D = 2\sin\frac{C + D}{2}.\sin\frac{D - C}{2}\right]$$

$$= \int\frac{2\sin\frac{3x}{2}.\sin\frac{x}{2}}{2\sin^2\frac{x}{2}}dx = \int\frac{\sin\frac{3x}{2}}{\sin\frac{x}{2}}dx = \int\frac{\sin 3\left(\frac{x}{2}\right)}{\sin\frac{x}{2}}dx$$

$$= \int\frac{3\sin\frac{x}{2} - 4\sin^3\frac{x}{2}}{\sin\frac{x}{2}}dx \quad [\sin 3x = 3\sin x - 4\sin^3 x]$$

$$= \int \frac{\sin\frac{x}{2}\left(3 - 4\sin^2\frac{x}{2}\right)}{\sin\frac{x}{2}} dx = \int\left(3 - 4\sin^2\frac{x}{2}\right) dx$$

$$= \int [3 - 2(1 - \cos x)]\, dx \qquad \left[\because 2\sin^2\frac{x}{2} = 1 - \cos x\right]$$

$$= \int (3 - 2 + 2\cos x)\, dx = \int (1 + 2\cos x)\, dx$$

$$= x + 2\sin x + C$$

Hence, $I = x + 2\sin x + C$.

Q26. $\displaystyle\int \frac{dx}{x\sqrt{x^4 - 1}}$ (Hint: Put $x^2 = \sec\theta$)

Sol. Let $\displaystyle I = \int \frac{dx}{x\sqrt{x^4 - 1}} = \int \frac{x\, dx}{x^2\sqrt{x^4 - 1}}$

Put $x^2 = \sec\theta$

$\therefore \quad 2x\, dx = \sec\theta \tan\theta\, d\theta$

$$x\, dx = \frac{1}{2}\sec\theta\tan\theta\, d\theta$$

$$\therefore \quad I = \frac{1}{2}\int \frac{\sec\theta\tan\theta}{\sec\theta\sqrt{\sec^2\theta - 1}} d\theta$$

$$= \frac{1}{2}\int \frac{\sec\theta\tan\theta}{\sec\theta.\tan\theta} d\theta = \frac{1}{2}\int 1\, d\theta = \frac{1}{2}\theta + C$$

So $\displaystyle I = \frac{1}{2}\sec^{-1}x^2 + C$

Hence, $\displaystyle I = \frac{1}{2}\sec^{-1}x^2 + C.$

Evaluate the following as a limit of sum:

Q27. $\displaystyle\int_0^2 (x^2 + 3)\, dx$

Sol. Let $\displaystyle I = \int_0^2 (x^2 + 3)\, dx$

Using the formula,

$$\int_a^b f(x)\, dx = \lim_{h \to 0} h\left[f(a) + f(a+h) + f(a+2h) + \ldots + f(a + \overline{n-1}h)\right]$$

where $h = \dfrac{b-a}{n}$

Here, $a = 0$ and $b = 2$

$\therefore \qquad h = \dfrac{2-0}{n} \quad \therefore nh = 2$

Here, $\quad f(x) = x^2 + 3$

$f(0) = 0 + 3 = 3$

$f(0+h) = (0+h)^2 + 3 = h^2 + 3$

$f(0+2h) = (0+2h)^2 + 3 = 4h^2 + 3$

....

....

$f(0+\overline{n-1}h) = (0+\overline{n-1}h)^2 + 3(n-1)^2h^2 + 3$

Now

$$\int_0^2 (x^2+3)\,dx$$

$$= \lim_{h\to 0} h\left[3 + h^2 + 3 + 4h^2 + 3 + \ldots + (n-1)^2h^2 + 3\right]$$

$$= \lim_{h\to 0} h\left[(3+3+3+\ldots+n) + \left\{h^2 + 4h^2 + \ldots + (n-1)^2h^2\right\}\right]$$

$$= \lim_{h\to 0} h\left[3n + h^2\left\{1 + 4 + \ldots + (n-1)^2\right\}\right]$$

$$= \lim_{h\to 0} h\left[3n + h^2\,\frac{n(n-1)(2n-1)}{6}\right]$$

$$\left[\because 1 + 4 + 9 + \ldots + (n-1)^2 = \frac{n(n-1)(2n-1)}{6}\right]$$

$$= \lim_{h\to 0}\left[3nh + \frac{h^3 n(n-1)(2n-1)}{6}\right]$$

$$= \lim_{h\to 0}\left[3nh + \frac{nh(nh-h)(2nh-h)}{6}\right]$$

$$= \left[3\times 2 + \frac{2(2-0)\,(2\times 2-0)}{6}\right] \qquad \left[\begin{aligned}\because nh &= 2\\ h &= 0\end{aligned}\right]$$

$$= \left[6 + \frac{2\times 2\times 4}{6}\right] = 6 + \frac{8}{3} = \frac{26}{3}$$

Hence, $\displaystyle\int_0^2 (x^2+3)\,dx = \frac{26}{3}$.

Q28. $\displaystyle\int_0^2 e^x\,dx$

Sol. Let $\qquad I = \displaystyle\int_0^2 e^x\,dx$

Here, $a = 0$ and $b = 2$ $\therefore h = \dfrac{b-a}{n} \Rightarrow h = \dfrac{2-0}{n}$ $\therefore nh = 2$

Here
$$\begin{aligned} f(x) &= e^x \\ f(0) &= e^0 = 1 \\ f(0+h) &= e^{0+h} = e^h \\ f(0+2h) &= e^{0+2h} = e^{2h} \end{aligned}$$

....

....

$$f(0+\overline{n-1}h) = e^{0+(n-1)h} = e^{(n-1)h}$$

Using $\int_a^b f(x)\,dx$

$$= \lim_{h \to 0} h\left[f(a) + f(a+h) + f(a+2h) + \ldots + f(a+\overline{n-1}h)\right]$$

$$\therefore \int_0^2 e^x\,dx = \lim_{h \to 0} h\left[1 + e^h + e^{2h} + \ldots + e^{(n-1)h}\right]$$

$$= \lim_{h \to 0} h\left[\frac{1(e^{nh}-1)}{e^h - 1}\right]$$

$$\left[\because a + ar + ar^2 + \ldots + ar^{n-1} = \frac{a(r^n - 1)}{r-1}\right]$$

$$= \lim_{h \to 0} \frac{e^{nh}-1}{\frac{e^h - 1}{h}} = \frac{e^2 - 1}{1} = e^2 - 1 \left[\because \lim_{x \to 0} \frac{e^x - 1}{x} = 1\right]$$

Hence, $I = e^2 - 1$.

Evaluate the following:

Q29. $\displaystyle\int_0^1 \frac{dx}{e^x + e^{-x}}$

Sol. Let $\quad I = \displaystyle\int_0^1 \frac{dx}{e^x + e^{-x}}$

$$= \int_0^1 \frac{dx}{e^x + \frac{1}{e^x}} = \int_0^1 \frac{dx}{\frac{e^{2x}+1}{e^x}} = \int_0^1 \frac{e^x\,dx}{e^{2x}+1}$$

Put $\quad e^x = t \Rightarrow e^x\,dx = dt$

Changing the limit, we have

When $x = 0$ $\quad \therefore t = e^0 = 1$

When $x = 1$ $\quad \therefore t = e^1 = e$

$$\therefore I = \int_1^e \frac{dt}{t^2+1} = [\tan^{-1} t]_1^e = \left[\tan^{-1} e - \tan^{-1}(1)\right] = \tan^{-1} e - \frac{\pi}{4}$$

Hence, $I = \tan^{-1} e - \frac{\pi}{4}$.

Q30. $\displaystyle\int_0^{\pi/2} \frac{\tan x}{1+m^2 \tan^2 x} dx$

Sol. Let $\displaystyle I = \int_0^{\pi/2} \frac{\tan x}{1+m^2 \tan^2 x} dx$

$$= \int_0^{\pi/2} \frac{\frac{\sin x}{\cos x}}{1+m^2 \frac{\sin^2 x}{\cos^2 x}} dx = \int_0^{\pi/2} \frac{\frac{\sin x}{\cos x}}{\frac{\cos^2 x + m^2 \sin^2 x}{\cos^2 x}} dx$$

$$= \int_0^{\pi/2} \frac{\sin x \cos x}{\cos^2 x + m^2 \sin^2 x} dx = \int_0^{\pi/2} \frac{\sin x \cos x}{1-\sin^2 x + m^2 \sin^2 x} dx$$

$$= \int_0^{\pi/2} \frac{\sin x \cos x}{1-\sin^2 x\,(1-m^2)} dx$$

Put $\sin^2 x = t$

$2 \sin x \cos x\, dx = dt$

$\sin x \cos x\, dx = \frac{dt}{2}$

Changing the limits we get,

When $x = 0$ $\therefore t = \sin^2 0 = 0$; When $x = \frac{\pi}{2}$ $\therefore$ $t = \sin^2 \frac{\pi}{2} = 1$

$$\therefore I = \frac{1}{2}\int_0^1 \frac{dt}{1-(1-m^2)t}$$

$$I = \frac{1}{2}\int_0^1 \frac{dt}{1+(m^2-1)t} = \frac{1}{2}\left[\frac{\log[1+(m^2-1)t]}{m^2-1}\right]_0^1$$

$$= \frac{1}{2(m^2-1)}\left[\log(1+m^2-1) - \log(1)\right] = \frac{\log|m^2|}{2(m^2-1)}$$

Hence, $I = \frac{\log|m^2|}{2(m^2-1)} = \frac{\log|m|}{m^2-1}$.

Q31. $\displaystyle\int_1^2 \frac{dx}{\sqrt{(x-1)(2-x)}}$

Sol. Let $\displaystyle I = \int_1^2 \frac{dx}{\sqrt{(x-1)(2-x)}}$

$$= \int_1^2 \frac{dx}{\sqrt{2x - x^2 - 2 + x}} = \int_1^2 \frac{dx}{\sqrt{-x^2 + 3x - 2}}$$

$$= \int_1^2 \frac{dx}{\sqrt{-(x^2 - 3x + 2)}}$$

$$= \int_1^2 \frac{dx}{\sqrt{-\left(x^2 - 3x + \frac{9}{4} - \frac{9}{4} + 2\right)}}$$ [Making perfect square]

$$= \int_1^2 \frac{dx}{\sqrt{-\left[\left(x - \frac{3}{2}\right)^2 - \frac{1}{4}\right]}} = \int_1^2 \frac{dx}{\sqrt{\frac{1}{4} - \left(x - \frac{3}{2}\right)^2}}$$

$$= \int_1^2 \frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2 - \left(x - \frac{3}{2}\right)^2}} = \left[\sin^{-1}\left(\frac{x - \frac{3}{2}}{\frac{1}{2}}\right)\right]_1^2$$

$$= \left[\sin^{-1}\left(\frac{2x-3}{1}\right)\right]_1^2 = \sin^{-1}(4-3) - \sin^{-1}(2-3)$$

$$= \sin^{-1}(1) - \sin^{-1}(-1) = \sin^{-1}(1) + \sin^{-1}(1)$$

$$= 2\sin^{-1}(1) = 2 \times \frac{\pi}{2} = \pi$$

Hence, $I = \pi$.

Q32. $\displaystyle\int_0^1 \frac{x\,dx}{\sqrt{1+x^2}}$

Sol. Let $\displaystyle I = \int_0^1 \frac{x\,dx}{\sqrt{1+x^2}}$

Put $1 + x^2 = t \Rightarrow 2x\,dx = dt \Rightarrow x\,dx = \dfrac{dt}{2}$

Changing the limits, we have

When $x = 0 \quad \therefore t = 1$

When $x = 1 \quad \therefore t = 2$

$$\therefore \quad I = \frac{1}{2}\int_1^2 \frac{dt}{\sqrt{t}} = \frac{1}{2}.2\left[t^{1/2}\right]_1^2 = \sqrt{2} - 1$$

Hence, $I = \sqrt{2} - 1$.

Q33. $\displaystyle\int_0^{\pi} x \sin x \cos^2 x\,dx$

Sol. Let $$I = \int_0^{\pi} x \sin x \cos^2 x \, dx \quad ...(i)$$

$$I = \int_0^{\pi} (\pi - x) \sin(\pi - x) \cos^2(\pi - x) \, dx$$

$$I = \int_0^{\pi} (\pi - x) \sin x \cos^2 x \, dx \quad ...(ii)$$

Adding (*i*) and (*ii*) we get,

$$2I = \int_0^{\pi} \left[x \sin x \cos^2 x + (\pi - x) \sin x \cos^2 x\right] dx$$

$$2I = \int_0^{\pi} \sin x \cos^2 x . (x + \pi - x) \, dx$$

$$2I = \int_0^{\pi} \pi \sin x \cos^2 x \, dx = \pi \int_0^{\pi} \sin x \cos^2 x \, dx$$

Put $\cos x = t \Rightarrow -\sin x \, dx = dt \Rightarrow \sin x \, dx = -dt$

Changing the limits, we have

When $x = 0$, $t = \cos 0 = 1$; When $x = \pi$, $t = \cos \pi = -1$

$$2I = \pi \int_1^{-1} -t^2 \, dt = -\pi \int_1^{-1} t^2 \, dt$$

$$2I = \pi \int_{-1}^{1} t^2 \, dt \qquad \left[\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx\right]$$

$$2I = \pi \left[\frac{t^3}{3}\right]_{-1}^{1} = \pi\left[\frac{1}{3} + \frac{1}{3}\right] = \pi\left(\frac{2}{3}\right)$$

$$\therefore \quad I = \frac{\pi}{3}$$

Q34. $\displaystyle\int_0^{1/2} \frac{dx}{(1 + x^2)\sqrt{1 - x^2}}$ (Hint: Let $x = \sin\theta$)

Sol. Let $$I = \int_0^{1/2} \frac{dx}{(1 + x^2)\sqrt{1 - x^2}}$$

Put $x = \sin\theta$

$\therefore$ $dx = \cos\theta \, d\theta$

Changing the limits, we get

When $x = 0$ $\therefore \sin\theta = 0$ $\therefore \theta = 0$

When $x = \frac{1}{2}$ $\therefore \sin\theta = \frac{1}{2}$ $\therefore \theta = \frac{\pi}{6}$

$$\therefore \quad I = \int_0^{\pi/6} \frac{\cos\theta\, d\theta}{(1+\sin^2\theta)\sqrt{1-\sin^2\theta}}$$

$$= \int_0^{\pi/6} \frac{\cos\theta\, d\theta}{(1+\sin^2\theta)\cos\theta} = \int_0^{\pi/6} \frac{1}{1+\sin^2\theta} d\theta$$

Now, dividing the numerator and denominator by $\cos^2\theta$, we get

$$= \int_0^{\pi/6} \frac{\frac{1}{\cos^2\theta}}{\frac{1}{\cos^2\theta}+\frac{\sin^2\theta}{\cos^2\theta}} d\theta = \int_0^{\pi/6} \frac{\sec^2\theta}{\sec^2\theta+\tan^2\theta} d\theta$$

$$= \int_0^{\pi/6} \frac{\sec^2\theta}{1+\tan^2\theta+\tan^2\theta} d\theta = \int_0^{\pi/6} \frac{\sec^2\theta}{2\tan^2\theta+1} d\theta$$

Put $\quad \tan\theta = t$

$\therefore \quad \sec^2\theta\, d\theta = dt$

Changing the limits, we get

When $\theta = 0 \quad \therefore t = \tan 0 = 0$

When $\theta = \frac{\pi}{6} \quad \therefore t = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$

$$\therefore \quad I = \int_0^{1/\sqrt{3}} \frac{dt}{2t^2+1} = \frac{1}{2}\int_0^{1/\sqrt{3}} \frac{dt}{t^2+\frac{1}{2}} = \frac{1}{2}\int_0^{1/\sqrt{3}} \frac{dt}{t^2+\left(\frac{1}{\sqrt{2}}\right)^2}$$

$$= \frac{1}{2}\times\frac{1}{1/\sqrt{2}}\left[\tan^{-1}\frac{t}{1/\sqrt{2}}\right]_0^{1/\sqrt{3}} = \frac{1}{\sqrt{2}}\tan^{-1}\left[\sqrt{2}\,t\right]_0^{1/\sqrt{3}}$$

$$= \frac{1}{\sqrt{2}}\left[\tan^{-1}\frac{\sqrt{2}}{\sqrt{3}} - \tan^{-1}0\right] = \frac{1}{\sqrt{2}}\tan^{-1}\sqrt{\frac{2}{3}}$$

LONG ANSWER TYPE QUESTIONS

Q35. $\int \frac{x^2}{x^4-x^2-12} dx$

Sol. Let $I = \int \frac{x^2}{x^4-x^2-12} dx = \int \frac{x^2}{x^4-4x^2+3x^2-12} dx$

$$= \int \frac{x^2}{x^2(x^2-4)+3(x^2-4)} dx = \int \frac{x^2}{(x^2-4)(x^2+3)} dx$$

Put $x^2 = t$ for the purpose of partial fraction.

We get $\frac{t}{(t-4)(t+3)}$

Let $\dfrac{t}{(t-4)(t+3)} = \dfrac{A}{t-4}+\dfrac{B}{t+3}$

[where A and B are arbitrary constants]

$$\frac{t}{(t-4)(t+3)} = \frac{A(t+3)+B(t-4)}{(t-4)(t+3)}$$

$\Rightarrow \qquad t = At+3A+Bt-4B$

Comparing the like terms, we get

$A+B=1 \quad$ and $\quad 3A-4B=0$

$\Rightarrow \qquad 3A = 4B$

$\therefore \qquad A = \dfrac{4}{3}B$

Now $\quad \dfrac{4}{3}B+B = 1$

$\dfrac{7}{3}B = 1 \quad \therefore B = \dfrac{3}{7}$ and $A = \dfrac{4}{3}\times\dfrac{3}{7} = \dfrac{4}{7}$

So, $\qquad A = \dfrac{4}{7}$ and $B = \dfrac{3}{7}$

$$\therefore \int \frac{x^2}{(x^2-4)(x^2+3)}dx$$

$$= \frac{4}{7}\int\frac{1}{x^2-4}dx+\frac{3}{7}\int\frac{1}{x^2+3}dx$$

$$= \frac{4}{7}\int\frac{1}{x^2-(2)^2}dx+\frac{3}{7}\int\frac{1}{x^2+(\sqrt{3})^2}dx$$

$$= \frac{4}{7}\times\frac{1}{2\times 2}\log\left|\frac{x-2}{x+2}\right|+\frac{3}{7}\times\frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}$$

$$= \frac{1}{7}\log\left|\frac{x-2}{x+2}\right|+\frac{\sqrt{3}}{7}\tan^{-1}\frac{x}{\sqrt{3}}+C$$

Hence, $\quad I = \dfrac{1}{7}\log\left|\dfrac{x-2}{x+2}\right|+\dfrac{\sqrt{3}}{7}\tan^{-1}\dfrac{x}{\sqrt{3}}+C.$

Q36. Evaluate: $\displaystyle\int\frac{x^2}{(x^2+a^2)(x^2+b^2)}dx$

Sol. Let $\quad I = \displaystyle\int\frac{x^2}{(x^2+a^2)(x^2+b^2)}dx$

Put $x^2 = t$ for the purpose of partial fraction.

We get $\dfrac{t}{(t+a^2)(t+b^2)}$

Put $\frac{t}{(t+a^2)(t+b^2)} = \frac{A}{t+a^2} + \frac{B}{t+b^2}$

$\Rightarrow \frac{t}{(t+a^2)(t+b^2)} = \frac{A(t+b^2)+B(t+a^2)}{(t+a^2)(t+b^2)}$

$\Rightarrow \quad t = At + Ab^2 + Bt + Ba^2$

Comparing the like terms, we get

$A + B = 1 \quad$ and $\quad Ab^2 + Ba^2 = 0$

$$A = \frac{-a^2}{b^2} B$$

$\therefore \quad \frac{-a^2}{b^2} B + B = 1$

$$B\left(\frac{-a^2}{b^2} + 1\right) = 1 \Rightarrow B\left(\frac{-a^2+b^2}{b^2}\right) = 1$$

$\Rightarrow \quad B = \frac{b^2}{b^2-a^2}$ and $A = \frac{-a^2}{b^2} \times \frac{b^2}{b^2-a^2} = \frac{a^2}{a^2-b^2}$

So $\quad A = \frac{a^2}{a^2-b^2}$ and $B = \frac{-b^2}{a^2-b^2}$

$$\therefore \int \frac{x^2}{(x^2+a^2)(x^2+b^2)} dx$$

$$= \frac{a^2}{a^2-b^2} \int \frac{1}{x^2+a^2} dx - \frac{b^2}{a^2-b^2} \int \frac{1}{x^2+b^2} dx$$

$$= \frac{a^2}{a^2-b^2} \times \frac{1}{a} \tan^{-1}\frac{x}{a} - \frac{b^2}{a^2-b^2} \cdot \frac{1}{b} \cdot \tan^{-1}\frac{x}{b}$$

$$= \frac{a}{a^2-b^2} \tan^{-1}\frac{x}{a} - \frac{b}{a^2-b^2} \tan^{-1}\frac{x}{b} + C$$

Hence, $\quad I = \frac{1}{a^2-b^2}\left[a \tan^{-1}\frac{x}{a} - b \tan^{-1}\frac{x}{b}\right] + C$

Q37. Evaluate: $\int_0^{\pi} \frac{x}{1+\sin x} dx$

Sol. Let $\quad I = \int_0^{\pi} \frac{x}{1+\sin x} dx \quad \ldots(i)$

$$= \int_0^{\pi} \frac{\pi - x}{1+\sin(\pi - x)} dx \quad \left[\text{using} \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^{\pi} \frac{\pi - x}{1+\sin x} dx \quad \ldots(ii)$$

Adding (*i*) and (*ii*), we get

$$2I = \int_0^{\pi}\left(\frac{x}{1+\sin x}+\frac{\pi-x}{1+\sin x}\right)dx$$

$$= \int_0^{\pi}\left(\frac{x+\pi-x}{1+\sin x}\right)dx = \int_0^{\pi}\frac{\pi}{1+\sin x}dx$$

$$= \pi\int_0^{\pi}\frac{1}{1+\sin x}dx = \pi\int_0^{\pi}\frac{1.(1-\sin x)}{(1+\sin x)(1-\sin x)}dx$$

$$= \pi\int_0^{\pi}\frac{1-\sin x}{1-\sin^2 x}dx = \pi\int_0^{\pi}\frac{1-\sin x}{\cos^2 x}dx$$

$$= \pi\int_0^{\pi}\left(\frac{1}{\cos^2 x}-\frac{\sin x}{\cos^2 x}\right)dx = \pi\int_0^{\pi}(\sec^2 x-\sec x\tan x)\,dx$$

$$= \pi\left[\tan x-\sec x\right]_0^{\pi} = \pi\left[(\tan\pi-\tan 0)-(\sec\pi-\sec 0)\right]$$

$$2I = \pi\left[0-(-1-1)\right] = \pi(2)$$

$\therefore \quad I = \pi$

Hence, $I = \pi$.

Q38. Evaluate: $\displaystyle\int\frac{2x-1}{(x-1)(x+2)(x-3)}dx$

Sol. Let $\displaystyle I = \int\frac{2x-1}{(x-1)(x+2)(x-3)}dx$

Resolving into partial fraction, we put

$$\frac{2x-1}{(x-1)(x+2)(x-3)} = \frac{A}{x-1}+\frac{B}{x+2}+\frac{C}{x-3}$$

$\Rightarrow \quad 2x-1 = A(x+2)(x-3)+B(x-1)(x-3)+C(x-1)(x+2)$

put $x=1$, $\quad 1 = A(3)(-2) \quad \Rightarrow A = -\dfrac{1}{6}$

put $x=-2$, $\quad -5 = B(-3)(-5) \quad \Rightarrow B = -\dfrac{1}{3}$

put $x=3$, $\quad 5 = C(2)(5) \quad \Rightarrow C = \dfrac{1}{2}$

$$\therefore \int\frac{2x-1}{(x-1)(x+2)(x-3)}dx$$

$$= -\frac{1}{6}\int\frac{1}{x-1}dx-\frac{1}{3}\int\frac{1}{x+2}dx+\frac{1}{2}\int\frac{1}{x-3}dx$$

$$= -\frac{1}{6}\log|x-1|-\frac{1}{3}\log|x+2|+\frac{1}{2}\log|x-3|+C$$

$$= -\log|x-1|^{1/6} - \log(x+2)^{1/3} + \log(x-3)^{1/2} + C$$

Hence, $\int \frac{2x-1}{(x-1)(x+2)(x-3)}\,dx = \log\left[\frac{\sqrt{x-3}}{(x-1)^{1/6}(x+2)^{1/3}}\right] + C.$

Q39. Evaluate: $\int e^{\tan^{-1}x}\left(\frac{1+x+x^2}{1+x^2}\right)dx$

Sol. Let $\quad I = \int e^{\tan^{-1}x}\left(\frac{1+x+x^2}{1+x^2}\right)dx$

Put $\quad \tan^{-1}x = t \Rightarrow \frac{1}{1+x^2}.dx = dt$

$$= \int e^t(1+\tan t+\tan^2 t)\,dt = \int e^t(\sec^2 t + \tan t)\,dt$$

Here $\quad f(t) = \tan t$

$\therefore \quad f'(t) = \sec^2 t$

$$= e^t.f(t) = e^t \tan t = e^{\tan^{-1}x}.x + C$$

$$\left[\because \int e^x[f(x)+f'(x)]\,dx = e^x f(x) + C\right]$$

Hence, $\quad I = e^{\tan^{-1}x}.x + C.$

Q40. Evaluate: $\int \sin^{-1}\sqrt{\frac{x}{a+x}}\,dx$ (Hint: Put $x = a\tan^2\theta$)

Sol. Let $\quad I = \int \sin^{-1}\sqrt{\frac{x}{a+x}}\,dx$

Put $\quad x = a\tan^2\theta$

$$dx = 2a\tan\theta\,.\,\sec^2\theta\,.\,d\theta$$

$$\therefore \quad I = \int \sin^{-1}\sqrt{\frac{a\tan^2\theta}{a+a\tan^2\theta}}.2a\tan\theta.\sec^2\theta\,d\theta$$

$$= \int \sin^{-1}\frac{\sqrt{a}\tan\theta}{\sqrt{a}\sec\theta}.2a\tan\theta.\sec\theta\,d\theta$$

$$= \int \sin^{-1}\left(\frac{\frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos\theta}}\right).2a\tan\theta.\sec^2\theta\,d\theta$$

$$= \int \sin^{-1}(\sin\theta).2a\tan\theta.\sec^2\theta\,d\theta$$

$$= 2a\int \theta\tan\theta.\sec^2\theta\,d\theta$$

$$= 2a\left[\theta\int \tan\theta.\sec^2\theta\, d\theta - \int\left[D(\theta).\int \tan\theta.\sec^2\theta\, d\theta\right]\right]$$

$$= 2a\left[\theta.\frac{\tan^2\theta}{2} - \int \frac{1.\tan^2\theta}{2}\, d\theta\right]$$

$$= 2a\left[\theta.\frac{\tan^2\theta}{2} - \frac{1}{2}\int(\sec^2\theta - 1)\, d\theta\right]$$

$$= 2a\left[\theta.\frac{\tan^2\theta}{2} - \frac{1}{2}(\tan\theta - \theta)\right]$$

$$= 2a\left[\theta.\frac{\tan^2\theta}{2} - \frac{1}{2}\tan\theta + \frac{1}{2}\theta\right]$$

$$= 2a\left[\tan^{-1}\sqrt{\frac{x}{a}}.\frac{x}{2a} - \frac{1}{2}\sqrt{\frac{x}{a}} + \frac{1}{2}\tan^{-1}\sqrt{\frac{x}{a}}\right] + C$$

$$= a\left[\frac{x}{a}\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{\frac{x}{a}} + \tan^{-1}\sqrt{\frac{x}{a}}\right] + C$$

Hence, $I = a\left[\frac{x}{a}\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{\frac{x}{a}} + \tan^{-1}\sqrt{\frac{x}{a}}\right] + C.$

Q41. Evaluate: $\int_{\pi/3}^{\pi/2} \frac{\sqrt{1+\cos x}}{(1-\cos x)^{5/2}}\, dx$

Sol. Let $I = \int_{\pi/3}^{\pi/2} \frac{\sqrt{1+\cos x}}{(1-\cos x)^{5/2}}\, dx = \int_{\pi/3}^{\pi/2} \frac{\sqrt{2\cos^2 x/2}}{(2\sin^2 x/2)^{5/2}}\, dx$

$$= \int_{\pi/3}^{\pi/2} \frac{\sqrt{2}\cos x/2}{(2)^{5/2}\sin^5 x/2}\, dx = \frac{1}{4}\int_{\pi/3}^{\pi/2} \frac{\cos x/2}{\sin^5 x/2}\, dx$$

Put $\sin\frac{x}{2} = t \Rightarrow \frac{1}{2}\cos\frac{x}{2}\, dx = dt \Rightarrow \cos\frac{x}{2}\, dx = 2dt$

Changing the limits, we have

When $x = \frac{\pi}{3}$, $\sin\frac{\pi}{6} = t \quad \therefore\ t = \frac{1}{2}$

When $x = \frac{\pi}{2}$, $\sin\frac{\pi}{4} = t \quad \therefore\ t = \frac{1}{\sqrt{2}}$

$$\therefore \quad I = \frac{1}{4}\times 2\int_{1/2}^{1/\sqrt{2}} \frac{dt}{t^5} = \frac{1}{2}\times\left(-\frac{1}{4}\right)[t^{-4}]_{1/2}^{1/\sqrt{2}} = -\frac{1}{8}\left[\frac{1}{t^4}\right]_{1/2}^{1/\sqrt{2}}$$

$$= -\frac{1}{8}\left[\frac{1}{(1/\sqrt{2})^4} - \frac{1}{(1/2)^4}\right] = -\frac{1}{8}[4-16]$$

$$= -\frac{1}{8}\times(-12) = \frac{3}{2}$$

Hence, $I = \frac{3}{2}$.

Q42. Evaluate: $\int e^{-3x}\cos^3 x\,dx$

Sol. Let $I = \int \underset{\text{II}}{e^{-3x}}\ \underset{\text{I}}{\cos^3} x\,dx$

$$= \cos^3 x.\int e^{-3x}\,dx - \int\left(D(\cos^3 x).\int e^{-3x}dx\right)dx$$

$$= \cos^3 x.\frac{e^{-3x}}{-3} - \int\left(3\cos^2 x.(-\sin x).\frac{e^{-3x}}{-3}\right)dx$$

$$= -\frac{1}{3}e^{-3x}\cos^3 x - \int \cos^2 x\sin x.e^{-3x}\,dx$$

$$= -\frac{1}{3}e^{-3x}\cos^3 x - \int (1-\sin^2 x)\sin x.e^{-3x}\,dx$$

$$= -\frac{1}{3}e^{-3x}\cos^3 x - \int \sin x.e^{-3x}\,dx + \int \underset{\text{I}}{\sin^3} x.\underset{\text{II}}{e^{-3x}}\,dx$$

$$= -\frac{1}{3}e^{-3x}\cos^3 x - \int \sin x.e^{-3x}\,dx + \sin^3 x\int e^{-3x}\,dx - \int\left(D(\sin^3 x)\int e^{-3x}dx\right)dx$$

$$= -\frac{1}{3}e^{-3x}\cos^3 x - \int \sin x.e^{-3x}dx + \sin^3 x.\frac{e^{-3x}}{-3} - \int 3\sin^2 x.\cos x.\frac{e^{-3x}}{-3}\,dx$$

$$= -\frac{1}{3}e^{-3x}\cos^3 x - \int \sin x.e^{-3x}dx - \frac{1}{3}e^{-3x}.\sin^3 x + \int \sin^2 x\cos x\, e^{-3x}\,dx$$

$$= -\frac{1}{3}e^{-3x}\cos^3 x - \int \sin x.e^{-3x}\,dx - \frac{1}{3}e^{-3x}\sin^3 x + \int (1-\cos^2 x)\cos x.e^{-3x}\,dx$$

$$I = -\frac{1}{3}e^{-3x}\cos^3 x - \left[\sin x.\frac{e^{-3x}}{-3} - \int \cos x.\frac{e^{-3x}}{-3}\right] - \frac{1}{3}e^{-3x}.\sin^3 x + \int \cos x.e^{-3x} - \int \cos^3 x.e^{-3x}\,dx$$

$$\boxed{I = -\frac{1}{3}e^{-3x}\cos^3 x + \sin x.\frac{e^{-3x}}{3} - \int \cos x.\frac{e^{-3x}}{3}\,dx - \frac{1}{3}e^{-3x}\sin^3 x + \int \cos x.e^{-3x} - I}$$

$$2I = \frac{e^{-3x}}{-3}[\cos^3 x + \sin^3 x] - \left[\sin x.\frac{e^{-3x}}{-3} - \int \cos x.\frac{e^{-3x}}{-3}dx\right] + \int \cos x.e^{-3x}\,dx$$

$$= \frac{e^{-3x}}{-3}\left[\cos^3 x + \sin^3 x\right] + \frac{1}{3}\sin x.e^{-3x} - \frac{1}{3}\int \cos x.e^{-3x}\,dx + \int \cos x.e^{-3x}\,dx$$

$$\therefore \quad 2I = \frac{e^{-3x}}{-3}\left[\cos^3 x + \sin^3 x\right] + \frac{1}{3}\sin x.e^{-3x} + \frac{2}{3}\int \cos x.e^{-3x}\,dx$$

Now, put

$$I_1 = \frac{2}{3}\int \underset{\text{I}}{\cos x}.\underset{\text{II}}{e^{-3x}}\,dx$$

$$= \frac{2}{3}\left[\cos x.\int e^{-3x}\,dx - \int\left(D(\cos x).\int e^{-3x}\,dx\right)dx\right]$$

$$= \frac{2}{3}\left[\cos x.\frac{e^{-3x}}{-3} - \int -\sin x.\frac{e^{-3x}}{-3}\,dx\right]$$

$$= \frac{2}{3}\left[\cos x.\frac{e^{-3x}}{-3} - \frac{1}{3}\int \sin x.e^{-3x}\,dx\right]$$

$$= -\frac{2}{9}\cos x.e^{-3x} - \frac{2}{9}\int \sin x.e^{-3x}\,dx$$

$$= -\frac{2}{9}\cos x.e^{-3x} - \frac{2}{9}\left[\sin x.\frac{e^{-3x}}{-3} - \int \cos x.\frac{e^{-3x}}{-3}\,dx\right]$$

$$I_1 = -\frac{2}{9}\cos x.e^{-3x} + \frac{2}{27}\sin x.e^{-3x} - \frac{2}{27}\int \cos x.e^{-3x}\,dx$$

$$I_1 = -\frac{2}{9}\cos x.e^{-3x} + \frac{2}{27}\sin x.e^{-3x} - \frac{1}{9}.\frac{2}{3}\int \cos x.e^{-3x}\,dx$$

$$I_1 = -\frac{2}{9}\cos x.e^{-3x} + \frac{2}{27}\sin x.e^{-3x} - \frac{1}{9}.I_1$$

$$I_1 + \frac{1}{9}I_1 = -\frac{2}{9}\cos x.e^{-3x} + \frac{2}{27}\sin x.e^{-3x}$$

$$\Rightarrow \frac{10I_1}{9} = -\frac{2}{9}\cos x.e^{-3x} + \frac{2}{27}\sin x.e^{-3x}$$

$$\therefore \quad I_1 = -\frac{1}{10}\cos x.e^{-3x} + \frac{1}{15}\sin x.e^{-3x}$$

So $\quad 2I = -\frac{1}{3}e^{-3x}[\sin^3 x + \cos^3 x] + \frac{1}{3}\sin x.e^{-3x} - \frac{1}{10}\cos x.e^{-3x}$

$$+\frac{1}{15}\sin x.e^{-3x}$$

$$\therefore \quad I = -\frac{1}{6}e^{-3x}[\sin^3 x+\cos^3 x]+\frac{1}{6}\sin x.e^{-3x}-\frac{1}{20}\cos x.e^{-3x}$$
$$+\frac{1}{30}\sin x.e^{-3x}$$

$$= -\frac{1}{6}e^{-3x}[\sin^3 x+\cos^3 x]+\frac{1}{5}\sin x.e^{-3x}-\frac{1}{20}\cos x.e^{-3x}$$

$$= \frac{e^{-3x}}{24}[\sin 3x-\cos 3x]+\frac{3e^{-3x}}{40}[\sin x-3\cos x]+C$$

$$\left[\begin{array}{l}\because \sin 3x = 3\sin x - 4\sin^3 x \\ \quad \cos 3x = 4\cos^3 x - 3\cos x\end{array}\right]$$

Hence, $I = \dfrac{e^{-3x}}{24}[\sin 3x-\cos 3x]+\dfrac{3e^{-3x}}{40}[\sin x-3\cos x]+C.$

Q43. Evaluate: $\int\sqrt{\tan x}\,dx$ (Hint: Put $\tan x = t^2$)

Sol. Let $I = \int\sqrt{\tan x}\,dx$

Put $\tan x = t^2$

$$\sec^2 x\,dx = 2t\,dt \Rightarrow dx = \frac{2t\,dt}{\sec^2 x} = \frac{2t\,dt}{1+\tan^2 x} \Rightarrow dx = \frac{2t\,dt}{1+t^4}$$

$$\therefore \quad I = \int\frac{t.2t}{1+t^4}dt = \int\frac{2t^2}{1+t^4}dt = \int\frac{2}{t^2+\frac{1}{t^2}}dt$$

[Dividing the numerator and denominator by t^2]

$$= \int\frac{1+\frac{1}{t^2}+1-\frac{1}{t^2}}{t^2+\frac{1}{t^2}+2-2}dt$$

$$= \int\frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}-2+2}dt+\int\frac{1-\frac{1}{t^2}}{t^2+\frac{1}{t^2}+2-2}dt$$

$$= \int\frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+(\sqrt{2})^2}dt+\int\frac{1-\frac{1}{t^2}}{\left(t+\frac{1}{t}\right)^2-(\sqrt{2})^2}dt$$

Put $\quad I_1 = \int \frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+(\sqrt{2})^2}\,dt$ and $I_2 = \int \frac{1-\frac{1}{t^2}}{\left(t+\frac{1}{t}\right)^2-(\sqrt{2})^2}\,dt$

$\therefore \quad I = I_1 + I_2 \qquad ...(i)$

Now $\quad I_1 = \int \frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+(\sqrt{2})^2}\,dt$

Put $\quad t - \frac{1}{t} = u$

$\therefore \quad \left(1+\frac{1}{t^2}\right)dt = du$

$$I_1 = \int \frac{du}{u^2+(\sqrt{2})^2} = \frac{1}{\sqrt{2}}\tan^{-1}\frac{u}{\sqrt{2}} + C_1$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\frac{t-\frac{1}{t}}{\sqrt{2}} + C_1 = \frac{1}{\sqrt{2}}\tan^{-1}\frac{t^2-1}{\sqrt{2}t} + C_1$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan x - 1}{\sqrt{2}\sqrt{\tan x}}\right) + C_1$$

Now $\quad I_2 = \int \frac{1-\frac{1}{t^2}}{\left(t+\frac{1}{t}\right)^2-(\sqrt{2})^2}\,dt$

Put $\quad t + \frac{1}{t} = v \quad \Rightarrow \quad \left(1-\frac{1}{t^2}\right)dt = dv$

$$= \int \frac{dv}{v^2-(\sqrt{2})^2} = \frac{1}{2.\sqrt{2}}\log\left|\frac{v-\sqrt{2}}{v+\sqrt{2}}\right| + C_2$$

$$= \frac{1}{2\sqrt{2}}\log\left|\frac{t+\frac{1}{t}-\sqrt{2}}{t+\frac{1}{t}+\sqrt{2}}\right| + C_2$$

$$= \frac{1}{2\sqrt{2}}\log\left|\frac{t^2-\sqrt{2}t+1}{t^2+\sqrt{2}t+1}\right| + C_2$$

$$= \frac{1}{2\sqrt{2}}\log\left|\frac{\tan x-\sqrt{2}\sqrt{\tan x}+1}{\tan x+\sqrt{2}\sqrt{\tan x}+1}\right| + C_2$$

So $\quad I = I_1 + I_2$

$$\Rightarrow \quad I = \frac{1}{\sqrt{2}} \tan^{-1}\left[\frac{\tan x - 1}{\sqrt{2 \tan x}}\right] + \frac{1}{2\sqrt{2}} \log \left|\frac{\tan x - \sqrt{2 \tan x} + 1}{\tan x + \sqrt{2 \tan x} + 1}\right| + C_1 + C_2$$

Hence, $$I = \frac{1}{\sqrt{2}} \tan^{-1}\left[\frac{\tan x - 1}{\sqrt{2 \tan x}}\right] + \frac{1}{2\sqrt{2}} \log \left|\frac{\tan x - \sqrt{2 \tan x} + 1}{\tan x + \sqrt{2 \tan x} + 1}\right| + C.$$

Q44. Evaluate: $\displaystyle\int_0^{\pi/2} \frac{dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^2}$

(Hint: Divide Numerator and Denominator by $\cos^4 x$)

Sol. Let $\quad I = \displaystyle\int_0^{\pi/2} \frac{dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^2}$

Dividing the numerator and denominator by $\cos^4 x$, we have

$$I = \int_0^{\pi/2} \frac{\sec^4 x}{\left(\frac{a^2 \cos^2 x}{\cos^2 x} + \frac{b^2 \sin^2 x}{\cos^2 x}\right)^2} dx$$

$$= \int_0^{\pi/2} \frac{\sec^2 x . \sec^2 x}{(a^2 + b^2 \tan^2 x)^2} dx = \int_0^{\pi/2} \frac{(1 + \tan^2 x) \sec^2 x}{(a^2 + b^2 \tan^2 x)^2} dx$$

Put $\quad \tan x = t \;\Rightarrow\; \sec^2 x\, dx = dt$

Changing the limits, we get

When $x = 0, \quad t = \tan 0 = 0$

When $x = \frac{\pi}{2}, \quad t = \tan \frac{\pi}{2} = \infty$

$$\therefore \quad I = \int_0^{\infty} \frac{1 + t^2}{(a^2 + b^2 t^2)^2} dt$$

Put $t^2 = u$ only for the purpose of partial fraction

$$\therefore \quad \frac{1 + u}{(a^2 + b^2 u)^2} = \frac{A}{(a^2 + b^2 u)} + \frac{B}{(a^2 + b^2 u)^2}$$

$$1 + u = A(a^2 + b^2 u) + B$$

Comparing the coefficients of like terms, we get

$a^2 A + B = 1$ and $b^2 A = 1 \Rightarrow A = \frac{1}{b^2}$

Now $a^2 . \frac{1}{b^2} + B = 1 \quad \Rightarrow B = 1 - \frac{a^2}{b^2} = \frac{b^2 - a^2}{b^2}$

$$\therefore \quad I = \int_0^\infty \frac{1+t^2}{(a^2+b^2t^2)^2} = \frac{1}{b^2}\int_0^\infty \frac{dt}{a^2+b^2t^2} + \frac{b^2-a^2}{b^2}\int_0^\infty \frac{dt}{(a^2+b^2t^2)^2}$$

$$= \frac{1}{b^2}\int_0^\infty \frac{dt}{b^2\left(\frac{a^2}{b^2}+t^2\right)} + \frac{b^2-a^2}{b^2}\int_0^\infty \frac{dt}{(a^2+b^2t^2)^2}$$

$$= \frac{1}{ab^3}\left[\tan^{-1}\frac{t}{a/b}\right]_0^\infty + \frac{b^2-a^2}{b^2}\left(\frac{\pi}{4}.\frac{1}{a^3b}\right)$$

$$= \frac{1}{ab^3}\left[\tan^{-1}\infty - \tan 0\right] + \frac{b^2-a^2}{b^2}\left(\frac{\pi}{4a^3b}\right)$$

$$= \frac{1}{ab^3}.\frac{\pi}{2} + \frac{\pi}{4}.\frac{b^2-a^2}{a^3b^3} = \frac{\pi}{2ab^3} + \frac{\pi}{4}.\frac{b^2-a^2}{a^3b^3}$$

$$= \pi\left[\frac{2a^2+b^2-a^2}{4a^3b^3}\right] = \frac{\pi}{4}\left(\frac{a^2+b^2}{a^3b^3}\right)$$

$$\text{Hence, } I = \frac{\pi}{4}\left(\frac{a^2+b^2}{a^3b^3}\right)$$

Q45. Evaluate: $\int_0^1 x \log|1+2x|\,dx$

Sol. Let $I = \int_0^1 \underset{\text{II}}{x} \log \underset{\text{I}}{|1+2x|}\,dx$

$$= \left[\log|1+2x|.\left(\frac{x^2}{2}\right)\right]_0^1 - \int_0^1\left(\frac{1.2}{1+2x}.\frac{x^2}{2}\right)dx$$

$$= \frac{1}{2}\left[x^2\log(1+2x)\right]_0^1 - \int_0^1 \frac{x^2}{1+2x}\,dx$$

$$= \frac{1}{2}[\log 3 - 0] - \int_0^1\left(\frac{x}{2} - \frac{x/2}{1+2x}\right)dx$$

$$= \frac{1}{2}\log 3 - \frac{1}{2}\int_0^1 x\,dx + \frac{1}{2}\int_0^1 \frac{x}{1+2x}\,dx$$

$$= \frac{1}{2}\log 3 - \frac{1}{2}\left[\frac{x^2}{2}\right]_0^1 + \frac{1}{2}.\frac{1}{2}\int_0^1 \frac{(2x+1-1)}{2x+1}\,dx$$

$$= \frac{1}{2}\log 3 - \frac{1}{4}[1-0] + \frac{1}{4}\int_0^1 1\,dx - \frac{1}{4}\int_0^1 \frac{1}{2x+1}\,dx$$

$$= \frac{1}{2}\log 3 - \frac{1}{4} + \frac{1}{4}[x]_0^1 - \frac{1}{4}.\frac{1}{2}\left[\log|2x+1|\right]_0^1$$

$$= \frac{1}{2}\log 3 - \frac{1}{4} + \frac{1}{4} - \frac{1}{8}[\log 3 - 0]$$

$$= \frac{1}{2}\log 3 - \frac{1}{8}\log 3 = \frac{3}{8}\log 3$$

Hence, $I = \frac{3}{8}\log 3.$

Q46. Evaluate: $\int_0^{\pi} x \log \sin x \, dx$

Sol. Let $$I = \int_0^{\pi} x \log \sin x \, dx \qquad ...(i)$$

$$= \int_0^{\pi} (\pi - x) \log \sin (\pi - x) \, dx$$

$$\left[\text{using } \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$I = \int_0^{\pi} (\pi - x) \log \sin x \, dx \qquad ...(ii)$$

Adding (*i*) and (*ii*), we get

$$2I = \int_0^{\pi} [(\pi - x)\log \sin x + x \log \sin x]\, dx$$

$$2I = \int_0^{\pi} \pi \log \sin x \, dx$$

$$2I = 2\pi \int_0^{\pi/2} \log \sin x \, dx \qquad \left[\because \int_0^a f(x)\,dx = 2\int_0^{a/2} f(x)\,dx\right]$$

$$\therefore \quad I = \pi \int_0^{\pi/2} \log \sin x \, dx \qquad ...(iii)$$

$$I = \pi \int_0^{\pi/2} \log \sin\left(\frac{\pi}{2} - x\right) dx$$

$$I = \pi \int_0^{\pi/2} \log \cos x \, dx \qquad ...(iv)$$

On adding (*iii*) and (*iv*), we get

$$2I = \pi \int_0^{\pi/2} (\log \sin x + \log \cos x)\, dx$$

$$2I = \pi \int_0^{\pi/2} \log \sin x \cos x \, dx = \pi \int_0^{\pi/2} \log \frac{2 \sin x \cos x}{2}\, dx$$

$$2I = \pi\int_0^{\pi/2} \log \sin 2x\, dx - \pi\int_0^{\pi/2} \log 2\, dx$$

Put $\quad 2x = t \;\Rightarrow\; 2\,dx = dt \;\Rightarrow\; dx = \dfrac{dt}{2}$

$$2I = \pi\int_0^{\pi} \log \sin t\, dt - \pi.\log 2\int_0^{\pi/2} 1\, dx \quad \text{[Changing the limit]}$$

$$2I = I - \pi.\log 2\,[x]_0^{\pi/2} \qquad \text{[from eqn. } (iii)\text{]}$$

$$2I - I = -\frac{\pi^2}{2}\log 2$$

So $\qquad I = \dfrac{\pi^2}{2}\log\left(\dfrac{1}{2}\right)$

Q47. Evaluate: $\displaystyle\int_{-\pi/4}^{\pi/4} \log|\sin x + \cos x|\, dx$

Sol. Let $\qquad I = \displaystyle\int_{-\pi/4}^{\pi/4} \log|\sin x + \cos x|\, dx \qquad ...(i)$

$$= \int_{-\pi/4}^{\pi/4} \log\left|\sin\left(\frac{\pi}{4} - \frac{\pi}{4} - x\right) + \cos\left(\frac{\pi}{4} - \frac{\pi}{4} - x\right)\right| dx$$

$$\left[\because \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx\right]$$

$$= \int_{-\pi/4}^{\pi/4} \log|\sin(-x) + \cos x|\, dx$$

$$= \int_{-\pi/4}^{\pi/4} \log|\cos x - \sin x|\, dx \qquad ...(ii)$$

Adding (i) and (ii), we get

$$2I = \int_{-\pi/4}^{\pi/4} \log|\cos x + \sin x|\, dx + \int_{-\pi/4}^{\pi/4} \log|\cos x - \sin x|\, dx$$

$$= \int_{-\pi/4}^{\pi/4} \log|(\cos x + \sin x)(\cos x - \sin x)|\, dx$$

$$= \int_{-\pi/4}^{\pi/4} \log|\cos^2 x - \sin^2 x|\, dx$$

$$\therefore \quad 2I = \int_{-\pi/4}^{\pi/4} \log \cos 2x\, dx$$

$$2I = 2\int_0^{\pi/4} \log \cos 2x \, dx$$

$$\left[\because \int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx \text{ if } f(-x) = f(x)\right]$$

$$\therefore \quad I = \int_0^{\pi/4} \log \cos 2x \, dx$$

Put $\quad 2x = t \quad \Rightarrow dx = \dfrac{dt}{2}$

Changing the limits we get

When $x = 0 \quad \therefore t = 0$; When $x = \dfrac{\pi}{4} \quad \therefore t = \dfrac{\pi}{2}$

$$I = \frac{1}{2}\int_0^{\pi/2} \log \cos t \, dt \qquad \text{...(iii)}$$

$$I = \frac{1}{2}\int_0^{\pi/2} \log \cos\left(\frac{\pi}{2} - t\right) dt$$

$$I = \frac{1}{2}\int_0^{\pi/2} \log \sin t \, dt \qquad \text{...(iv)}$$

On adding (*iii*) and (*iv*), we get,

$$2I = \frac{1}{2}\int_0^{\pi/2} (\log \cos t + \log \sin t) \, dt$$

$$\Rightarrow \quad 2I = \frac{1}{2}\int_0^{\pi/2} \log \sin t \cos t \, dt$$

$$\Rightarrow \quad 2I = \frac{1}{2}\int_0^{\pi/2} \log \frac{2 \sin t \cos t}{2} \, dt$$

$$\Rightarrow \quad 2I = \frac{1}{2}\int_0^{\pi/2} (\log \sin 2t - \log 2) \, dt$$

$$\Rightarrow \quad 4I = \int_0^{\pi/2} \log \sin 2t \, dt - \int_0^{\pi/2} \log 2 \, dt$$

Put $\quad 2t = u \Rightarrow 2\,dt = du \Rightarrow dt = \dfrac{du}{2}$

$$\therefore \quad 4I = \frac{1}{2}\int_0^{\pi} \log \sin u \, du - \int_0^{\pi/2} \log 2 \, . \, dt$$

$$\Rightarrow \quad 4I = \frac{1}{2} \times 2 \int_0^{\pi/2} \log \sin u \, du - \log 2 [t]_0^{\pi/2}$$

$$\Rightarrow \quad 4I = \int_0^{\pi/2} \log \sin u \, du - \log 2 . \frac{\pi}{2}$$

$$\Rightarrow \quad 4I = 2I - \frac{\pi}{2} \log 2 \qquad \text{[From eq. (ii)]}$$

$$\Rightarrow \quad 2I = -\frac{\pi}{2} \log 2 \quad \Rightarrow \quad I = \frac{\pi}{4} \log \frac{1}{2}$$

Hence, $I = \frac{\pi}{4} \log \frac{1}{2}$.

OBJECTIVE TYPE QUESTIONS

Choose the correct option from given four options in each of the Exercises from 48 to 58.

Q48. $\int \frac{\cos 2x - \cos 2\theta}{\cos x - \cos \theta} dx$ is equal to

(a) $2(\sin x + x \cos \theta) + C$ (b) $2(\sin x - x \cos \theta) + C$

(c) $2(\sin x + 2x \cos \theta) + C$ (d) $2(\sin x - 2x \cos \theta) + C$

Sol. Let $\quad I = \int \frac{\cos 2x - \cos 2\theta}{\cos x - \cos \theta} dx$

$$= \int \frac{(2\cos^2 x - 1) - (2\cos^2 \theta - 1)}{\cos x - \cos \theta} dx$$

$$= \int \frac{2\cos^2 x - 1 - 2\cos^2 \theta + 1}{\cos x - \cos \theta} dx$$

$$= \int \frac{2\cos^2 x - 2\cos^2 \theta}{\cos x - \cos \theta} dx = 2\int \frac{\cos^2 x - \cos^2 \theta}{\cos x - \cos \theta} dx$$

$$= 2\int \frac{(\cos x + \cos \theta)(\cos x - \cos \theta)}{(\cos x - \cos \theta)} dx$$

$$= 2\int (\cos x + \cos \theta) \, dx$$

$\therefore \quad I = 2(\sin x + \cos \theta . x) + C.$

Hence, correct option is (a).

Q49. $\int \frac{dx}{\sin(x-a) . \sin(x-b)}$ is equal to—

(a) $\sin(b-a) \log \left| \frac{\sin(x-b)}{\sin(x-a)} \right| + C$

(b) $\text{cosec}(b-a) \log \left| \frac{\sin(x-a)}{\sin(x-b)} \right| + C$

(*c*) $\operatorname{cosec}(b-a)\log\left|\dfrac{\sin(x-b)}{\sin(x-a)}\right|+C$

(*d*) $\sin(b-a)\log\left|\dfrac{\sin(x-a)}{\sin(x-b)}\right|+C$

Sol. Let $\quad I = \displaystyle\int \frac{dx}{\sin(x-a).\sin(x-b)}$

Multiplying and dividing by $\sin(b-a)$ we get,

$$I = \frac{1}{\sin(b-a)}\int \frac{\sin(b-a)}{\sin(x-a).\sin(x-b)}dx$$

$$= \frac{1}{\sin(b-a)}\int \frac{\sin(x+b-x-a)}{\sin(x-a).\sin(x-b)}dx$$

$$= \frac{1}{\sin(b-a)}\int \frac{\sin[(x-a)-(x-b)]}{\sin(x-a).\sin(x-b)}dx$$

$$= \frac{1}{\sin(b-a)}\int \frac{\sin(x-a)\cos(x-b)-\cos(x-a)\sin(x-b)}{\sin(x-a).\sin(x-b)}dx$$

$$= \frac{1}{\sin(b-a)}\int \frac{\sin(x-a).\cos(x-b)}{\sin(x-a).\sin(x-b)}-\frac{\cos(x-a).\sin(x-b)}{\sin(x-a).\sin(x-b)}dx$$

$$= \frac{1}{\sin(b-a)}\int\left[\frac{\cos(x-b)}{\sin(x-b)}-\frac{\cos(x-a)}{\sin(x-a)}\right]dx$$

$$= \frac{1}{\sin(b-a)}\int[\cot(x-b)-\cot(x-a)]\,dx$$

$$= \frac{1}{\sin(b-a)}[\log\sin(x-b)-\log\sin(x-a)]+C$$

$$= \frac{1}{\sin(b-a)}.\log\left|\frac{\sin(x-b)}{\sin(x-a)}\right|+C$$

$$I = \operatorname{cosec}(b-a).\log\left|\frac{\sin(x-b)}{\sin(x-a)}\right|+C$$

Hence, the correct option is (*c*).

Q50. $\displaystyle\int \tan^{-1}\sqrt{x}\,dx$ is equal to

(*a*) $(x+1)\tan^{-1}\sqrt{x}-\sqrt{x}+C$ (*b*) $x\tan^{-1}\sqrt{x}-\sqrt{x}+C$

(*c*) $\sqrt{x}-x\tan^{-1}\sqrt{x}+C$ (*d*) $\sqrt{x}-(x+1)\tan^{-1}\sqrt{x}+C$

Sol. Let $\quad I = \displaystyle\int \tan^{-1}\sqrt{x}\,dx$

Put $\sqrt{x} = \tan\theta \Rightarrow x = \tan^2\theta \Rightarrow dx = 2\tan\theta\sec^2\theta\, d\theta$

$\therefore \quad I = \int \tan^{-1}(\tan\theta).2\tan\theta\sec^2\theta\, d\theta = 2\int \theta.\tan\theta.\sec^2\theta\, d\theta$

$= 2\left[\theta.\int \tan\theta.\sec^2\theta\, d\theta - \int\left(D(\theta).\int \tan\theta\sec^2\theta\, d\theta\right)d\theta\right]$

Let us take

$I_1 = \int \tan\theta\sec^2\theta\, d\theta$

Put $\tan\theta = t \Rightarrow \sec^2\theta\, d\theta = dt$

$\therefore \quad I_1 = \int t\, dt = \frac{1}{2}t^2 = \frac{1}{2}\tan^2\theta$

$\therefore \quad I = 2\left[\theta.\frac{1}{2}\tan^2\theta - \int\left(1.\frac{1}{2}\tan^2\theta\right)d\theta\right]$

$= \theta\tan^2\theta - \int \tan^2\theta\, d\theta = \theta\tan^2\theta - \int(\sec^2\theta - 1)\, d\theta$

$= \theta\tan^2\theta - (\tan\theta - \theta) + C = \theta\tan^2\theta - \tan\theta + \theta + C$

$\therefore \quad I = \tan^{-1}\sqrt{x}.x - \sqrt{x} + \tan^{-1}\sqrt{x} + C$

$= (x+1)\tan^{-1}\sqrt{x} - \sqrt{x} + C$

Hence, the correct option is (*a*).

Q51. $\int e^x\left(\frac{1-x}{1+x^2}\right)^2 dx$ is equal to

(*a*) $\frac{e^x}{1+x^2} + C$ (*b*) $\frac{-e^x}{1+x^2} + C$

(*c*) $\frac{e^x}{(1+x^2)^2} + C$ (*d*) $\frac{-e^x}{(1+x^2)^2} + C$

Sol. Let $I = \int e^x\left(\frac{1-x}{1+x^2}\right)^2 dx$

$= \int e^x\left[\frac{1+x^2-2x}{(1+x^2)^2}\right]dx = \int e^x\left[\frac{(1+x^2)}{(1+x^2)^2} - \frac{2x}{(1+x^2)^2}\right]dx$

$= \int e^x\left[\frac{1}{1+x^2} - \frac{2x}{(1+x^2)^2}\right]dx$

Here $f(x) = \frac{1}{1+x^2} \quad \therefore f'(x) = \frac{-2x}{(1+x^2)^2}$

Using $\int e^x[f(x) + f'(x)]\, dx = e^x.f(x) + C$

$\therefore \quad I = e^x.\frac{1}{(1+x^2)} + C = \frac{e^x}{1+x^2} + C$

Hence, the correct option is (*a*).

Q52. $\int \frac{x^9}{(4x^2+1)^6}dx$ is equal to

(a) $\frac{1}{5x}\left(4+\frac{1}{x^2}\right)^{-5}+C$ (b) $\frac{1}{5}\left(4+\frac{1}{x^2}\right)^{-5}+C$

(c) $\frac{1}{10x}(1+4)^{-5}+C$ (d) $\frac{1}{10}\left(\frac{1}{x^2}+4\right)^{-5}+C$

Sol. Let $I = \int \frac{x^9}{(4x^2+1)^6}dx = \int \frac{x^9}{x^{12}\left(4+\frac{1}{x^2}\right)^6}dx = \int \frac{1}{x^3\left(4+\frac{1}{x^2}\right)^6}dx$

Put $\left(4+\frac{1}{x^2}\right) = t \Rightarrow \frac{-2}{x^3}dx = dt \Rightarrow \frac{dx}{x^3} = -\frac{1}{2}dt$

$\therefore \quad I = -\frac{1}{2}\int \frac{dt}{t^6}$

$= -\frac{1}{2} \times -\frac{1}{5}t^{-5}+C = \frac{1}{10}t^{-5}+C = \frac{1}{10}\left(4+\frac{1}{x^2}\right)^{-5}+C$

Hence, the correct option is (*d*).

Q53. If $\int \frac{dx}{(x+2)(x^2+1)} = a\log\left|1+x^2\right| + b\tan^{-1}x + \frac{1}{5}\log|x+2|+C$ then

(a) $a=-\frac{1}{10}, b=\frac{-2}{5}$ (b) $a=\frac{1}{10}, b=\frac{-2}{5}$

(c) $a=-\frac{1}{10}, b=\frac{2}{5}$ (d) $a=\frac{1}{10}, b=\frac{2}{5}$

Sol. Let $\quad I = \int \frac{dx}{(x+2)(x^2+1)}$

Let us resolve the given integrand into partial fractions

Put $\quad \frac{1}{(x+2)(x^2+1)} = \frac{A}{(x+2)} + \frac{Bx+C}{(x^2+1)}$

$1 = A(x^2+1)+(x+2)(Bx+C)$

$1 = Ax^2+A+Bx^2+Cx+2Bx+2C$

$1 = (A+B)x^2+(C+2B)x+(A+2C)$

Comparing the like terms, we have

$A+B=0$ ...(*i*)

$2B+C=0$ ...(*ii*)

$A+2C=1$ ...(*iii*)

Subtracting (*i*) from (*iii*) we get

$2C-B=1 \quad \therefore B=2C-1$

Putting the value of B in eqn. (*ii*) we have

$$2(2C-1)+C=0 \Rightarrow 4C-2+C=0$$

$$5C=2 \quad \therefore C=\frac{2}{5}$$

$$\therefore \qquad B=2\left(\frac{2}{5}\right)-1=-\frac{1}{5} \text{ and } A=\frac{1}{5}$$

$$\therefore \int \frac{1}{(x+2)(x^2+1)}\,dx=\int \frac{\frac{1}{5}}{(x+2)}\,dx+\int \frac{-\frac{1}{5}x+\frac{2}{5}}{(x^2+1)}\,dx$$

$$=\frac{1}{5}\int \frac{1}{(x+2)}\,dx-\frac{1}{5}\int \frac{x-2}{(x^2+1)}\,dx$$

$$=\frac{1}{5}\int \frac{1}{x+2}\,dx-\frac{1}{5}\int \frac{x}{x^2+1}\,dx+\frac{2}{5}\int \frac{1}{x^2+1}\,dx$$

$$=\frac{1}{5}\int \frac{1}{x+2}\,dx-\frac{1}{10}\int \frac{2x}{x^2+1}\,dx+\frac{2}{5}\int \frac{1}{x^2+1}\,dx$$

$$\therefore \qquad I=\frac{1}{5}\log|x+2|-\frac{1}{10}\log|x^2+1|+\frac{2}{5}\tan^{-1}x+C$$

Putting the given value of I

$$\therefore\ a\log|1+x^2|+b\tan^{-1}x+\frac{1}{5}\log|x+2|+C$$

$$=\frac{1}{5}\log|x+2|-\frac{1}{10}\log|x^2+1|+\frac{2}{5}\tan^{-1}x+C$$

$$\therefore\ a=-\frac{1}{10} \quad \text{and} \quad b=\frac{2}{5}$$

Hence, the correct option is (*c*).

Q54. $\int \frac{x^3}{x+1}\,dx$ is equal to

(*a*) $x+\frac{x^2}{2}+\frac{x^3}{3}-\log|1-x|+C$ (*b*) $x+\frac{x^2}{2}-\frac{x^3}{3}-\log|1-x|+C$

(*c*) $x-\frac{x^2}{2}-\frac{x^3}{3}-\log|1+x|+C$ (*d*) $x-\frac{x^2}{2}+\frac{x^3}{3}-\log|1+x|+C$

Sol. Let $I=\int \frac{x^3}{x+1}\,dx$

$$\therefore \qquad I=\int\left(x^2-x+1-\frac{1}{x+1}\right)dx=\frac{x^3}{3}-\frac{x^2}{2}+x-\log|x+1|+C$$

$$=x-\frac{x^2}{2}+\frac{x^3}{3}-\log|x+1|+C$$

Hence, the correct option is (*d*).

Q55. $\int \frac{x+\sin x}{1+\cos x}\,dx$ is equal to

(*a*) $\log|1+\cos x|+C$ (*b*) $\log|x+\sin x|+C$

(*c*) $x-\tan\frac{x}{2}+C$ (*d*) $x.\tan\frac{x}{2}+C$

Sol. Let $I = \int \frac{x+\sin x}{1+\cos x}\,dx$

$$= \int \frac{x}{1+\cos x}\,dx + \int \frac{\sin x}{1+\cos x}\,dx$$

$$= \int \frac{x}{2\cos^2\frac{x}{2}}\,dx + \int \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\,dx$$

$$= \frac{1}{2}\int x.\sec^2\frac{x}{2}\,dx + \int \tan\frac{x}{2}\,dx$$

$$= \frac{1}{2}\left[x.\int \sec^2\frac{x}{2}\,dx - \int\left(D(x).\int \sec^2\frac{x}{2}dx\right)dx\right] + \int \tan\frac{x}{2}\,dx$$

$$= \frac{1}{2}\left[x.2\tan\frac{x}{2} - \int 2\tan\frac{x}{2}\,dx\right] + \int \tan\frac{x}{2}\,dx$$

$$= x\tan\frac{x}{2} - \int \tan\frac{x}{2}\,dx + \int \tan\frac{x}{2}.dx + C$$

$\therefore \quad I = x\tan\frac{x}{2} + C$

Hence, the correct option is (*d*).

Q56. If $\int \frac{x^3}{\sqrt{1+x^2}}\,dx = a(1+x^2)^{3/2} + b\sqrt{1+x^2} + C$, then

(*a*) $a = \frac{1}{3}, b = 1$ (*b*) $a = -\frac{1}{3}, b = 1$

(*c*) $a = -\frac{1}{3}, b = -1$ (*d*) $a = \frac{1}{3}, b = -1$

Sol. Let $I = \int \frac{x^3}{\sqrt{1+x^2}}\,dx$

Put $1+x^2 = t \Rightarrow 2x\,dx = dt \Rightarrow x\,dx = \frac{dt}{2}$

$\therefore \; I = \frac{1}{2}\int \frac{(t-1)}{\sqrt{t}}\,dt$

$$= \frac{1}{2}\int \frac{t}{\sqrt{t}}\,dt - \frac{1}{2}\int \frac{1}{\sqrt{t}}\,dt = \frac{1}{2}\int \sqrt{t}\,dt - \frac{1}{2}\int t^{-1/2}\,dt$$

$$= \frac{1}{2} \times \frac{2}{3}(t)^{3/2} - \frac{1}{2}.2\sqrt{t} + C = \frac{1}{3}(1+x^2)^{3/2} - \sqrt{1+x^2} + C$$

But $I = a(1+x^2)^{3/2} + b\sqrt{1+x^2} + C$

Comparing the like terms we get, $a = \frac{1}{3}$ and $b = -1$

Hence, the correct option is (*d*).

Q57. $\int\limits_{-\pi/4}^{\pi/4} \frac{dx}{1+\cos 2x}$ is equal to

(*a*) 1 (*b*) 2 (*c*) 3 (*d*) 4

Sol. Let $I = \int\limits_{-\pi/4}^{\pi/4} \frac{dx}{1+\cos 2x}$

$$= \int\limits_{-\pi/4}^{\pi/4} \frac{dx}{2\cos^2 x} = \frac{1}{2}\int\limits_{-\pi/4}^{\pi/4} \sec^2 x\, dx = \frac{1}{2}[\tan x]_{-\pi/4}^{\pi/4}$$

$$= \frac{1}{2}\left[\tan\frac{\pi}{4} - \tan\left(-\frac{\pi}{4}\right)\right] = \frac{1}{2}[1+1] = \frac{1}{2} \times 2 = 1$$

Hence, the correct option is (*a*).

Q58. $\int\limits_{0}^{\pi/2} \sqrt{1-\sin 2x}\, dx$ is equal to

(*a*) $2\sqrt{2}$ (*b*) $2(\sqrt{2}+1)$ (*c*) 2 (*d*) $2(\sqrt{2}-1)$

Sol. Let $I = \int\limits_{0}^{\pi/2} \sqrt{1-\sin 2x}\, dx = \int\limits_{0}^{\pi/2} \sqrt{(\sin^2 x + \cos^2 x - 2\sin x \cos x)}\, dx$

$$= \int\limits_{0}^{\pi/2} \sqrt{(\sin x - \cos x)^2}\, dx = \int\limits_{0}^{\pi/2} \pm(\sin x - \cos x)\, dx$$

$$= \int\limits_{0}^{\pi/4} -(\sin x - \cos x)\, dx + \int\limits_{\pi/4}^{\pi/2} (\sin x - \cos x)\, dx$$

$$= \int\limits_{0}^{\pi/4} (\cos x - \sin x)\, dx + \int\limits_{\pi/4}^{\pi/2} (\sin x - \cos x)\, dx$$

$$= [\sin x + \cos x]_0^{\pi/4} + [-\cos x - \sin x]_{\pi/4}^{\pi/2}$$

$$= \left[\left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right) - (\sin 0 - \cos 0)\right] - \left[\left(\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right) - \left(\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\right)\right]$$

$$= \left[\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)-(+1)\right]-\left[(0+1)-\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)\right]$$

$$= \left(\frac{2}{\sqrt{2}}-1\right)-\left(1-\frac{2}{\sqrt{2}}\right) = \frac{2}{\sqrt{2}}-1-1+\frac{2}{\sqrt{2}}$$

$$= \frac{4}{\sqrt{2}}-2 = 2\sqrt{2}-2 = 2(\sqrt{2}-1)$$

Hence, the correct option is (*d*).

Fill in the blanks in each of the following Exercises 59 to 63:

Q59. $\int_0^{\pi/2} \cos x . e^{\sin x} dx$ is equal to ________.

Sol. Let $\quad I = \int_0^{\pi/2} \cos x . e^{\sin x} dx$

Put $\sin x = t \Rightarrow \cos x\, dx = dt$

When $x = 0$ then $t = \sin 0 = 0$; When $x = \frac{\pi}{2}$ then $t = \sin\frac{\pi}{2} = 1$

$\therefore \quad I = \int_0^1 e^t\, dt = [e^t]_0^1 = (e^1 - e^0) = e - 1$

Hence, $I = e - 1$.

Q60. $\int \frac{x+3}{(x+4)^2} . e^x\, dx =$ ________

Sol. Let $\quad I = \int \frac{x+3}{(x+4)^2} . e^x\, dx = \int \frac{x+4-1}{(x+4)^2} . e^x\, dx$

$$= \int \left[\frac{x+4}{(x+4)^2} - \frac{1}{(x+4)^2}\right] e^x\, dx = \int \left[\frac{1}{x+4} - \frac{1}{(x+4)^2}\right] e^x\, dx$$

Put $\quad \frac{1}{x+4} = t \Rightarrow -\frac{1}{(x+4)^2} dx = dt$

Let $\quad f(x) = \frac{1}{x+4} \quad \therefore f'(x) = -\frac{1}{(x+4)^2}$

Using $\int e^x [f(x) + f'(x)]\, dx = e^x f(x) + C$

$\therefore \quad I = e^x . \frac{1}{x+4} + C$

Hence, $I = \frac{e^x}{x+4} + C$.

Q61. If $\int_0^a \frac{1}{1+4x^2} dx = \frac{\pi}{8}$, then $a =$ ________.

Sol. Given that: $\int_0^a \frac{1}{1+4x^2}dx = \frac{\pi}{8}$

$$\Rightarrow \quad \frac{1}{4}\int_0^a \frac{1}{\left(\frac{1}{4}+x^2\right)}dx = \frac{\pi}{8} \Rightarrow \int_0^a \frac{1}{\left[\left(\frac{1}{2}\right)^2+x^2\right]}dx = \frac{\pi}{2}$$

$$\Rightarrow \quad \frac{1}{1/2}\left[\tan^{-1}\frac{x}{1/2}\right]_0^a = \frac{\pi}{2} \Rightarrow 2\left[\tan^{-1}2a - \tan^{-1}0\right] = \frac{\pi}{2}$$

$$\Rightarrow \tan^{-1}2a = \frac{\pi}{4} \Rightarrow 2a = \tan\frac{\pi}{4} \Rightarrow 2a = 1 \Rightarrow a = \frac{1}{2}$$

Hence, the value of $a = \frac{1}{2}$.

Q62. $\int \frac{\sin x}{3+4\cos^2 x}dx =$ ________.

Sol. Let $\quad I = \int \frac{\sin x}{3+4\cos^2 x}dx$

Put $\quad \cos x = t$

$\therefore \quad -\sin x\, dx = dt \Rightarrow \sin x\, dx = -dt$

$$\therefore \; I = -\int \frac{dt}{3+4t^2} = -\frac{1}{4}\int \frac{dt}{\frac{3}{4}+t^2} = -\frac{1}{4}\int \frac{dt}{\left(\frac{\sqrt{3}}{2}\right)^2+t^2}$$

$$= -\frac{1}{4}\times\frac{1}{\sqrt{3}/2}\tan^{-1}\left(\frac{t}{\sqrt{3}/2}\right)+C$$

$$= -\frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{2t}{\sqrt{3}}\right)+C = -\frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{2\cos x}{\sqrt{3}}\right)+C$$

Hence, $\; I = -\frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}\cos x\right)+C.$

Q63. The value of $\int_{-\pi}^{\pi} \sin^3 x.\cos^2 x\, dx$ is ________.

Sol. Let $\quad I = \int_{-\pi}^{\pi} \sin^3 x.\cos^2 x\, dx$

Let $\quad f(x) = \sin^3 x \cos^2 x$

$f(-x) = \sin^3(-x).\cos^2(-x) = -\sin^3 x \cos^2 x = -f(x)$

$\therefore \int_{-\pi}^{\pi} \sin^3 x.\cos^2 x\, dx$ is an odd function

$\therefore \quad I = 0$

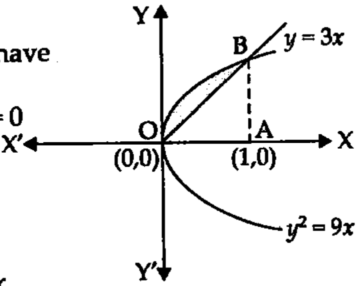
8 Application of Integrals

8.3 EXERCISE

SHORT ANSWER TYPE QUESTIONS

Q1. Find the area of the region bounded by the curves $y^2 = 9x$, $y = 3x$

Sol. We have, $y^2 = 9x$, $y = 3x$



Solving the two equations, we have

$$(3x)^2 = 9x$$

$\Rightarrow \quad 9x^2 - 9x = 0 \Rightarrow 9x(x-1) = 0$

$\therefore \quad x = 0, 1$

Area of the shaded region

$= \text{ar (region OAB)} - \text{ar}(\Delta OAB)$

$$= -\int_0^1 y_1 . dx = \int_0^1 \sqrt{9x}\, dx - \int_0^1 3x\, dx$$

$$= 3\int_0^1 \sqrt{x}\, dx - 3\int_0^1 x\, dx = 3 \times \frac{2}{3}\left[x^{3/2}\right]_0^1 - 3\left[\frac{x^2}{2}\right]_0^1$$

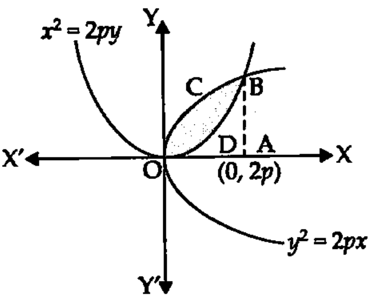
$$= 2\left[(1)^{3/2} - 0\right] - \frac{3}{2}[(1)^2 - 0] = 2(1) - \frac{3}{2}(1) = 2 - \frac{3}{2} = \frac{1}{2} \text{ sq. units}$$

Hence, the required area $= \frac{1}{2}$ sq. units.

Q2. Find the area of the region bounded by the parabola $y^2 = 2px$ and $x^2 = 2py$.

Sol. We are given that: $x^2 = 2py$ ...(i)

and $\quad y^2 = 2px \quad$ ...(ii)



From eqn. (i) we get $y = \frac{x^2}{2p}$

Putting the value of y in eqn. (ii) we have

$$\left(\frac{x^2}{2p}\right)^2 = 2px \Rightarrow \frac{x^4}{4p^2} = 2px$$

$\Rightarrow x^4 = 8p^3x \quad \Rightarrow \quad x^4 - 8p^3x = 0$

$\Rightarrow x(x^3 - 8p^3) = 0 \quad \therefore x = 0, 2p$

Required area = Area of the region (OCBA – ODBA)

$$= \int_0^{2p} \sqrt{2px}\, dx - \int_0^{2p} \frac{x^2}{2p} dx = \sqrt{2p} \int_0^{2p} \sqrt{x}\, dx - \frac{1}{2p} \int_0^{2p} x^2 dx$$

$$= \sqrt{2p} \cdot \frac{2}{3} [x^{3/2}]_0^{2p} - \frac{1}{2p} \cdot \frac{1}{3} [x^3]_0^{2p}$$

$$= \frac{2\sqrt{2}}{3} \sqrt{p}\, [(2p)^{3/2} - 0] - \frac{1}{6p} [(2p)^3 - 0]$$

$$= \frac{2\sqrt{2}}{3} \sqrt{p} \cdot 2\sqrt{2}\, p^{\frac{3}{2}} - \frac{1}{6p} \cdot 8p^3$$

$$= \frac{8}{3} \cdot p^2 - \frac{8}{6} p^2 = \frac{8}{6} p^2 = \frac{4}{3} p^2 \text{ sq. units}$$

Hence, the required area $= \frac{4}{3} p^2$ sq. units.

Q3. Find the area of the region bounded by the curve $y = x^3$, $y = x + 6$ and $x = 0$.

Sol. We are given that: $y = x^3$, $y = x + 6$ and $x = 0$

Solving $y = x^3$ and $y = x + 6$, we get

$$x + 6 = x^3$$

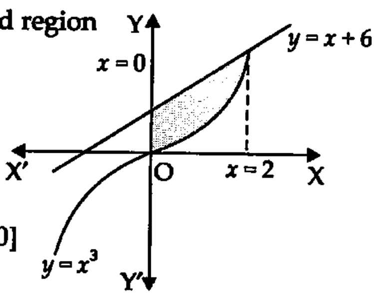
$$\Rightarrow \quad x^3 - x - 6 = 0$$

$$\Rightarrow \quad x^2 (x - 2) + 2x (x - 2) + 3 (x - 2) = 0$$

$$\Rightarrow \quad (x - 2)(x^2 + 2x + 3) = 0$$

$x^2 + 2x + 3 = 0$ has no real roots. $\quad \therefore x = 2$

$\therefore$ Required area of the shaded region



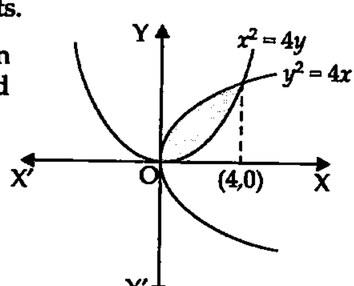
$$= \int_0^2 (x + 6)\, dx - \int_0^2 x^3 dx$$

$$= \left[\frac{x^2}{2} + 6x\right]_0^2 - \frac{1}{4} [x^4]_0^2$$

$$= \left(\frac{4}{2} + 12\right) - (0 + 0) - \frac{1}{4} [(2)^4 - 0]$$

$$= 14 - \frac{1}{4} \times 16 = 14 - 4 = 10 \text{ sq. units.}$$

Q4. Find the area of the region bounded by the curve $y^2 = 4x$ and $x^2 = 4y$.



Sol. We have $y^2 = 4x$ and $x^2 = 4y$.

$$y = \frac{x^2}{4}$$

$$\Rightarrow \quad \left(\frac{x^2}{4}\right)^2 = 4x$$

$\Rightarrow \quad \dfrac{x^4}{16} = 4x$

$\Rightarrow \quad x^4 = 64x \Rightarrow x^4 - 64x = 0$

$\Rightarrow \quad x(x^3 - 64) = 0$

$\therefore x = 0, x = 4$

Required area $= \int_0^4 \sqrt{4x}\, dx - \int_0^4 \dfrac{x^2}{4} dx = 2\int_0^4 \sqrt{x}\, dx - \dfrac{1}{4}\int_0^4 x^2\, dx$

$= 2 . \dfrac{2}{3}[x^{3/2}]_0^4 - \dfrac{1}{4} . \dfrac{1}{3}[x^3]_0^4$

$= \dfrac{4}{3}[(4)^{3/2} - 0] - \dfrac{1}{12}[(4)^3 - 0] = \dfrac{4}{3}[8] - \dfrac{1}{12}[64]$

$= \dfrac{32}{3} - \dfrac{16}{3} = \dfrac{16}{3}$ sq. units

Hence, the required area $= \dfrac{16}{3}$ sq. units.

Q5. Find the area of the region included between $y^2 = 9x$ and $y = x$.

Sol. Given that: $y^2 = 9x$ ...(i)

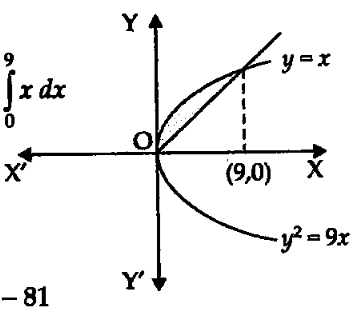
and $y = x$ ...(ii)

Solving eqns. (i) and (ii) we have

$x^2 = 9x \Rightarrow x^2 - 9x = 0$

$x(x-9) = 0 \quad \therefore \; x = 0, 9$

Required area



$= \int_0^9 \sqrt{9x}\, dx - \int_0^9 x\, dx = 3\int_0^9 \sqrt{x}\, dx - \int_0^9 x\, dx$

$= 3 . \dfrac{2}{3}[x^{3/2}]_0^9 - \dfrac{1}{2}[x^2]_0^9$

$= 2[(9)^{3/2} - 0] - \dfrac{1}{2}[(9)^2 - 0]$

$= 2(27) - \dfrac{1}{2}(81) = 54 - \dfrac{81}{2} = \dfrac{108 - 81}{2}$

$= \dfrac{27}{2}$ sq. units

Hence, the required area $= \dfrac{27}{2}$ sq. units.

Q6. Find the area of the region enclosed by the parabola $x^2 = y$ and the line $y = x + 2$.

Sol. Here, $x^2 = y$ and $y = x + 2$

$\therefore \quad x^2 = x + 2$

$\Rightarrow \quad x^2 - x - 2 = 0$

$\Rightarrow \quad x^2 - 2x + x - 2 = 0$

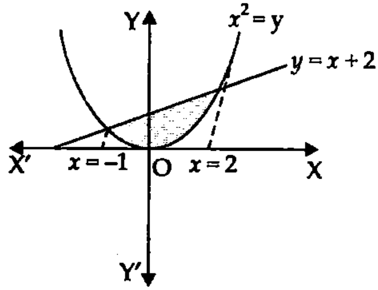
$\Rightarrow \quad x(x-2) + 1(x-2) = 0$

$\Rightarrow \quad (x-2)(x+1) = 0$

$\therefore x = -1, 2$

Graph of $y = x + 2$

x	0	–2
y	2	0



Area of the required region

$$= \int_{-1}^{2} (x+2)\,dx - \int_{-1}^{2} x^2\,dx = \left[\frac{x^2}{2} + 2x\right]_{-1}^{2} - \frac{1}{3}[x^3]_{-1}^{2}$$

$$= \left[\left(\frac{4}{2} + 4\right) - \left(\frac{1}{2} - 2\right)\right] - \frac{1}{3}[8 - (-1)]$$

$$= \left(6 + \frac{3}{2}\right) - \frac{1}{3}(9) = \frac{15}{2} - 3 = \frac{9}{2} \text{ sq. units}$$

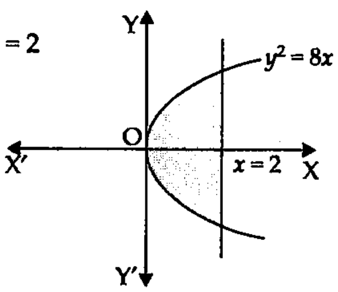
Hence, the required area $= \frac{9}{2}$ sq. units.

Q7. Find the area of the region bounded by the line $x = 2$ and parabola $y^2 = 8x$.

Sol. Here, $\quad y^2 = 8x$ and $x = 2$

$y^2 = 8(2) = 16$

$\therefore \quad y = \pm 4$



Required area

$$= 2\int_{0}^{2} \sqrt{8x}\,dx = 2 \times 2\sqrt{2} \int_{0}^{2} \sqrt{x}\,dx$$

$$= 4\sqrt{2} \times \frac{2}{3}[x^{3/2}]_0^2$$

$$= \frac{8\sqrt{2}}{3}[(2)^{3/2}] = \frac{8\sqrt{2}}{3} \times 2\sqrt{2} = \frac{32}{3} \text{ sq. units}$$

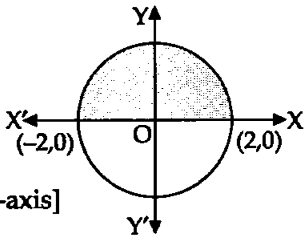
Hence, the area of the region $= \frac{32}{3}$ sq. units.

Q8. Sketch the region $\{(x, 0) : y = \sqrt{4 - x^2}\}$ and x-axis. Find the area of the region using integration.

Sol. Given that $\{(x, 0) : y = \sqrt{4 - x^2}\}$

$\Rightarrow \quad y^2 = 4 - x^2$

$\Rightarrow \quad x^2 + y^2 = 4$ which is a circle.



Required area

$$= 2 . \int_0^2 \sqrt{4 - x^2}\, dx$$

[Since circle is symmetrical about y-axis]

$$= 2 . \int_0^2 \sqrt{(2)^2 - x^2}\, dx$$

$$= 2 . \left[\frac{x}{2}\sqrt{4 - x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2$$

$$= 2\left[\left(\frac{2}{2}\sqrt{4-4} + 2\sin^{-1}(1)\right) - (0+0)\right]$$

$$= 2\left[2 . \frac{\pi}{2}\right] = 2\pi \text{ sq. units}$$

Hence, the required area = 2π sq. units.

Q9. Calculate the area under the curve $y = 2\sqrt{x}$ included between the lines $x = 0$ and $x = 1$.

Sol. Given the curves $y = 2\sqrt{x}$, $x = 0$ and $x = 1$.

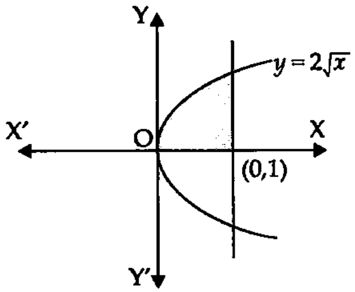
$$y = 2\sqrt{x} \Rightarrow y^2 = 4x \text{ (Parabola)}$$

Required area $= \int_0^1 (2\sqrt{x})\, dx$

$$= 2 \times \frac{2}{3}[x^{3/2}]_0^1$$

$$= \frac{4}{3}[(1)^{3/2} - 0]$$

$$= \frac{4}{3} \text{ sq. units}$$



Hence, required area = $\frac{4}{3}$ sq. units.

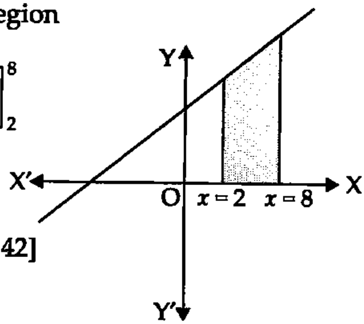
Q10. Using integration, find the area of the region bounded by the line $2y = 5x + 7$, x-axis and the lines $x = 2$ and $x = 8$.

Sol. Given that: $2y = 5x + 7$, x-axis, $x = 2$ and $x = 8$.

Let us draw the graph of $2y = 5x + 7 \Rightarrow y = \dfrac{5x+7}{2}$

x	1	–1
y	6	1

Area of the required shaded region



$$= \int_2^8 \left(\frac{5x+7}{2}\right) dx = \frac{1}{2}\left[\frac{5}{2}x^2 + 7x\right]_2^8$$

$$= \frac{1}{2}\left[\frac{5}{2}(64-4) + 7(8-2)\right]$$

$$= \frac{1}{2}\left[\frac{5}{2} \times 60 + 7 \times 6\right] = \frac{1}{2}[150 + 42]$$

$$= \frac{1}{2} \times 192 = 96 \text{ sq. units}$$

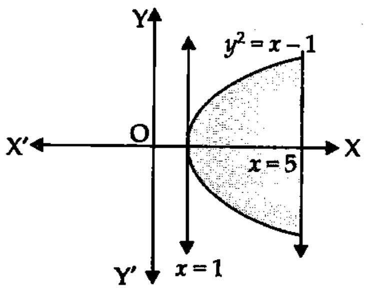
Hence, the required area = 96 sq. units.

Q11. Draw a rough sketch of the curve $y = \sqrt{x-1}$ in the interval [1, 5]. Find the area under the curve and between the lines $x = 1$ and $x = 5$.

Sol. Here, we have $y = \sqrt{x-1}$

$\Rightarrow$ $y^2 = x - 1$ (Parabola)

Area of the required region



$$= \int_1^5 \sqrt{x-1}\, dx$$

$$= \frac{2}{3}\left[(x-1)^{3/2}\right]_1^5$$

$$= \frac{2}{3}\left[(5-1)^{3/2} - 0\right] = \frac{2}{3} \times (4)^{3/2}$$

$$= \frac{2}{3} \times 8 = \frac{16}{3} \text{ sq. units}$$

Hence, the required area $= \frac{16}{3}$ sq. units.

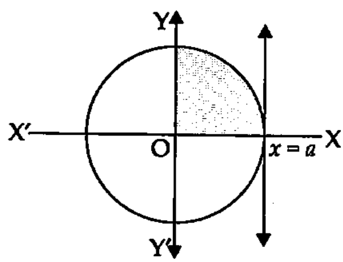
Q12. Determine the area under the curve $y = \sqrt{a^2 - x^2}$ included between the lines $x = 0$ and $x = a$.

Sol. Here, we are given $y = \sqrt{a^2 - x^2}$

$\Rightarrow$ $y^2 = a^2 - x^2$

$\Rightarrow$ $x^2 + y^2 = a^2$

Area of the shaded region



$$= 2\left[(1)^{3/2} - 0\right] - \frac{3}{2}[(1)^2 - 0]$$

$$= \left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \left[\frac{a}{2}\sqrt{a^2 - a^2} + \frac{a^2}{2}\sin^{-1}\frac{a}{a} - 0 - 0\right]$$

$$= \frac{a^2}{2}\sin^{-1}(1) = \frac{a^2}{2}\cdot\frac{\pi}{2} = \frac{\pi a^2}{4}$$

Hence, the required area $= \frac{\pi a^2}{4}$ sq. units.

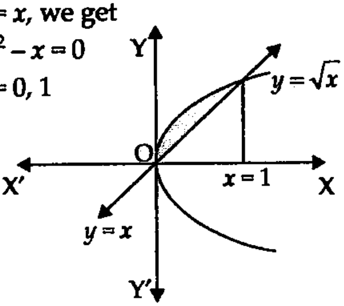
Q13. Find the area of the region bounded by $y = \sqrt{x}$ and $y = x$.

Sol. We are given the equations of curve $y = \sqrt{x}$ and line $y = x$.

Solving $y = \sqrt{x} \Rightarrow y^2 = x$ and $y = x$, we get

$$x^2 = x \Rightarrow x^2 - x = 0$$

$$\Rightarrow \quad x(x-1) = 0 \quad \therefore x = 0, 1$$



Required area of the shaded region

$$= \int_0^1 \sqrt{x}\,dx - \int_0^1 x\,dx$$

$$= \frac{2}{3}\left[x^{3/2}\right]_0^1 - \frac{1}{2}\left[x^2\right]_0^1$$

$$= \frac{2}{3}[(1)^{3/2} - 0] - \frac{1}{2}[(1)^2 - 0]$$

$$= \frac{2}{3} - \frac{1}{2} \Rightarrow \frac{4-3}{6} \Rightarrow \frac{1}{6} \text{ sq. units}$$

Hence, the required area $= \frac{1}{6}$ sq. units.

Q14. Find the area enclosed by the curve $y = -x^2$ and the straight line $x + y + 2 = 0$.

Sol. We are given that $y = -x^2$ or $x^2 = -y$

and the line $x + y + 2 = 0$

Solving the two equations, we get

$$x - x^2 + 2 = 0$$

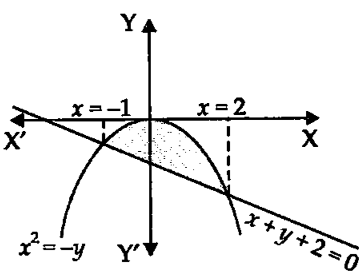
$$\Rightarrow \quad x^2 - x - 2 = 0$$

$$\Rightarrow \quad x^2 - 2x + x - 2 = 0$$

$$\Rightarrow \quad x(x-2) + 1(x-2) = 0$$

$$\Rightarrow \quad (x-2)(x+1) = 0$$

$$\therefore \quad x = -1, 2$$



Area of the required shaded region

$$= \left| \int_{-1}^{2} (-x-2)\, dx - \int_{-1}^{2} -x^2\, dx \right|$$

$$\Rightarrow \quad \left| -\left[\frac{x^2}{2} + 2x\right]_{-1}^{2} + \frac{1}{3}[x^3]_{-1}^{2} \right|$$

$$\Rightarrow \quad \left| -\left[\left(\frac{4}{2} + 4\right) - \left(\frac{1}{2} - 2\right)\right] + \frac{1}{3}(8+1) \right|$$

$$\Rightarrow \quad \left| -\left(6 + \frac{3}{2}\right) + \frac{1}{3}(9) \right| \Rightarrow \left| -\frac{15}{2} + 3 \right|$$

$$\Rightarrow \quad \left| \frac{-15+6}{2} \right| = \left| \frac{-9}{2} \right| = \frac{9}{2} \text{ sq. units}$$

Q15. Find the area bounded by the curve $y = \sqrt{x}$, $x = 2y + 3$ in the first quadrant and x-axis.

Sol. Given that: $y = \sqrt{x}$, $x = 2y + 3$, first quadrant and x-axis.

Solving $y = \sqrt{x}$ and $x = 2y + 3$, we get

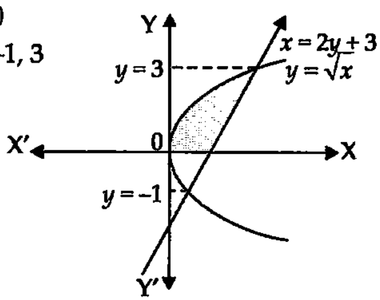
$$y = \sqrt{2y+3} \Rightarrow y^2 = 2y + 3$$

$$\Rightarrow \quad y^2 - 2y - 3 = 0 \Rightarrow y^2 - 3y + y - 3 = 0$$

$$\Rightarrow \quad y(y-3) + 1\,(y-3) = 0$$

$$\Rightarrow \quad (y+1)\,(y-3) = 0$$

$$\therefore \quad y = -1, 3$$



Area of shaded region

$$= \int_0^3 (2y+3)\, dy - \int_0^3 y^2 dy$$

$$= \left[2\frac{y^2}{2} + 3y\right]_0^3 - \frac{1}{3}\left[y^3\right]_0^3$$

$$= [(9+9) - (0+0)] - \frac{1}{3}[27 - 0]$$

$$= 18 - 9 = 9 \text{ sq. units}$$

Hence, the required area = 9 sq. units.

LONG ANSWER TYPE QUESTIONS

Q16. Find the area of the region bounded by the curve $y^2 = 2x$ and $x^2 + y^2 = 4x$.

Sol. Equations of the curves are given by

$$x^2 + y^2 = 4x \qquad ...(i)$$

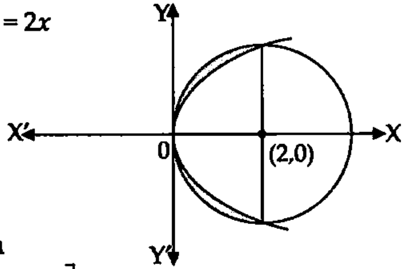
and

$$y^2 = 2x \qquad ...(ii)$$

$$\Rightarrow \quad x^2 - 4x + y^2 = 0$$
$$\Rightarrow \quad x^2 - 4x + 4 - 4 + y^2 = 0$$
$$\Rightarrow \quad (x-2)^2 + y^2 = 4$$

Clearly it is the equation of a circle having its centre (2, 0) and radius 2.

Solving $x^2 + y^2 = 4x$ and $y^2 = 2x$



$$x^2 + 2x = 4x$$
$$\Rightarrow \quad x^2 + 2x - 4x = 0$$
$$\Rightarrow \quad x^2 - 2x = 0$$
$$\Rightarrow \quad x(x-2) = 0$$
$$\therefore \quad x = 0, 2$$

Area of the required region

$$= 2\left[\int_0^2 \sqrt{4-(x-2)^2}\, dx - \int_0^2 \sqrt{2x}\, dx\right]$$

[$\because$ Parabola and circle both are symmetrical about x-axis.]

$$= 2\left[\frac{x-2}{2}\sqrt{4-(x-2)^2} + \frac{4}{2}\sin^{-1}\frac{x-2}{2}\right]_0^2 - 2.\sqrt{2}.\frac{2}{3}\left[x^{3/2}\right]_0^2$$

$$= 2\left[(0+0) - (0 + 2\sin^{-1}(-1))\right] - \frac{4\sqrt{2}}{3}\left[2^{3/2} - 0\right]$$

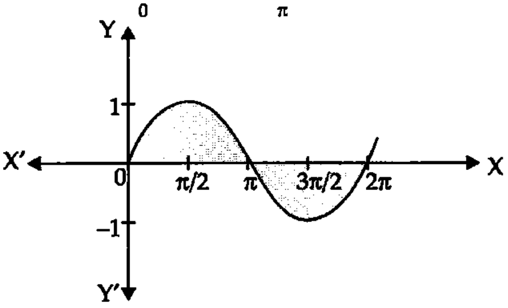
$$= -2 \times 2.\left(-\frac{\pi}{2}\right) - \frac{4\sqrt{2}}{3}.2\sqrt{2}$$

$$= 2\pi - \frac{16}{3} = 2\left(\pi - \frac{8}{3}\right) \text{ sq. units}$$

Hence, the required area $= 2\left(\pi - \frac{8}{3}\right)$ sq. units.

Q17. Find the area bounded by the curve $y = \sin x$ between $x = 0$ and $x = 2\pi$.

Sol. Required area $= \int_0^{\pi} \sin x\, dx + \int_{\pi}^{2\pi} |\sin x|\, dx$

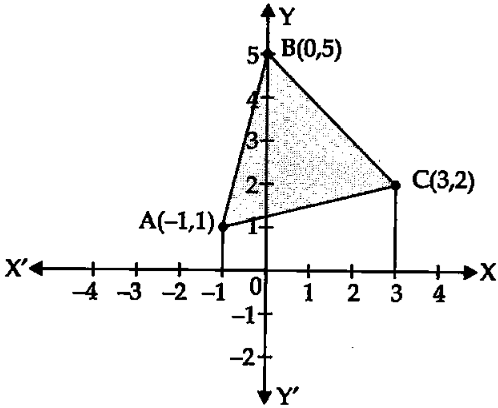


$$= -[\cos x]_0^{\pi} + |(-\cos x)|_{\pi}^{2\pi} = -[\cos\pi - \cos 0] + [\cos 2\pi - \cos\pi]$$

$$= -[-1-1] + [1+1] = 2 + 2 = 4 \text{ sq. units}$$

Q18. Find the area of the region bounded by the triangle whose vertices are (–1, 1), (0, 5) and (3, 2), using integration.

Sol. The coordinates of the vertices of ΔABC are given by A(–1, 1), B (0, 5) and C (3, 2).



Equation of AB is $y - 1 = \dfrac{5-1}{0+1}(x+1)$

$\Rightarrow \quad y - 1 = 4x + 4$

$\therefore \quad y = 4x + 4 + 1 \Rightarrow y = 4x + 5 \qquad ...(i)$

Equation of BC is $y - 5 = \dfrac{2-5}{3-0}(x-0)$

$\Rightarrow \quad y - 5 = -x$

$\therefore \quad y = 5 - x \qquad ...(ii)$

Equation of CA is

$$y - 1 = \frac{2-1}{3+1}(x+1)$$

$\Rightarrow \quad y - 1 = \dfrac{1}{4}x + \dfrac{1}{4} \Rightarrow y = \dfrac{1}{4}x + \dfrac{1}{4} + 1$

$\therefore \quad y = \dfrac{1}{4}x + \dfrac{5}{4} = \dfrac{1}{4}(5+x)$

Area of ΔABC

$$= \int_{-1}^{0}(4x+5)\,dx + \int_{0}^{3}(5-x)\,dx - \int_{-1}^{3}\frac{1}{4}(5+x)\,dx$$

$$= \frac{4}{2}[x^2]_{-1}^{0} + 5[x]_{-1}^{0} + 5[x]_0^3 - \frac{1}{2}[x^2]_0^3 - \frac{1}{4}\left[5x + \frac{x^2}{2}\right]_{-1}^{3}$$

$$= 2(0-1)+5(0+1)+5(3-0)-\frac{1}{2}(9-0) -\frac{1}{4}\left[\left(15+\frac{9}{2}\right)-\left(-5+\frac{1}{2}\right)\right]$$

$$= -2+5+15-\frac{9}{2}-\frac{1}{4}\left(\frac{39}{2}+\frac{9}{2}\right)$$

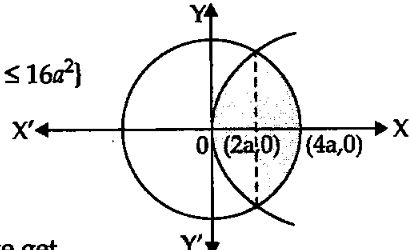
$$= 18-\frac{9}{2}-\frac{1}{4}\times\frac{48}{2} = 18-\frac{9}{2}-6 = 12-\frac{9}{2} = \frac{15}{2} \text{ sq. units}$$

Hence, the required area $= \frac{15}{2}$ sq. units.

Q19. Draw a rough sketch of the region $\{(x, y) : y^2 \le 6ax \text{ and } x^2 + y^2 \le 16a^2\}$. Also find the area of the region sketched using method of integration.

Sol. Given that:

$\{(x, y) : y^2 \le 6ax \text{ and } x^2 + y^2 \le 16a^2\}$



Equation of Parabola is

$$y^2 = 6ax \quad ...(i)$$

and equation of circle is

$$x^2 + y^2 \le 16a^2 \quad ...(ii)$$

Solving eqns. (*i*) and (*ii*) we get

$$x^2 + 6ax = 16a^2$$

$$\Rightarrow \quad x^2 + 6ax - 16a^2 = 0$$

$$\Rightarrow \quad x^2 + 8ax - 2ax - 16a^2 = 0$$

$$\Rightarrow \quad x(x+8a) - 2a(x+8a) = 0$$

$$\Rightarrow \quad (x+8a)(x-2a) = 0$$

$\therefore x = 2a$ and $x = -8a$. (Rejected as it is out of region)

Area of the required shaded region

$$= 2\left[\int_0^{2a} \sqrt{6ax}\, dx + \int_{2a}^{4a} \sqrt{16a^2 - x^2}\, dx\right]$$

$$= 2\left[\sqrt{6a}\int_0^{2a} \sqrt{x}\, dx + \int_{2a}^{4a} \sqrt{(4a)^2 - x^2}\, dx\right]$$

$$= 2\sqrt{6a}\cdot\frac{2}{3}\cdot\left[x^{3/2}\right]_0^{2a} + 2\left[\frac{x}{2}\sqrt{(4a)^2 - x^2} + \frac{16a^2}{2}\sin^{-1}\frac{x}{4a}\right]_{2a}^{4a}$$

$$= \frac{4\sqrt{6}}{3}\cdot\sqrt{a}\left[(2a)^{3/2} - 0\right] + \left[x\sqrt{(4a)^2 - x^2} + 16a^2\sin^{-1}\frac{x}{4a}\right]_{2a}^{4a}$$

$$= \frac{4\sqrt{6}}{3}\sqrt{a}\,.\,2\sqrt{2}\,.\,a^{3/2} + \left[0 + 16a^2\sin^{-1}\left(\frac{4a}{4a}\right) - 2a\sqrt{16a^2 - 4a^2} - 16a^2\sin^{-1}\frac{2a}{4a}\right]$$

$$= \frac{8\sqrt{12}}{3}a^2 + \left[16a^2.\sin^{-1}(1) - 2a\sqrt{12a^2} - 16a^2\sin^{-1}\frac{1}{2}\right]$$

$$= \frac{16\sqrt{3}}{3}a^2 + \left[16a^2.\frac{\pi}{2} - 2a\,.\,2\sqrt{3}a - 16a^2.\frac{\pi}{6}\right]$$

$$= \frac{16\sqrt{3}}{3}a^2 + 8\pi a^2 - 4\sqrt{3}a^2 - \frac{8}{3}\pi a^2$$

$$= \left(\frac{16\sqrt{3}}{3} - 4\sqrt{3}\right)a^2 + \frac{16}{3}\pi a^2 = \frac{4\sqrt{3}}{3}a^2 + \frac{16}{3}\pi a^2$$

$$= \frac{4}{3}\left(\sqrt{3} + 4\pi\right)a^2$$

Hence, required area $= \frac{4}{3}\left(\sqrt{3} + 4\pi\right)a^2$ sq. units.

Q20. Compute the area bounded by the lines $x + 2y = 2$, $y - x = 1$ and $2x + y = 7$.

Sol. Given that:

$$x + 2y = 2 \quad \text{...(i)}$$
$$y - x = 1 \quad \text{...(ii)}$$
and
$$2x + y = 7 \quad \text{...(iii)}$$

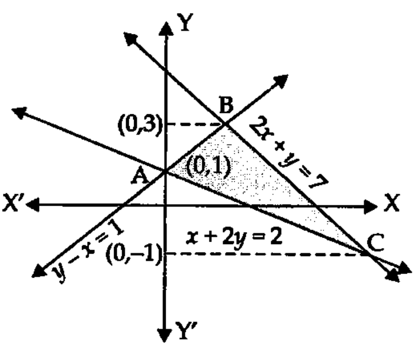
x	0	2
y	1	0

x	0	–1
y	1	0

x	0	7/2
y	7	0

Solving eqns. (*ii*) and (*iii*) we get

$$y = 1 + x$$
$$\therefore \quad 2x + 1 + x = 7$$
$$3x = 6$$
$$\Rightarrow \quad x = 2$$
$$\therefore \quad y = 1 + 2 = 3$$



Coordinates of B = (2, 3)

Solving eqns. (*i*) and (*iii*) we get

$$x + 2y = 2$$
$$\therefore \quad x = 2 - 2y$$
$$2x + y = 7$$
$$2(2 - 2y) + y = 7$$
$$\Rightarrow \quad 4 - 4y + y = 7 \Rightarrow -3y = 3$$
$$\therefore \quad y = -1 \text{ and } x = 4$$

$\therefore$ Coordinates of $C = (4, -1)$ and coordinates of $A = (0, 1)$.
Taking the limits on y-axis, we get

$$\int_{-1}^{3} x_{BC}\, dy - \int_{-1}^{1} x_{AC}\, dy - \int_{1}^{3} x_{AB}\, dy$$

$$= \int_{-1}^{3} \frac{7-y}{2}\, dy - \int_{-1}^{1} (2-2y)\, dy - \int_{1}^{3} (y-1)\, dy$$

$$= \frac{1}{2}\left[7y - \frac{y^2}{2}\right]_{-1}^{3} - 2\left[y - \frac{y^2}{2}\right]_{-1}^{1} - \left[\frac{y^2}{2} - y\right]_{1}^{3}$$

$$= \frac{1}{2}\left[\left(21 - \frac{9}{2}\right) - \left(-7 - \frac{1}{2}\right)\right] - 2\left[\left(1 - \frac{1}{2}\right) - \left(-1 - \frac{1}{2}\right)\right] - \left[\left(\frac{9}{2} - 3\right) - \left(\frac{1}{2} - 1\right)\right]$$

$$= \frac{1}{2}\left[\frac{33}{2} + \frac{15}{2}\right] - 2\left[\frac{1}{2} + \frac{3}{2}\right] - \left[\frac{3}{2} + \frac{1}{2}\right]$$

$$= \frac{1}{2} \times 24 - 2 \times 2 - 2 \Rightarrow 12 - 4 - 2 = 6 \text{ sq.units}$$

Hence, the required area = 6 sq. units.

Q21. Find the area bounded by the lines $y = 4x + 5$, $y = 5 - x$ and $4y = x + 5$.

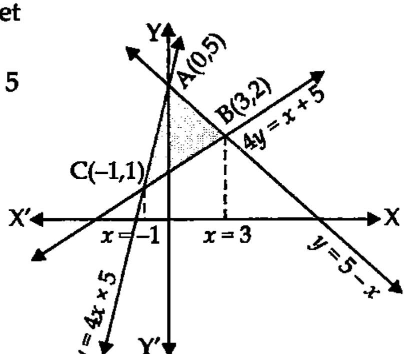
Sol. Given that

$$y = 4x + 5 \qquad \ldots(i)$$
$$y = 5 - x \qquad \ldots(ii)$$
and
$$4y = x + 5 \qquad \ldots(iii)$$

x	0	$-5/4$
y	5	0

x	0	5
y	5	0

x	0	-5
y	$5/4$	0



Solving eq. (*i*) and (*ii*) we get

$$4x + 5 = 5 - x$$
$$\Rightarrow \qquad x = 0 \text{ and } y = 5$$

$\therefore$ Coordinates of A = (0, 5)

Solving eq. (ii) and (iii)

$$y = 5 - x$$
$$4y = x + 5$$
$$5y = 10$$

$\therefore y = 2$ and $x = 3$

$\therefore$ Coordinates of B = (3, 2)

Solving eq. (*i*) and (*iii*)

$$y = 4x + 5$$

$$4y = x + 5$$

$$\Rightarrow \quad 4(4x+5) = x+5$$

$$\Rightarrow \quad 16x + 20 = x + 5 \Rightarrow 15x = -15$$

$$\therefore \quad x = -1 \text{ and } y = 1$$

$\therefore$ Coordinates of C = (–1, 1).

$\therefore$ Area of required regions

$$= \int_{-1}^{0} y_{AC}\, dx + \int_{0}^{3} y_{AB}\, dx - \int_{-1}^{3} y_{CB}\, dx$$

$$= \int_{-1}^{0} (4x+5)\, dx + \int_{0}^{3} (5-x)\, dx - \int_{-1}^{3} \frac{x+5}{4}\, dx$$

$$= \left[4\frac{x^2}{2} + 5x\right]_{-1}^{0} + \left[5x - \frac{x^2}{2}\right]_{0}^{3} - \frac{1}{4}\left[\frac{x^2}{2} + 5x\right]_{-1}^{3}$$

$$= [(0+0)-(2-5)] + \left[\left(15 - \frac{9}{2}\right) - (0-0)\right] - \frac{1}{4}\left[\left(\frac{9}{2} + 15\right) - \left(\frac{1}{2} - 5\right)\right]$$

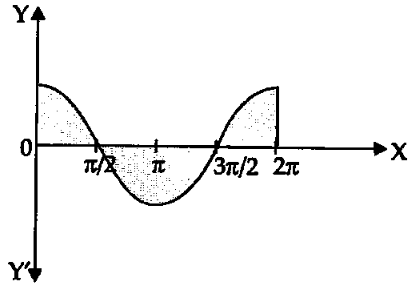
$$= 3 + \frac{21}{2} - \frac{1}{4}\left[\frac{39}{2} + \frac{9}{2}\right] = 3 + \frac{21}{2} - \frac{1}{4} \times 24 \Rightarrow 3 + \frac{21}{2} - 6$$

$$= \frac{15}{2} \text{ sq. units}$$

Hence, the required area $= \dfrac{15}{2}$ sq. units.

Q22. Find the area bounded by the curve $y = 2\cos x$ and the x-axis from $x = 0$ to $x = 2\pi$.

Sol. Given equation of the curve is $y = 2\cos x$



$\therefore$ Area of the shaded region

$$\int_{0}^{2\pi} 2\cos x\, dx = \int_{0}^{\pi/2} 2\cos x\, dx + \int_{\pi/2}^{3\pi/2} |2\cos x|\, dx + \int_{3\pi/2}^{2\pi} 2\cos x\, dx$$

$$= 2\left[\sin x\right]_0^{\pi/2} + \left|\left[2\sin x\right]_{\pi/2}^{3\pi/2}\right| + 2\left[\sin x\right]_{3\pi/2}^{2\pi}$$

$$= 2\left[\sin\frac{\pi}{2} - \sin 0\right] + \left|2\left(\sin\frac{3\pi}{2} - \sin\frac{\pi}{2}\right)\right|$$

$$+ 2\left[\sin 2\pi - \sin\frac{3\pi}{2}\right]$$

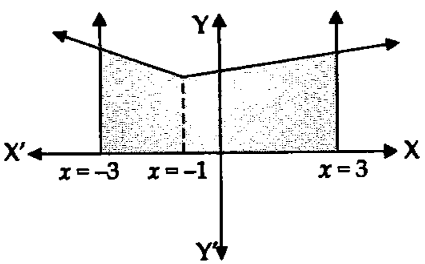
$$= 2(1) + |2(-1-1)| + 2(0+1) = 2 + 4 + 2 = 8 \text{ sq. units}$$

Q23. Draw a rough sketch of the given curve $y = 1 + |x+1|$, $x = -3$, $x = 3$, $y = 0$ and find the area of the region bounded by them, using integration.

Sol. Given equations are

$y = 1 + |x+1|$, $x = -3$

and $x = 3$, $y = 0$



Taking $y = 1 + |x+1|$

$\Rightarrow \quad y = 1 + x + 1$

$\Rightarrow \quad y = x + 2$

and $\quad y = 1 - x - 1 \quad \Rightarrow \quad y = -x$

On solving we get $x = -1$

Area of the required regions

$$= \int_{-3}^{-1} -x\,dx + \int_{-1}^{3} (x+2)\,dx$$

$$= -\left[\frac{x^2}{2}\right]_{-3}^{-1} + \left[\frac{x^2}{2} + 2x\right]_{-1}^{3} = -\left[\frac{1}{2} - \frac{9}{2}\right] + \left[\left(\frac{9}{2} + 6\right) - \left(\frac{1}{2} - 2\right)\right]$$

$$= -(-4) + \left[\frac{21}{2} + \frac{3}{2}\right] = 4 + 12 = 16 \text{ sq. units}$$

Hence, the required area = 16 sq. units.

OBJECTIVE TYPE QUESTIONS

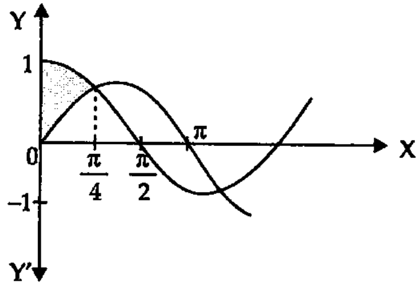
Choose the correct answer from the given four options in each of the Exercises 24 to 34.

Q24. The area of the region bounded by the y-axis, $y = \cos x$ and $y = \sin x$, where $0 \le x \le \frac{\pi}{2}$ is

(*a*) $\sqrt{2}$ sq. units (*b*) $(\sqrt{2} + 1)$ sq. units

(*c*) $(\sqrt{2} - 1)$ sq. units (*d*) $(2\sqrt{2} - 1)$ sq. units

Sol. Given that y-axis, $y = \cos x$, $y = \sin x$, $0 \le x \le \frac{\pi}{2}$



$$\text{Required area} = \int_0^{\pi/4} \cos x\, dx - \int_0^{\pi/4} \sin x\, dx$$

$$= [\sin x]_0^{\pi/4} - [-\cos x]_0^{\pi/4}$$

$$= \left[\sin\frac{\pi}{4} - \sin 0\right] + \left[\cos\frac{\pi}{4} - \cos 0\right]$$

$$= \left[\frac{1}{\sqrt{2}} - 0 + \frac{1}{\sqrt{2}} - 1\right] = \frac{2}{\sqrt{2}} - 1$$

$$= (\sqrt{2} - 1) \text{ sq. units}$$

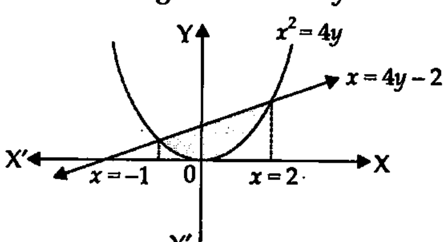
Hence, the correct option is (*c*).

Q25. The area of the region bounded by the curve $x^2 = 4y$ and the straight line $x = 4y - 2$ is

(*a*) $\frac{3}{8}$ sq. units (*b*) $\frac{5}{8}$ sq. units

(*c*) $\frac{7}{8}$ sq. units (*d*) $\frac{9}{8}$ sq. units

Sol. Given that: The equation of parabola is $x^2 = 4y$ ...(*i*)
and equation of straight line is $x = 4y - 2$ ...(*ii*)



Solving eqn. (*i*) and (*ii*) we get

$$y = \frac{x^2}{4}$$

$$x = 4\left(\frac{x^2}{4}\right) - 2$$

$$\Rightarrow \quad x = x^2 - 2$$

$$\Rightarrow \quad x^2 - x - 2 = 0 \Rightarrow x^2 - 2x + x - 2 = 0$$

$$\Rightarrow \quad x(x-2) + 1(x-2) = 0 \Rightarrow (x-2)(x+1) = 0 \therefore x = -1, x = 2$$

Required area $= \displaystyle\int_{-1}^{2} \frac{x+2}{4}\,dx - \int_{-1}^{2} \frac{x^2}{4}\,dx$

$$= \frac{1}{4}\left[\frac{x^2}{2} + 2x\right]_{-1}^{2} - \frac{1}{4}\cdot\frac{1}{3}[x^3]_{-1}^{2}$$

$$= \frac{1}{4}\left[\left(\frac{4}{2} + 4\right) - \left(\frac{1}{2} - 2\right)\right] - \frac{1}{12}[8+1]$$

$$= \frac{1}{4}\left[6 + \frac{3}{2}\right] - \frac{1}{12}[9] = \frac{1}{4} \times \frac{15}{2} - \frac{3}{4}$$

$$= \frac{15}{8} - \frac{3}{4} = \frac{9}{8} \text{ sq. units}$$

Hence, the correct option is (*d*).

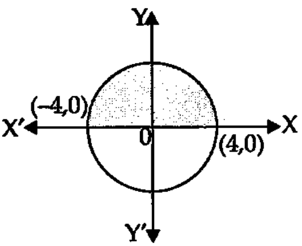
Q26. The area of the region bounded by the curve $y = \sqrt{16 - x^2}$ and x-axis is

(*a*) 8π sq. units (*b*) 20π sq. units

(*c*) 16π sq. units (*d*) 256π sq. units

Sol. Here, equation of curve is $y = \sqrt{16 - x^2}$

Required area



$$= 2\left[\int_0^4 \sqrt{16 - x^2}\,dx\right]$$

$$= 2\left[\frac{x}{2}\sqrt{16 - x^2} + \frac{16}{2}\sin^{-1}\frac{x}{4}\right]_0^4$$

$$= 2\left[\left(0 + 8\sin^{-1}\frac{4}{4}\right) - (0+0)\right]$$

$$= 2\,[8\sin^{-1}(1)] = 16.\frac{\pi}{2} = 8\pi \text{ sq. units}$$

Hence, the correct option is (*a*).

Q27. Area of the region in the first quadrant enclosed by the x-axis, the line $y = x$ and the circle $x^2 + y^2 = 32$ is

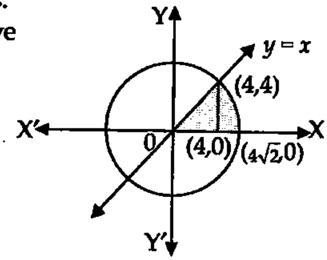
(*a*) 16π sq. units (*b*) 4π sq. units

(*c*) 32π sq. units (*d*) 24 sq. units

Sol. Given equation of circle is $x^2+y^2=32 \Rightarrow x^2+y^2=(4\sqrt{2})^2$ and the line is $y=x$ and the x-axis.

Solving the two equations we have

$$x^2+x^2=32$$
$$\Rightarrow \quad 2x^2=32$$
$$\Rightarrow \quad x^2=16$$
$$\therefore \quad x=\pm 4$$



Required area

$$=\int_0^4 x\,dx+\int_4^{4\sqrt{2}}\sqrt{(4\sqrt{2})^2-x^2}\,dx$$

$$=\frac{1}{2}[x^2]_0^4+\left[\frac{x}{2}\sqrt{(4\sqrt{2})^2-x^2}+\frac{32}{2}\sin^{-1}\frac{x}{4\sqrt{2}}\right]_4^{4\sqrt{2}}$$

$$=\frac{1}{2}[16-0]+\left[0+16\sin^{-1}\left(\frac{4\sqrt{2}}{4\sqrt{2}}\right)-2\sqrt{32-16}-16\sin^{-1}\frac{4}{4\sqrt{2}}\right]$$

$$=8+\left[16\sin^{-1}(1)-8-16\sin^{-1}\frac{1}{\sqrt{2}}\right]$$

$$=8+16\cdot\frac{\pi}{2}-8-16\cdot\frac{\pi}{4}=8\pi-4\pi=4\pi \text{ sq. units}$$

Hence, the correct option is (*b*).

Q28. Area of the region bounded by the curve $y=\cos x$ between $x=0$ and $x=\pi$ is

(*a*) 2 sq. units (*b*) 4 sq. units

(*c*) 3 sq. units (*d*) 1 sq. units

Sol. Given that: $y=\cos x$, $x=0$, $x=\pi$

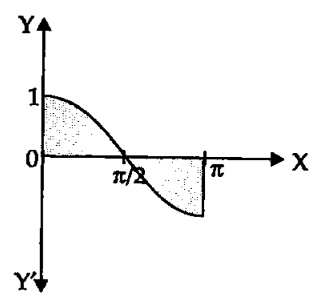
Required area

$$=\int_0^{\pi/2}\cos x\,dx+\left|\int_{\pi/2}^{\pi}\cos x\,dx\right|$$

$$=[\sin x]_0^{\pi/2}+\left|(\sin x)_{\pi/2}^{\pi}\right|$$

$$=\left[\sin\frac{\pi}{2}-\sin 0\right]+\left|\left[\sin\pi-\sin\frac{\pi}{2}\right]\right|$$

$$=(1-0)+|0-1|=1+1=2 \text{ sq. units}$$



Hence, the correct option is (*a*).

Q29. The area of the region bounded by parabola $y^2=x$ and the straight line $2y=x$ is

(a) $\frac{4}{3}$ sq. units (b) 1 sq. unit

(c) $\frac{2}{3}$ sq. units (d) $\frac{1}{3}$ sq. units

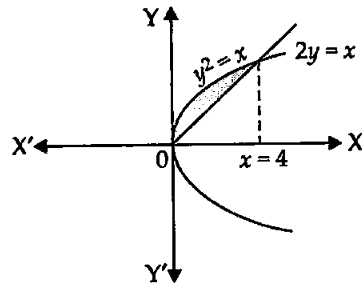
Sol. Given equation of parabola is $y^2 = x$ ...(i)

and equation of straight line is $2y = x$ ...(ii)

Solving eqns. (i) and (ii) we get

$$\left(\frac{x}{2}\right)^2 = x \Rightarrow \frac{x^2}{4} = x \Rightarrow x^2 = 4x$$

$\Rightarrow x(x-4) = 0 \quad \therefore x = 0, 4$



Required area

$$= \int_0^4 \sqrt{x}\,dx - \int_0^4 \frac{x}{2}\,dx$$

$$= \frac{2}{3}[x^{3/2}]_0^4 - \frac{1}{2}\cdot\frac{1}{2}[x^2]_0^4$$

$$= \frac{2}{3}\left[(4)^{3/2} - 0\right] - \frac{1}{4}\left[(4)^2 - 0\right] = \frac{2}{3}\times 8 - \frac{1}{4}\times 16$$

$$= \frac{16}{3} - 4 = \frac{4}{3} \text{ sq. units}$$

Hence, the correct answer is (a).

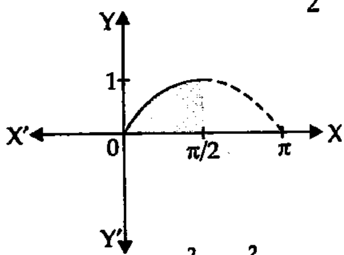
Q30. The area of the region bounded by the curve $y = \sin x$, between the ordinates $x = 0$ and $x = \frac{\pi}{2}$ and the x-axis is

(a) 2 sq. units (b) 4 sq. units

(c) 3 sq. units (d) 1 sq. units

Sol. Given equation of curve is $y = \sin x$ between $x = 0$ and $x = \frac{\pi}{2}$

Area of required region



$$= \int_0^{\pi/2} \sin x\,dx = -[\cos x]_0^{\pi/2}$$

$$= -\left[\cos\frac{\pi}{2} - \cos 0\right]$$

$$= -[0-1] = 1 \text{ sq. unit}$$

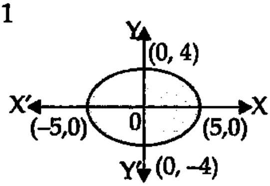
Hence, the correct answer is (d).

Q31. The area of the region bounded by the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ is

(a) 20π sq. units (b) $20\pi^2$ sq. units

(c) $16\pi^2$ sq. units (d) 25π sq. units

Sol. Given equation of ellipse is $\dfrac{x^2}{25}+\dfrac{y^2}{16}=1$



$\Rightarrow \dfrac{y^2}{16}=1-\dfrac{x^2}{25} \Rightarrow y^2=\dfrac{16}{25}(25-x^2)$

$\therefore \quad y=\dfrac{4}{5}\sqrt{25-x^2}$

$\therefore$ Since the ellipse is symmetrical about the axes.

$\therefore$ Required area $=4\times\int_0^5 \dfrac{4}{5}\sqrt{25-x^2}\,dx = 4\times\dfrac{4}{5}\int_0^5\sqrt{(5)^2-x^2}\,dx$

$$=\frac{16}{5}\left[\frac{x}{2}\sqrt{(5)^2-x^2}+\frac{25}{2}\sin^{-1}\frac{x}{5}\right]_0^5$$

$$=\frac{16}{5}\left[0+\frac{25}{2}.\sin^{-1}\left(\frac{5}{5}\right)-0-0\right]=\frac{16}{5}\left[\frac{25}{2}.\sin^{-1}(1)\right]$$

$$=\frac{16}{5}\left[\frac{25}{2}.\frac{\pi}{2}\right]=20\pi \text{ sq. units}$$

Hence, the correct answer is (a).

Q32. The area of the region bounded by the circle $x^2+y^2=1$ is

(*a*) 2π sq. units (*b*) π sq. units

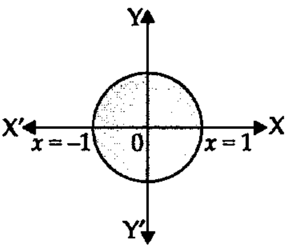
(*c*) 3π sq. units (*d*) 4π sq. units

Sol. Given equation of circle is

$x^2+y^2=1 \quad \Rightarrow \quad y=\sqrt{1-x^2}$

Since the circle is symmetrical about the axes.

$\therefore$ Required area $=4\times\int_0^1\sqrt{1-x^2}\,dx$



$$=4\left[\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}x\right]_0^1$$

$$=4\left[0+\frac{1}{2}\sin^{-1}(1)-0-0\right]$$

$$=4\times\frac{1}{2}\times\frac{\pi}{2}=\pi \text{ sq. units}$$

Hence, the correct answer is (*b*).

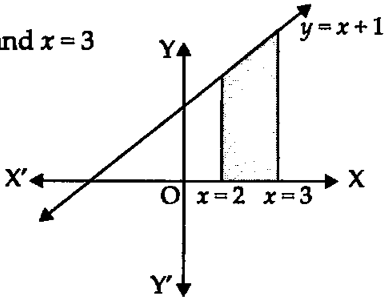
Q33. The area of the region bounded by the curve $y=x+1$ and the lines $x=2$ and $x=3$ is

(*a*) $\dfrac{7}{2}$ sq. units (*b*) $\dfrac{9}{2}$ sq. units

(*c*) $\dfrac{11}{2}$ sq. units (*d*) $\dfrac{13}{2}$ sq. units

Sol. Given equation of lines are

$$y = x + 1,\ x = 2 \text{ and } x = 3$$



Required area

$$= \int_2^3 (x+1)\,dx = \left[\frac{x^2}{2} + x\right]_2^3$$

$$= \left(\frac{9}{2} + 3\right) - \left(\frac{4}{2} + 2\right)$$

$$= \frac{15}{2} - 4 = \frac{7}{2} \text{ sq. units}$$

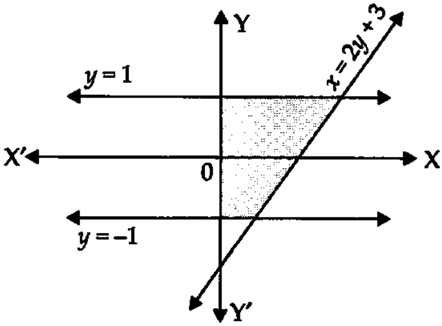
Hence, the correct option is (*a*).

Q34. The area of the region bounded by the curve $x = 2y + 3$ and the lines $y = 1$ and $y = -1$ is

(*a*) 4 sq. units (*b*) $\frac{3}{2}$ sq. units

(*c*) 6 sq. units (*d*) 8 sq. units

Sol. Given equations of lines are $x = 2y + 3$, $y = 1$ and $y = -1$



$$\text{Required area } = \int_{-1}^{1} (2y + 3)\,dy$$

$$= 2 \cdot \frac{1}{2}\left[y^2\right]_{-1}^{1} + 3\left[y\right]_{-1}^{1}$$

$$= (1 - 1) + 3(1 + 1) = 6 \text{ sq. units}$$

Hence, the correct answer is (*c*).

❑❑❑

9 Differential Equations

9.3 EXERCISE

SHORT ANSWER TYPE QUESTIONS

Q1. Find the solution of $\frac{dy}{dx} = 2^{y-x}$.

Sol. The given differential equation is

$$\frac{dy}{dx} = 2^{y-x} \Rightarrow \frac{dy}{dx} = \frac{2^y}{2^x}$$

Separating the variables, we get

$$\frac{dy}{2^y} = \frac{dx}{2^x} \quad \Rightarrow 2^{-y}dy = 2^{-x}dx$$

Integrating both sides, we get

$$\int 2^{-y}dy = \int 2^{-x}dx$$

$$\frac{-2^{-y}}{\log 2} = \frac{-2^{-x}}{\log 2} + c \quad \Rightarrow -2^{-y} = -2^{-x} + c\log 2$$

$$\Rightarrow \quad -2^{-y} + 2^{-x} = c\log 2$$

$$\Rightarrow \quad 2^{-x} - 2^{-y} = k \qquad [\text{where } c\log 2 = k]$$

Q2. Find the differential equation of all non vertical lines in a plane.

Sol. Equation of all non vertical lines are $y = mx + c$

Differentiating with respect to x, we get $\frac{dy}{dx} = m$

Again differentiating w.r.t. x we have $\frac{d^2y}{dx^2} = 0$

Hence, the required equation is $\frac{d^2y}{dx^2} = 0$.

Q3. Given that $\frac{dy}{dx} = e^{-2y}$ and $y = 0$ when $x = 5$. Find the value of x when $y = 3$.

Sol. Given equation is

$$\frac{dy}{dx} = e^{-2y}$$

$$\Rightarrow \quad \frac{dy}{e^{-2y}} = dx \quad \Rightarrow e^{2y}.dy = dx$$

Integrating both sides, we get

$$\int e^{2y} dy = \int dx \quad \Rightarrow \frac{1}{2} e^{2y} = x + c$$

Put $y = 0$ and $x = 5$

$$\Rightarrow \frac{1}{2} e^0 = 5 + c \Rightarrow c = \frac{1}{2} - 5 = -\frac{9}{2}$$

$\therefore$ The equation becomes $\frac{1}{2} e^{2y} = x - \frac{9}{2}$

Now putting $y = 3$, we get

$$\frac{1}{2} e^6 = x - \frac{9}{2} \quad \Rightarrow x = \frac{1}{2} e^6 + \frac{9}{2}$$

Hence the required value of $x = \frac{1}{2}(e^6 + 9)$.

Q4. Solve the differential equation $(x^2 - 1)\frac{dy}{dx} + 2xy = \frac{1}{x^2 - 1}$.

Sol. Given differential equation is

$$(x^2 - 1)\frac{dy}{dx} + 2xy = \frac{1}{x^2 - 1}$$

Dividing by $(x^2 - 1)$, we get

$$\frac{dy}{dx} + \frac{xy}{x^2 - 1} = \frac{}{(x^2 - 1)^2}$$

It is a linear differential equation of first order and first degree.

$$\therefore \quad P = \frac{2x}{x^2 - 1} \text{ and } Q = \frac{1}{(x^2 - 1)^2}$$

Integrating factor I.F. $= e^{\int P dx} = e^{\int \frac{2x}{x^2-1} dx} = e^{\log (x^2 - 1)} = (x^2 - 1)$.

$\therefore$ Solution of the equation is

$$y \times \text{I.F.} = \int Q \,.\, \text{I.F.}\, dx + C$$

$$\Rightarrow \quad y \times (x^2 - 1) = \int \frac{1}{(x^2 - 1)^2} \times (x^2 - 1)\, dx + C$$

$$\Rightarrow \quad y(x^2 - 1) = \int \frac{1}{x^2 - 1} dx + C \quad \Rightarrow y(x^2 - 1) = \frac{1}{2} \log \left|\frac{x - 1}{x + 1}\right| + C$$

Hence the required solution is $y(x^2 - 1) = \frac{1}{2} \log \left|\frac{x - 1}{x + 1}\right| + C$.

Q5. Solve the differential equation $\frac{dy}{dx} + 2xy = y$.

Sol. Given equation is $\frac{dy}{dx} + 2xy = y$.

$$\Rightarrow \quad \frac{dy}{dx} = y - 2xy \quad \Rightarrow \frac{dy}{dx} = y(1-2x) \quad \Rightarrow \frac{dy}{y} = (1-2x)\,dx$$

Integrating both sides, we have

$$\int \frac{dy}{dy} = \int (1-2x)dx \Rightarrow \log y = x - 2.\frac{x^2}{2} + \log c$$

$$\Rightarrow \quad \log y = x - x^2 + \log c \Rightarrow \log y - \log c = x - x^2$$

$$\Rightarrow \quad \log \frac{y}{c} = x - x^2 \Rightarrow \frac{y}{c} = e^{x-x^2}$$

$$\therefore \quad y = y = c\,.\,e^{x-x^2}$$

Hence, the required solution is $y = c\,.\,e^{x-x^2}$.

Q6. Find the general solution of $\frac{dy}{dx} + ay = e^{mx}$.

Sol. Given equation is $\frac{dy}{dx} + ay = e^{mx}$.

Here, $P = a$ and $Q = e^{mx}$

$\therefore$ I.F $= e^{\int P dx} = e^{\int a.dx} = e^{ax}$.

Solution of equation is $y \times$ I.F $= \int Q$ I.F $dx + c$

$$\Rightarrow \quad y.e^{ax} = \int e^{mx}.e^{ax}dx + c \Rightarrow y\,.\,e^{ax} = \int e^{(m+a)x}dx + c$$

$$\Rightarrow \quad y.e^{ax} = \frac{e^{(m+a)x}}{(m+a)} + c \Rightarrow y = \frac{e^{(m+a)x}}{(m+a)}.e^{-ax} + c.e^{-ax}$$

$$\therefore \quad y = \frac{e^{mx}}{(m+a)} + c\,.\,e^{-ax}$$

Hence the required solution is $y = \frac{e^{mx}}{(m+a)} + c\,.\,e^{-ax}$.

Q7. Solve the differential equation $\frac{dy}{dx} + 1 = e^{x+y}$.

Sol. Given that: $\frac{dy}{dx} + 1 = e^{x+y}$

Put $x + y = t$

$$\therefore \quad 1 + \frac{dy}{dx} = \frac{dt}{dx}$$

$$\therefore \quad \frac{dt}{dx} = e^t \quad \Rightarrow \frac{dt}{e^t} = dx \quad \Rightarrow e^{-t}dt = dx$$

Integrating both sides, we have

$$\int e^{-t}dt = \int dx \quad \Rightarrow -e^{-t} = x + c$$

$$\Rightarrow \quad -e^{-(x+y)} = x + c \Rightarrow \frac{-1}{e^{x+y}} = x + c \Rightarrow (x+c)\, e^{x+y} = -1$$

Hence, the required solution is $(x+c).e^{x\ \ y} + 1 = 0$.

Q8. Solve: $ydx - xdy = x^2y\, dx$.

Sol. Given equation is $ydx - xdy = x^2y\, dx$.

$$\Rightarrow \quad y\, dx - x^2y\, dx = xdy$$

$$\Rightarrow \quad y(1-x^2)\, dx = xdy$$

$$\Rightarrow \quad \left(\frac{1-x^2}{x}\right)dx = \frac{dy}{y} \quad \Rightarrow \left(\frac{1}{x} - x\right)dx = \frac{dy}{y}$$

Integrating both sides we get

$$\int\left(\frac{1}{x} - x\right)dx = \int\frac{dy}{y}$$

$$\Rightarrow \log x - \frac{x^2}{2} = \log y + \log c$$

$$\Rightarrow \log x - \frac{x^2}{2} = \log yc \Rightarrow \log x - \log c = \frac{x^2}{2} \quad \Rightarrow \log\frac{x}{yc} = \frac{x^2}{2}$$

$$\Rightarrow \quad \frac{x}{yc} = e^{x^2/2} \quad \Rightarrow \frac{yc}{x} = e^{-x^2/2} \Rightarrow yc = xe^{-x^2/2}$$

$$\therefore \quad y = \frac{1}{c}.xe^{-x^2/2} \quad \Rightarrow y = kxe^{-x^2/2} \qquad \left[\because k = \frac{1}{c}\right]$$

Hence, the required solution is $y = kxe^{-x^2/2}$.

Q9. Solve the differential equation $\dfrac{dy}{dx} = 1 + x + y^2 + xy^2$, when $y = 0, x = 0$.

Sol. Given equation is

$$\frac{dy}{dx} = 1 + x + y^2 + xy^2$$

$$\Rightarrow \quad \frac{dy}{dx} = 1(1+x) + y^2(1+x)$$

$$\Rightarrow \quad \frac{dy}{dx} = (1+x)(1+y^2) \Rightarrow \frac{dy}{1+y^2} = (1+x)\,dx$$

Integrating both sides, we get

$$\int \frac{dy}{1+y^2} = \int (1+x)\,dx \;\Rightarrow\; \tan^{-1} y = x + \frac{x^2}{2} + c$$

Put $x = 0$ and $y = 0$, we get $\tan^{-1}(0) = 0 + 0 + c \Rightarrow c = 0$

$$\therefore \qquad \tan^{-1} y = x + \frac{x^2}{2} \;\Rightarrow y = \tan\left(x + \frac{x^2}{2}\right)$$

Hence, the required solution is $y = \tan\left(x + \dfrac{x^2}{2}\right)$.

Q10. Find the general solution of $(x + 2y^3)\dfrac{dy}{dx} = y$.

Sol. Given equation is $(x + 2y^3)\dfrac{dy}{dx} = y$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{y}{x + 2y^3} \;\Rightarrow \frac{dx}{dy} = \frac{x + 2y^3}{y}$$

$$\Rightarrow \qquad \frac{dx}{dy} = \frac{x}{y} + \frac{2y^3}{y} \;\Rightarrow \frac{dx}{dy} - \frac{x}{y} = 2y^2$$

Here $P = -\dfrac{1}{y}$ and $Q = 2y^2$.

$\therefore$ Integrating factor I.F. $= e^{\int P dy} = e^{\int -\frac{1}{y} dy} = e^{-\log y} = e^{\log \frac{1}{y}} = \dfrac{1}{y}$.

So the solution of the equation is

$$x.\text{I.F.} = \int \text{Q.I.F.}\,dy + c$$

$$x.\frac{1}{y} = \int 2y^2.\frac{1}{y}\,dy + c$$

$$\Rightarrow \qquad \frac{x}{y} = 2\int y\,dy + c \Rightarrow \frac{x}{y} = 2.\frac{y^2}{2} + c \Rightarrow \frac{x}{y} = y^2 + c$$

So $\quad x = y^3 + cy = y(y^2 + c)$

Hence, the required solution is $x = y\,(y^2 + c)$.

Q11. If $y(x)$ is a solution of $\left(\dfrac{2 + \sin x}{1 + y}\right)\dfrac{dy}{dx} = -\cos x$ and $y\,(0) = 1$,

then find the value of $y\left(\dfrac{\pi}{2}\right)$.

Sol. Given equation is

$$\left(\frac{2+\sin x}{1+y}\right)\frac{dy}{dx}=-\cos x$$

$$\Rightarrow \quad \left(\frac{2+\sin x}{\cos x}\right)\frac{dy}{dx}=-(1+y) \quad \Rightarrow \quad \frac{dy}{(1+y)}=-\left(\frac{\cos x}{2+\sin x}\right)dx$$

Integrating both sides, we get

$$\int\frac{dy}{1+y}=-\int\frac{\cos x}{2+\sin x}dx$$

$$\Rightarrow \quad \log|1+y|=-\log|2+\sin x|+\log c$$

$$\Rightarrow \log|1+y|+\log|2+\sin x|=\log c$$

$$\Rightarrow \quad \log(1+y)(2+\sin x)=\log c \Rightarrow (1+y)(2+\sin x)=c$$

Put $x=0$ and $y=1$, we get

$(1+1)(2+\sin 0)=c \quad \Rightarrow \quad 4=c$

$\therefore$ equation is $(1+y)(2+\sin x)=4$

Now put $x=\dfrac{\pi}{2}$

$$\therefore \quad (1+y)\left(2+\sin\frac{\pi}{2}\right)=4$$

$$\Rightarrow \quad (1+y)(2+1)=4 \Rightarrow 1+y=\frac{4}{3} \Rightarrow y=\frac{4}{3}-1\Rightarrow\frac{1}{3}$$

So, $y\left(\dfrac{\pi}{2}\right)=\dfrac{1}{3}$

Hence, the required solution is $y\left(\dfrac{\pi}{2}\right)=\dfrac{1}{3}$.

Q12. If $y(t)$ is a solution of $(1+t)\dfrac{dy}{dt}-ty=1$ and $y(0)=-1$, then show that $y(1)=-\dfrac{1}{2}$.

Sol. Given equation is

$$(1+t)\frac{dy}{dt}-ty=1 \quad \Rightarrow \quad \frac{dy}{dt}-\left(\frac{t}{1+t}\right)y=\frac{1}{1+t}$$

Here, $P=\dfrac{-t}{1+t}$ and $Q=\dfrac{1}{1+t}$

$$\therefore \text{ Integrating factor I.F. }=e^{\int P dt}=e^{\int\frac{-t}{1+t}dt}=e^{-\int\frac{1+t-1}{1+t}dt}$$

$$=e^{-\int\left(1-\frac{1}{1+t}\right)dt}=e^{-[t-\log(1+t)]}$$

$$=e^{-t+\log(1+t)}=e^{-t}.e^{\log(1+t)}$$

$\therefore$ I.F. $=e^{-t}.(1+t)$

Required solution of the given differential equation is

$$y\,.\,\text{I.F.} = \int Q\,.\,\text{I.F.}\,dt + c$$

$$\Rightarrow \quad y\,.\,e^{-t}(1+t) = \int \frac{1}{(1+t)}\,.\,e^{-t}\,.\,(1+t)\,dt + c$$

$$\Rightarrow \quad y\,.\,e^{-t}(1+t) = \int e^{-t}\,dt + c$$

$$\Rightarrow \quad y\,.\,e^{-t}(1+t) = -e^{-t} + c$$

Put $t = 0$ and $y = -1$ $\quad [\because y(0) = -1]$

$$\Rightarrow \quad -1\,.\,e^{0}\,.\,1 = -e^{0} + c$$

$$\Rightarrow \quad -1 = -1 + c \quad \Rightarrow c = 0$$

So the equation becomes

$$ye^{-t}(1+t) = -e^{-t}$$

Now put $t = 1$

$$\therefore \quad y\,.\,e^{-1}(1+1) = -e^{-1}$$

$$\Rightarrow \quad 2y = -1 \Rightarrow y = -\frac{1}{2}$$

Hence $y(1) = -\dfrac{1}{2}$ is verified.

Q13. Form the differential equation having $y = (\sin^{-1}x)^2 + A\cos^{-1}x + B$ where A and B are arbitrary constants, as its general solution.

Sol. Given equation is $y = (\sin^{-1}x)^2 + A\cos^{-1}x + B$

$$\frac{dy}{dx} = 2\sin^{-1}x\,.\,\frac{1}{\sqrt{1-x^2}} + A\,.\left(\frac{-1}{\sqrt{1-x^2}}\right)$$

Multiplying both sides by $\sqrt{1-x^2}$, we get

$$\sqrt{1-x^2}\,\frac{dy}{dx} = 2\sin^{-1}x - A$$

Again differentiating w.r.t x, we get

$$\sqrt{1-x^2}\,\frac{d^2y}{dx^2} + \frac{dy}{dx}\,.\,\frac{1\times(-2x)}{2\sqrt{1-x^2}} = \frac{2}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad \sqrt{1-x^2}\,\frac{d^2y}{dx^2} - \frac{x}{\sqrt{1-x^2}}\,\frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}}$$

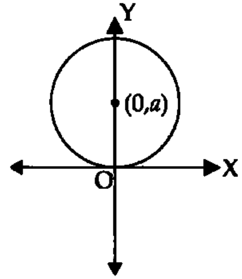
Multiplying both sides by $\sqrt{1-x^2}$, we get

$$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 2 = 0$$

Which is the required differential equation.

Q14. Form the differential equation of all circles which pass through origin and whose centres lie on y-axis.

Sol. Equation of circle which passes through the origin and whose centre lies on y-axis is



$$(x-0)^2+(y-a)^2=a^2$$

$$\Rightarrow \quad x^2+y^2+a^2-2ay=a^2$$

$$\Rightarrow \quad x^2+y^2-2ay=0 \quad \ldots(i)$$

Differentiating both sides w.r.t. x we get

$$\Rightarrow \quad 2x+2y\cdot\frac{dy}{dx}-2a\cdot\frac{dy}{dx}=0$$

$$\Rightarrow \quad x+y\frac{dy}{dx}-a\cdot\frac{dy}{dx}=0 \Rightarrow x+(y-a)\cdot\frac{dy}{dx}=0$$

$$y-a=\frac{x}{dy/dx}$$

$$\therefore \quad a=y+\frac{-x}{dy/dx}$$

Putting the value of a in eq. (i), we get

$$x^2+y^2-2\left(y+\frac{x}{dy/dx}\right)y=0$$

$$\Rightarrow \quad x^2+y^2-2y^2-\frac{2xy}{\frac{dy}{dx}}=0 \Rightarrow x^2-y^2=\frac{2xy}{\frac{dy}{dx}}$$

$$\therefore \quad (x^2-y^2)\frac{dy}{dx}-2xy=0$$

Hence, the required differential equation is

$$(x^2-y^2)\frac{dy}{dx}-2xy=0$$

Q15. Find the equation of a curve passing through origin and satisfying the differential equation $(1+x^2)\frac{dy}{dx}+2xy=4x^2$.

Sol. Given equation is

$$(1+x^2)\frac{dy}{dx}+2xy=4x^2$$

$$\Rightarrow \quad \frac{dy}{dx}+\frac{2x}{1+x^2}\cdot y=\frac{4x^2}{1+x^2}$$

Here, $P = \frac{2x}{1+x^2}$ and $Q = \frac{4x^2}{1+x^2}$

Integrating factor I.F. $= e^{\int P dx} = e^{\int \frac{2x}{1+x^2}dx} = e^{\log(1+x^2)} = 1 + x^2$

$\therefore$ Solution is

$$y \times \text{I.F.} = \int Q \times \text{I.F.}\, dx + c$$

$$\Rightarrow \quad y(1+x^2) = \int \frac{4x^2}{1+x^2} \times (1+x^2)\, dx + c$$

$$\Rightarrow \quad y(1+x^2) = \int 4x^2 dx + c$$

$$\Rightarrow \quad y(1+x^2) = \frac{4}{3}x^3 + c \qquad \text{...(i)}$$

Since the curve is passing through origin *i.e.*, (0, 0)

$\therefore$ Put $y = 0$ and $x = 0$ in eq. (i)

$0(1+0) = \frac{4}{3}(0)^3 + c \Rightarrow c = 0$

$\therefore$ Equation is $y(1+x^2) = \frac{4}{3}x^3 \Rightarrow y = \frac{4x^3}{3(1+x^2)}$

Hence, the required solution is $y = \frac{4x^3}{3(1+x^2)}$.

Q16. Solve : $x^2 . \frac{dy}{dx} = x^2 + xy + y^2$

Sol. Given equation is $x^2 \frac{dy}{dx} = x^2 + xy + y^2$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2}$$

Put $y = vx$ [$\because$ it is a homogeneous differential equation]

$$\therefore \quad \frac{dy}{dx} = v + x . \frac{dv}{dx}$$

$$\therefore \quad v + x . \frac{dv}{dx} = \frac{x^2 + vx^2 + v^2x^2}{x^2}$$

$$\Rightarrow \quad v + x . \frac{dv}{dx} = \frac{x^2(1 + v + v^2)}{x^2}$$

$$\Rightarrow \quad v + x . \frac{dv}{dx} = 1 + v + v^2 \Rightarrow x . \frac{dv}{dx} = 1 + v + v^2 - v$$

$$\Rightarrow \quad x.\frac{dv}{dx} = 1+v^2 \Rightarrow \frac{dv}{1+v^2} = \frac{dx}{x}$$

Integrating both sides, we get

$$\int \frac{dv}{1+v^2} = \int \frac{dx}{x}$$

$$\Rightarrow \quad \tan^{-1}v = \log x + c \Rightarrow \tan^{-1}\left(\frac{y}{x}\right) = \log x + c$$

Hence, the required solution is $\tan^{-1}\left(\frac{y}{x}\right) = \log|x| + c.$

Q17. Find the general solution of the differential equation

$$(1+y^2)+(x-e^{\tan^{-1}y})\frac{dy}{dx}=0$$

Sol. Given equation is

$$(1+y^2)+(x-e^{\tan^{-1}y})\frac{dy}{dx}=0$$

$$\Rightarrow (x-e^{\tan^{-1}y})\frac{dy}{dx} = -(1+y^2) \Rightarrow \frac{dy}{dx} = \frac{-(1+y^2)}{x-e^{\tan^{-1}y}}$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{x-e^{\tan^{-1}y}}{-(1+y^2)} \Rightarrow \frac{dx}{dy} = -\frac{x}{(1+y^2)}+\frac{e^{\tan^{-1}y}}{1+y^2}$$

$$\Rightarrow \quad \frac{dx}{dy}+\frac{x}{(1+y^2)} = \frac{e^{\tan^{-1}y}}{1+y^2}$$

Here, $P = \frac{1}{1+y^2}$ and $Q = \frac{e^{\tan^{-1}y}}{1+y^2}$

$\therefore$ Integrating factor I.F. $= e^{\int P dy} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$

$\therefore$ Solution is

$$x\,.\,\text{I.F.} = \int Q\,.\text{I.F.}\,dy + c$$

$$\Rightarrow \quad x\,.\,e^{\tan^{-1}y} = \int \frac{e^{\tan^{-1}y}}{1+y^2}.e^{\tan^{-1}y}dy + c$$

Put $e^{\tan^{-1}y} = t$

$$\therefore \quad e^{\tan^{-1}y}.\frac{1}{1+y^2}dy = dt$$

$$\therefore \quad x\,.\,e^{\tan^{-1}y} = \int t\,.\,dt + c$$

$$\Rightarrow \quad x\,.\,e^{\tan^{-1}y} = \frac{1}{2}t^2 + c$$

$$\Rightarrow \quad x \cdot e^{\tan^{-1} y} = \frac{1}{2}(e^{\tan^{-1} y})^2 + c \quad \Rightarrow x = \frac{1}{2}(e^{\tan^{-1} y}) + \frac{c}{e^{\tan^{-1} y}}$$

$$\Rightarrow \quad 2x = e^{\tan^{-1} y} + \frac{2c}{e^{\tan^{-1} y}}$$

$$\Rightarrow 2x \cdot e^{\tan^{-1} y} = (e^{\tan^{-1} y})^2 + 2c$$

Hence, this is the required general solution.

Q18. Find the general solution of $y^2 dx + (x^2 - xy + y^2)\, dy = 0$.

Sol. The given equation is $y^2 dx + (x^2 - xy + y^2)\, dy = 0$.

$$\Rightarrow \quad y^2 dx = -(x^2 - xy + y^2)\, dy$$

$$\Rightarrow \quad \frac{dx}{dy} = -\frac{x^2 - xy + y^2}{y^2}$$

Since it is a homogeneous differential equation

$\therefore$ Put $x = vy \Rightarrow \dfrac{dx}{dy} = v + y \cdot \dfrac{dv}{dy}$

So, $$v + y \cdot \frac{dv}{dy} = -\left(\frac{v^2 y^2 - vy^2 + y^2}{y^2}\right)$$

$$\Rightarrow \quad v + y \cdot \frac{dv}{dy} = -\frac{y^2(v^2 - v + 1)}{y^2}$$

$$\Rightarrow \quad v + y \cdot \frac{dv}{dy} = (-v^2 + v - 1) \quad \Rightarrow y \cdot \frac{dv}{dy} = -v^2 + v - 1 - v$$

$$\Rightarrow \quad y \cdot \frac{dv}{dy} = -v^2 - 1 \quad \Rightarrow \frac{dv}{(v^2 + 1)} = -\frac{dy}{y}$$

Integrating both sides, we get

$$\Rightarrow \quad \int \frac{dv}{(v^2 + 1)} = -\int \frac{dy}{y} \quad \Rightarrow \tan^{-1} v = -\log y + c$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{x}{y}\right) + \log y + c$$

Hence the required solution is $\tan^{-1}\left(\dfrac{x}{y}\right) + \log y = c$.

Q19. Solve: $(x + y)(dx - dy) = dx + dy$. [Hint: Substitute $x + y = z$ after separating dx and dy]

Sol. Given differential equation is

$$(x + y)(dx - dy) = dx + dy$$

$$\Rightarrow \quad (x + y)\, dx - (x + y)\, dy = dx + dy$$

$$\Rightarrow \quad -(x + y)\, dy - dy = dx - (x + y)\, dx$$

$$\Rightarrow \quad -(x+y+1)\,dy = -(x+y-1)\,dx$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x+y-1}{x+y+1}$$

Put $x+y=z$

$$\therefore \quad 1+\frac{dy}{dx} = \frac{dz}{dx}$$

$$\frac{dy}{dx} = \frac{dz}{dx}-1$$

So, $$\frac{dz}{dx}-1 = \frac{z-1}{z+1}$$

$$\Rightarrow \quad \frac{dz}{dx} = \frac{z-1}{z+1}+1 \quad \Rightarrow \frac{dz}{dx} = \frac{z-1+z+1}{z+1}$$

$$\Rightarrow \quad \frac{dz}{dx} = \frac{2z}{z+1} \quad \Rightarrow \frac{z+1}{z}dz = 2\,.\,dx$$

Integrating both sides, we get

$$\int\frac{z+1}{z}dz = 2\int dx$$

$$\Rightarrow \quad \int\left(1+\frac{1}{z}\right)dz = 2\int dx$$

$$\Rightarrow \quad z+\log|z| = 2x+\log|c|$$

$$\Rightarrow \quad x+y+\log|x+y| = 2x+\log|c|$$

$$\Rightarrow \quad y+\log|x+y| = x+\log|c|$$

$$\Rightarrow \quad \log|x+y| = x-y+\log|c|$$

$$\Rightarrow \quad \log|x+y|-\log|c| = (x-y)$$

$$\Rightarrow \quad \log\left|\frac{x+y}{c}\right| = (x-y) \quad \Rightarrow \frac{x+y}{c} = e^{x-y}$$

$$\therefore \quad x+y = c\,.\,e^{x-y}$$

Hence, the required solution is $x+y=c\,.\,e^{x-y}$.

Q20. Solve: $2(y+3)-xy\,.\,\frac{dy}{dx}=0$, given that $y(1)=-2$.

Sol. Given differential equation is

$$2(y+3)-xy\,.\,\frac{dy}{dx} = 0$$

$$\Rightarrow \quad xy\,.\,\frac{dy}{dx} = 2y+6$$

$$\Rightarrow \quad \left(\frac{y}{2y+6}\right)dy = \frac{dx}{x} \quad \Rightarrow \frac{1}{2}\left(\frac{y}{y+3}\right)dy = \frac{dx}{x}$$

Integrating both sides, we get

$$\Rightarrow \quad \frac{1}{2}\int \frac{y}{y+3}\cdot dy = \int \frac{dx}{x} \Rightarrow \frac{1}{2}\int \frac{y+3-3}{y+3}dy = \int \frac{dx}{x}$$

$$\Rightarrow \quad \frac{1}{2}\int \left(1-\frac{3}{y+3}\right)dy = \int \frac{dx}{x}$$

$$\Rightarrow \frac{1}{2}\int 1 . dy - \frac{3}{2}\int \frac{1}{y+3}dy = \int \frac{dx}{x}$$

$$\Rightarrow \quad \frac{1}{2}y - \frac{3}{2}\log|y+3| = \log x + c$$

Put $x = 1, y = -2$

$$\Rightarrow \quad \frac{1}{2}(-2) - \frac{3}{2}\log|-2+3| = \log(1) + c$$

$$\Rightarrow \quad -1 - \frac{3}{2}\log(1) = \log(1) + c$$

$$\Rightarrow \quad -1 - 0 = 0 + c \qquad [\because \log(1) = 0]$$

$$\therefore \quad c = -1$$

$\therefore$ equation is

$$\frac{1}{2}y - \frac{3}{2}\log|y+3| = \log x - 1$$

$$\Rightarrow \quad y - 3\log|y+3| = 2\log x - 2$$

$$\Rightarrow \quad y - \log|(y+3)^3| = \log x^2 - 2$$

$$\Rightarrow \log|(y+3)^3| + \log x^2 = y + 2$$

$$\Rightarrow \quad \log|x^2(y+3)^3| = y + 2 \quad \Rightarrow x^2(y+3)^3 = e^{y+2}$$

Hence, the required solution is $x^2(y+3)^3 = e^{y+2}$.

Q21. Solve the differential equation $dy = \cos x\,(2 - y\,\text{cosec}\,x)\,dx$ given that $y = 2$ when $x = \frac{\pi}{2}$.

Sol. The given differential equation is

$$dy = \cos x\,(2 - y\,\text{cosec}\,x)\,dx$$

$$\Rightarrow \frac{dy}{dx} = \cos x\,(2 - y\,\text{cosec}\,x) \Rightarrow \frac{dy}{dx} = 2\cos x - y\cos x . \text{cosec}\,x$$

$$\Rightarrow \frac{dy}{dx} = 2\cos x - y\cot x \quad \Rightarrow \frac{dy}{dx} + y\cot x = 2\cos x$$

Here, $P = \cot x$ and $Q = 2\cos x$.

$\therefore$ Integrating factor I.F. $= e^{\int P dx} = e^{\int \cot x dx} = e^{\log \sin x} = \sin x$

$\therefore$ Required solution is $y \times$ I.F $= \int Q \times \text{I.F.}\,dx + c$

$\Rightarrow \quad y \,.\, \sin x = \int 2 \cos x \,.\, \sin x \, dx + c$

$\Rightarrow \quad y \,.\, \sin x = \int \sin 2x \, dx + c \quad \Rightarrow y \,.\, \sin x = -\frac{1}{2} \cos 2x + c$

Put $x = \frac{\pi}{2}$ and $y = 2$, we get

$$2 \sin \frac{\pi}{2} = -\frac{1}{2} \cos \pi + c$$

$\Rightarrow \quad 2(1) = -\frac{1}{2}(-1) + c \quad \Rightarrow 2 = \frac{1}{2} + c \Rightarrow c = 2 - \frac{1}{2} = \frac{3}{2}$

$\therefore$ The equation is $y \sin x = -\frac{1}{2} \cos 2x + \frac{3}{2}$.

Q22. Form the differential equation by eliminating A and B in $Ax^2 + By^2 = 1$.

Sol. Given that $Ax^2 + By^2 = 1$

Differentiating w.r.t. x, we get

$$2A \,.\, x + 2By \frac{dy}{dx} = 0$$

$\Rightarrow \quad Ax + By \,.\, \frac{dy}{dx} = 0 \quad \Rightarrow \quad By \,.\, \frac{dy}{dx} = -Ax$

$\therefore \quad \frac{y}{x} \,.\, \frac{dy}{dx} = -\frac{A}{B}$

Differentiating both sides again w.r.t. x, we have

$$\frac{y}{x} . \frac{d^2y}{dx^2} + \frac{dy}{dx}\left(\frac{x . \frac{dy}{dx} - y.1}{x^2}\right) = 0$$

$\Rightarrow \frac{yx^2}{x} . \frac{d^2y}{dx^2} + x . \left(\frac{dy}{dx}\right)^2 - y . \frac{dy}{dx} = 0$

$\Rightarrow \quad xy . \frac{d^2y}{dx^2} + x . \left(\frac{dy}{dx}\right)^2 - y . \frac{dy}{dx} = 0 \Rightarrow \quad xy . y'' + x . (y')^2 - y . y' = 0$

Hence, the required equation is

$$xy . y'' + x . (y')^2 - y . y' = 0$$

Q23. Solve the differential equation $(1 + y^2) \tan^{-1}x \, dx + 2y \, (1 + x^2) \, dy = 0$.

Sol. Given differential equation is

$(1 + y^2) \tan^{-1}x \, dx + 2y \, (1 + x^2) \, dy = 0$

$\Rightarrow \quad 2y \, (1 + x^2) \, dy = -(1 + y^2) \,.\, \tan^{-1}x \,.\, dx$

$\Rightarrow \quad \frac{2y}{1+y^2} dy = -\frac{\tan^{-1} x}{1+x^2} . \, dx$

Integrating both sides, we get

$$\int \frac{2y}{1+y^2} dy = -\int \frac{\tan^{-1} x}{1+x^2} . dx$$

$$\Rightarrow \quad \log\left|1+y^2\right| = -\frac{1}{2}(\tan^{-1} x)^2 + c$$

$$\Rightarrow \quad \frac{1}{2}(\tan^{-1} x)^2 + \log\left|1+y^2\right| = c$$

Which is the required solution.

Q24. Find the differential equation of system of concentric circles with centre (1, 2).

Sol. Family of concentric circles with centre (1, 2) and radius 'r' is

$$(x-1)^2 + (y-2)^2 = r^2$$

Differentiating both sides w.r.t., x we get

$$2(x-1) + 2(y-2)\frac{dy}{dx} = 0 \Rightarrow (x-1) + (y-2)\frac{dy}{dx} = 0$$

Which is the required equation.

LONG ANSWER TYPE QUESTIONS

Q25. Solve: $y + \frac{d}{dx}(xy) = x(\sin x + \log x)$

Sol. The given differential equation is

$$y + \frac{d}{dx}(xy) = x(\sin x + \log x)$$

$$\Rightarrow \quad y + x.\frac{dy}{dx} + y = x(\sin x + \log x)$$

$$\Rightarrow \quad x\frac{dy}{dx} = x(\sin x + \log x) - 2y$$

$$\Rightarrow \quad \frac{dy}{dx} = (\sin x + \log x) - \frac{2y}{x} \Rightarrow \frac{dy}{dx} + \frac{2}{x}y = (\sin x + \log x)$$

Here, $P = \frac{2}{x}$ and $Q = (\sin x + \log x)$

Integrating factor I.F. $= e^{\int P dx} = e^{\int \frac{2}{x} dx} = e^{2\log x} = e^{\log x^2} = x^2$

$\therefore$ Solution is

$$y \times \text{I.F.} = \int \text{Q.I.F.}\, dx + c$$

$$\Rightarrow \quad y . x^2 = \int (\sin x + \log x)\, x^2 dx + c \quad \text{...(1)}$$

Let I

$$= \int (\sin x + \log x)\, x^2 dx$$

$$= \int \underset{\text{I}}{x^2} \underset{\text{II}}{\sin x}\, dx + \int \underset{\text{II}}{x^2} \underset{\text{I}}{\log x}\, dx$$

$$= \left[x^2.\int \sin x\, dx - \int \left(D(x^2).\int \sin x\, dx\right) dx\right] +$$

$$\left[\log x.\int x^2\, dx - \int \left(D(\log x).\int x^2 dx\right) dx\right]$$

$$= \left[x^2(-\cos x) - 2\int -x\cos x\, dx\right] + \left[\log x.\frac{x^3}{3} - \int \frac{1}{x}.\frac{x^3}{3} dx\right]$$

$$= \left[-x^2\cos x + 2\left(x\sin x - \int 1.\sin x\, dx\right)\right] + \left[\frac{x^3}{3}\log x - \frac{1}{3}\int x^2 dx\right]$$

$$= -x^2\cos x + 2x\sin x + 2\cos x + \frac{x^3}{3}\log x - \frac{1}{9}x^3$$

Now from eq (1) we get,

$$y.x^2 = -x^2\cos x + 2x\sin x + 2\cos x + \frac{x^3}{3}\log x - \frac{1}{9}x^3 + c$$

$$\therefore \quad y = -\cos x + \frac{2\sin x}{x} + \frac{2\cos x}{x^2} + \frac{x\log x}{3} - \frac{1}{9}x + c.x^{-2}$$

Hence, the required solution is

$$y = -\cos x + \frac{2\sin x}{x} + \frac{2\cos x}{x^2} + \frac{x\log x}{3} - \frac{1}{9}x + c.x^{-2}$$

Q26. Find the general solution of $(1 + \tan y)(dx - dy) + 2xdy = 0$.

Sol. Given that: $(1 + \tan y)(dx - dy) + 2xdy = 0$

$$\Rightarrow \quad (1 + \tan y)\, dx - (1 + \tan y)\, dy + 2xdy = 0$$

$$\Rightarrow \quad (1 + \tan y)\, dx - (1 + \tan y - 2x)\, dy = 0$$

$$\Rightarrow \quad (1 + \tan y)\frac{dx}{dy} = (1 + \tan y - 2x) \quad \Rightarrow \quad \frac{dx}{dy} = \frac{1 + \tan y - 2x}{1 + \tan y}$$

$$\Rightarrow \quad \frac{dx}{dy} = 1 - \frac{2x}{1 + \tan y} \quad \Rightarrow \quad \frac{dx}{dy} + \frac{2x}{1 + \tan y} = 1$$

Here, $P = \dfrac{2}{1 + \tan y}$ and $Q = 1$

Integrating factor I.F.

$$= e^{\int \frac{2}{1+\tan y} dy} = e^{\int \frac{2\cos y}{\sin y + \cos y} dy}$$

$$= e^{\int \frac{\sin y + \cos y - \sin y + \cos y}{(\sin y + \cos y)} dy} = e^{\int \left(1 + \frac{\cos y - \sin y}{\sin y + \cos y}\right) dy}$$

$$= e^{\int 1.dy}.e^{\int \frac{\cos y - \sin y}{\sin y + \cos y} dy}$$

$$= e^y.e^{\log(\sin y + \cos y)} = e^y.(\sin y + \cos y)$$

So, the solution is $x \times \text{I.F.} = \int Q \times \text{I.F.}\, dy + c$

$\Rightarrow \quad x \, . \, e^y (\sin y + \cos y) = \int 1 \, . \, e^y (\sin y + \cos y) dy + c$

$\Rightarrow \quad x \, . \, e^y (\sin y + \cos y) = e^y \, . \, \sin y + c$

$$\left[\because \int e^x [f(x) + f'(x)]\, dx = e^x f(x) + c\right]$$

$\Rightarrow \quad x(\sin y + \cos y) = \sin y + c \, . \, e^{-y}$

Hence, the required solution is $x(\sin y + \cos y) = \sin y + c \, . \, e^{-y}$.

Q27. Solve : $\dfrac{dy}{dx} = \cos(x+y) + \sin(x+y)$. [Hint: Substitute $x + y = z$]

Sol. Given that : $\dfrac{dy}{dx} = \cos(x+y) + \sin(x+y)$

Put $\quad x + y = v$, on differentiating w.r.t. x, we get,

$$1 + \frac{dy}{dx} = \frac{dv}{dx}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dv}{dx} - 1$$

$$\therefore \quad \frac{dv}{dx} - 1 = \cos v + \sin v$$

$$\Rightarrow \quad \frac{dv}{dx} = \cos v + \sin v + 1$$

$$\Rightarrow \quad \frac{dv}{\cos v + \sin v + 1} = dx$$

Integrating both sides, we have

$$\int \frac{dv}{\cos v + \sin v + 1} = \int 1 \, . \, dx$$

$$\Rightarrow \quad \int \frac{dv}{\left(\dfrac{1 - \tan^2 \dfrac{v}{2}}{1 + \tan^2 \dfrac{v}{2}} + \dfrac{2 \tan \dfrac{v}{2}}{1 + \tan^2 \dfrac{v}{2}} + 1\right)} = \int 1 \, . \, dx$$

$$\Rightarrow \int \frac{\left(1 + \tan^2 \dfrac{v}{2}\right)}{1 - \tan^2 \dfrac{v}{2} + 2 \tan \dfrac{v}{2} + 1 + \tan^2 \dfrac{v}{2}} dv = \int 1 \, . \, dx$$

$$\Rightarrow \quad \int \frac{\sec^2 \dfrac{v}{2}}{2 + 2 \tan \dfrac{v}{2}} dv = \int 1 \, . \, dx$$

Put $\quad 2 + 2 \tan \dfrac{v}{2} = t$

$$2 \cdot \frac{1}{2} \sec^2 \frac{v}{2} dv = dt \Rightarrow \sec^2 \frac{v}{2} dv = dt$$

$$\Rightarrow \quad \int \frac{dt}{t} = \int 1 \,.\, dx$$

$$\Rightarrow \quad \log |t| = x + c$$

$$\Rightarrow \quad \log \left| 2 + 2 \tan \frac{v}{2} \right| = x + c$$

$$\Rightarrow \log \left| 2 + 2 \tan \left(\frac{x+y}{2} \right) \right| = x + c \Rightarrow \log 2 \left[1 + \tan \left(\frac{x+y}{2} \right) \right] = x + c$$

$$\Rightarrow \quad \log 2 + \log \left[1 + \tan \left(\frac{x+y}{2} \right) \right] = x + c$$

$$\Rightarrow \quad \log \left[1 + \tan \left(\frac{x+y}{2} \right) \right] = x + c - \log 2$$

Hence, the required solution is

$$\log \left[1 + \tan \left(\frac{x+y}{2} \right) \right] = x + K \qquad [c - \log 2 = K]$$

Q28. Find the general solution of $\frac{dy}{dx} - 3y = \sin 2x$.

Sol. Given equation is $\frac{dy}{dx} - 3y = \sin 2x$.

Here, $P = -3$ and $Q = \sin 2x$

$\therefore$ Integrating factor I.F. $= e^{\int P dx} = e^{\int -3 dx} = e^{-3x}$

$\therefore$ Solution is

$$y \times \text{I.F.} = \int Q \,.\, \text{I.F.}\, dx + c$$

$$\Rightarrow \quad y \,.\, e^{-3x} = \int \sin 2x \,.\, e^{-3x}\, dx + c$$

Let $\quad I = \int \underset{\text{I}}{\sin 2x} \,.\, \underset{\text{II}}{e^{-3x}}\, dx$

$$\Rightarrow \quad I = \sin 2x \,.\, \int e^{-3x} dx - \int \left(D(\sin 2x) \,.\, \int e^{-3x} dx \right) dx$$

$$\Rightarrow \quad I = \sin 2x \,.\, \frac{e^{-3x}}{-3} - \int 2 \cos 2x \,.\, \frac{e^{-3x}}{-3}\, dx$$

$$\Rightarrow \quad I = \frac{e^{-3x}}{-3} \sin 2x + \frac{2}{3} \int \underset{\text{I}}{\cos 2x} \,.\, \underset{\text{II}}{e^{-3x}}\, dx$$

$$\Rightarrow \quad I = \frac{e^{-3x}}{-3} \sin 2x + \frac{2}{3} \left[\cos 2x \,.\, \int e^{-3x} dx - \int [D \cos 2x \,.\, \int e^{-3x} dx]\, dx \right]$$

$$\Rightarrow \quad I = \frac{e^{-3x}}{-3}\sin 2x + \frac{2}{3}\left[\cos 2x \cdot \frac{e^{-3x}}{-3} - \int 2\sin 2x \cdot \frac{e^{-3x}}{-3}\right]dx$$

$$\Rightarrow \quad I = \frac{e^{-3x}}{-3}\sin 2x - \frac{2}{9}\cos 2x \cdot e^{-3x} - \frac{4}{9}\int \sin 2x \cdot e^{-3x}\,dx$$

$$\Rightarrow \quad = \frac{e^{-3x}}{-3}\sin 2x - \frac{2}{9}e^{-3x}\cos 2x - \frac{4}{9}I$$

$$\Rightarrow \quad I + \frac{4}{9}I = \frac{e^{-3x}}{-3}\sin 2x - \frac{2}{9}e^{-3x}\cos 2x$$

$$\Rightarrow \quad \frac{13\,I}{9} = -\frac{1}{9}\left[3\,e^{-3x}\sin 2x + 2\,e^{-3x}\cos 2x\right]$$

$$\Rightarrow \quad I = -\frac{1}{13}e^{-3x}[3\sin 2x + 2\cos 2x]$$

$\therefore$ The equation becomes

$$y \cdot e^{-3x} = -\frac{1}{13}e^{-3x}[3\sin 2x + 2\cos 2x] + c$$

$$\therefore \quad y = -\frac{1}{13}[3\sin 2x + 2\cos 2x] + c \cdot e^{3x}$$

Hence, the required solution is

$$y = -\left[\frac{3\sin 2x + 2\cos 2x}{13}\right] + c \cdot e^{3x}$$

Q29. Find the equation of a curve passing through (2, 1) if the slope of the tangent to the curve at any point (x, y) is $\frac{x^2 + y^2}{2xy}$.

Sol. Given that the slope of tangent to a curve at (x, y) is

$$\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$$

It is a homogeneous differential equation

So, put $y = vx \Rightarrow \frac{dy}{dx} = v + x \cdot \frac{dv}{dx}$

$$v + x \cdot \frac{dv}{dx} = \frac{x^2 + v^2x^2}{2x \cdot vx}$$

$$\Rightarrow v + x \cdot \frac{dv}{dx} = \frac{1+v^2}{2v}$$

$$\Rightarrow \quad x \cdot \frac{dv}{dx} = \frac{1+v^2}{2v} - v \quad \Rightarrow x \cdot \frac{dv}{dx} = \frac{1+v^2-2v^2}{2v}$$

$$\Rightarrow \quad x \cdot \frac{dv}{dx} = \frac{1-v^2}{2v} \quad \Rightarrow \frac{2v}{1-v^2} dv = \frac{dx}{x}$$

Integrating both sides, we get

$$\int \frac{2v}{1-v^2} dv = \int \frac{dx}{x} \Rightarrow -\log|1-v^2| = \log x + \log c$$

$$\Rightarrow \quad -\log\left|1-\frac{y^2}{x^2}\right| = \log x + \log c \Rightarrow -\log\left|\frac{x^2-y^2}{x^2}\right| = \log x + \log c$$

$$\Rightarrow \log\left|\frac{x^2}{x^2-y^2}\right| = \log|xc| \quad \Rightarrow \frac{x^2}{x^2-y^2} = xc$$

Since, the curve is passing through the point (2, 1)

$$\therefore \quad \frac{(2)^2}{(2)^2-(1)^2} = 2c \quad \Rightarrow \frac{4}{3} = 2c \quad \Rightarrow \quad c = \frac{2}{3}$$

Hence, the required equation is

$$\frac{x^2}{x^2-y^2} = \frac{2}{3}x \quad \Rightarrow 2(x^2-y^2) = 3x$$

Q30. Find the equation of the curve through the point (1, 0) if the slope of the tangent to the curve at any point (x, y) is $\frac{y-1}{x^2+x}$.

Sol. Given that the slope of the tangent to the curve at (x, y) is

$$\frac{dy}{dx} = \frac{y-1}{x^2+x} \quad \Rightarrow \frac{dy}{y-1} = \frac{dx}{x^2+x}$$

Integrating both sides, we have

$$\int \frac{dy}{y-1} = \int \frac{dx}{x^2+x}$$

$$\Rightarrow \quad \int \frac{dy}{y-1} = \int \frac{dx}{x^2+x+\frac{1}{4}-\frac{1}{4}} \quad \text{[making perfect square]}$$

$$\Rightarrow \quad \int \frac{dy}{y-1} = \int \frac{dx}{\left(x+\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}$$

$$\Rightarrow \quad \log|y-1| = \frac{1}{2\times\frac{1}{2}}\log\left|\frac{x+\frac{1}{2}-\frac{1}{2}}{x+\frac{1}{2}+\frac{1}{2}}\right| + \log c$$

$$\Rightarrow \quad \log|y-1| = \log\left|\frac{x}{x+1}\right| + \log c$$

$$\Rightarrow \quad \log|y-1| = \log\left|c\left(\frac{x}{x+1}\right)\right|$$

$$\therefore \quad y-1 = \frac{cx}{x+1} \Rightarrow (y-1)(x+1) = cx$$

Since, the line is passing through the point (1, 0), then

$(0-1)(1+1) = c(1) \Rightarrow c = 2.$

Hence, the required solution is $(y-1)(x+1) = 2x$.

Q31. Find the equation of a curve passing through origin if the slope of the tangent to the curve at any point (x, y) is equal to the square of the difference of the abscissa and ordinate of the point.

Sol. Here, slope of the tangent of the curve $= \frac{dy}{dx}$

and the difference between the abscissa and ordinate $= x - y$.

$\therefore$ As per the condition, $\frac{dy}{dx} = (x-y)^2$

Put $x - y = v$

$$1 - \frac{dy}{dx} = \frac{dv}{dx}$$

$$\therefore \quad \frac{dy}{dx} = 1 - \frac{dv}{dx}$$

$\therefore$ the equation becomes

$$1 - \frac{dv}{dx} = v^2 \Rightarrow \frac{dv}{dx} = 1 - v^2 \Rightarrow \frac{dv}{1-v^2} = dx$$

Integrating both sides, we get

$$\int \frac{dv}{1-v^2} = \int dx$$

$$\Rightarrow \quad \frac{1}{2}\log\left|\frac{1+v}{1-v}\right| = x + c \Rightarrow \frac{1}{2}\log\left|\frac{1+x-y}{1-x+y}\right| = x + c \qquad ...(1)$$

Since, the curve is passing through (0, 0)

then $\frac{1}{2}\log\left|\frac{1+0-0}{1-0+0}\right| = 0 + c \Rightarrow c = 0$

$\therefore$ On putting $c = 0$ in eq. (1) we get

$$\frac{1}{2}\log\left|\frac{1+x-y}{1-x+y}\right| = x \Rightarrow \log\left|\frac{1+x-y}{1-x+y}\right| = 2x$$

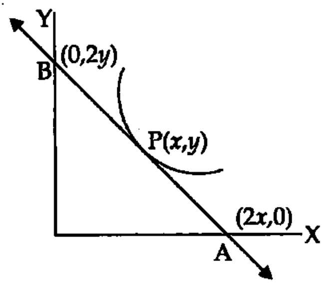
$$\therefore \qquad \frac{1+x-y}{1-x+y} = e^{2x}$$

$$\Rightarrow \qquad (1+x-y) = e^{2x}(1-x+y)$$

Hence, the required equation is $(1+x-y) = e^{2x}(1-x+y)$.

Q32. Find the equation of a curve passing through the point (1, 1), if the tangent drawn at any point P (x, y) on the curve meets the coordinate axes at A and B such that P is the mid point of AB.

Sol. Let P (x, y) be any point on the curve and AB be the tangent to the given curve at P.



P is the mid point of AB (given)

$\therefore$ Coordinates of A and B are $(2x, 0)$ and $(0, 2y)$ respectively.

$\therefore$ Slope of the tangent

AB = $\dfrac{2y-0}{0-2x} = -\dfrac{y}{x}$

$$\therefore \qquad \frac{dy}{dx} = -\frac{y}{x} \Rightarrow \frac{dy}{y} = -\frac{dx}{x}$$

Integrating both sides, we get

$$\int\frac{dy}{y} = -\int\frac{dx}{x} \Rightarrow \log y = -\log x + \log c$$

$$\Rightarrow \qquad \log y + \log x = \log c \Rightarrow \log yx = \log c$$

$$\therefore \qquad yx = c$$

Since, the curve passes through (1, 1)

$$\therefore \qquad 1 \times 1 = c \quad \therefore \quad c = 1$$

$$\Rightarrow \qquad yx = 1$$

Hence, the required equation is $xy = 1$.

Q33. Solve: $x\dfrac{dy}{dx} = y(\log y - \log x + 1)$

Sol. Given that: $x\dfrac{dy}{dx} = y(\log y - \log x + 1)$

$$\Rightarrow \quad x\frac{dy}{dx} = y\left[\log\left(\frac{y}{x}\right)+1\right] \Rightarrow \frac{dy}{dx} = \frac{y}{x}\left[\log\left(\frac{y}{x}\right)+1\right]$$

Since, it is a homogeneous differential equation.

$\therefore$ Put $y = vx \Rightarrow \dfrac{dy}{dx} = v + x.\dfrac{dv}{dx}$

$$\therefore \quad v + x.\frac{dv}{dx} = \frac{vx}{x}\left[\log\left(\frac{vx}{x}\right) + 1\right]$$

$$\Rightarrow \quad v + x.\frac{dv}{dx} = v\,[\log v + 1]$$

$$\Rightarrow \quad x.\frac{dv}{dx} = v\,[\log v + 1] - v \quad \Rightarrow x.\frac{dv}{dx} = v\,[\log v + 1 - 1]$$

$$\Rightarrow \quad x.\frac{dv}{dx} = v.\log v \quad \Rightarrow \frac{dv}{v\log v} = \frac{dx}{x}$$

Integrating both sides, we get

$$\int \frac{dv}{v\log v} = \int \frac{dx}{x}$$

Put $\log v = t$ on L.H.S.

$$\frac{1}{v}dv = dt$$

$$\therefore \quad \int \frac{dt}{t} = \int \frac{dx}{x}$$

$$\log|t| = \log|x| + \log c$$

$$\Rightarrow \quad \log|\log v| = \log xc \quad \Rightarrow \log v = xc$$

$$\Rightarrow \quad \log\left(\frac{y}{x}\right) = xc$$

Hence, the required solution is $\log\left(\dfrac{y}{x}\right) = xc$.

OBJECTIVE TYPE QUESTIONS

Choose the correct answer from the given four options in each of the Exercises from 34 to 75 (M.C.Q.)

Q34. The degree of the differential equation

$\left(\dfrac{d^2y}{dx^2}\right)^2 + \left(\dfrac{dy}{dx}\right)^2 = x\sin\left(\dfrac{dy}{dx}\right)$ is

(*a*) 1 (*b*) 2 (*c*) 3 (*d*) not defined

Sol. The degree of the given differential equation is not defined because the value of $\sin\left(\dfrac{dy}{dx}\right)$ on expansion will be in increasing power of $\left(\dfrac{dy}{dx}\right)$.

Hence, the correct option is (*d*).

Q35. The degree of the differential equation $\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2} = \dfrac{d^2y}{dx^2}$ is (*a*) 4 (*b*) $\frac{3}{2}$ (*c*) not defined (*d*) 2

Sol. The given differential equation is

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} = \left(\frac{d^2y}{dx^2}\right)$$

Squaring both sides, we have

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3} = \left(\frac{d^2y}{dx^2}\right)^2$$

So, the degree of the given differential equation is 2.

Hence, the correct option is (*d*).

Q36. The order and degree of the differential equation

$\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^{\frac{1}{4}}+x^{\frac{1}{5}}=0$ respectively are

(*a*) 2 and not defined (*b*) 2 and 2

(*c*) 2 and 3 (*d*) 3 and 3

Sol. Given differential equation is

$$\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^{\frac{1}{4}}+x^{\frac{1}{5}}=0$$

$$\Rightarrow \quad \frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^{\frac{1}{4}}=-x^{\frac{1}{5}}$$

Since the degree of $\frac{dy}{dx}$ is in fraction.

So, the degree of the differential equation is not defined as the order is 2.

Hence, the correct option is (*a*).

Q37. If $y = e^{-x}$ (A cos x + B sin x), then y is a solution of

(*a*) $\frac{d^2y}{dx^2}+2\frac{dy}{dx}=0$ (*b*) $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+2y=0$

(*c*) $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+2y=0$ (*d*) $\frac{d^2y}{dx^2}+2y=0$

Sol. Given equation is $y = e^{-x}$ (A cos x + B sin x)

Differentiating both sides, w.r.t. x, we get

$$\frac{dy}{dx} = e^{-x}(-A\sin x + B\cos x) - e^{-x}(A\cos x + B\sin x)$$

$$\frac{dy}{dx} = e^{-x}(-A\sin x + B\cos x) - y$$

Again differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = e^{-x}(-A\cos x - B\sin x) - e^{-x}(-A\sin x + B\cos x) - \frac{dy}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -e^{-x}(A\cos x + B\sin x) - \left[\frac{dy}{dx} + y\right] - \frac{dy}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -y - \frac{dy}{dx} - y - \frac{dy}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -2\frac{dy}{dx} - 2y \quad \Rightarrow \quad \frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$$

Hence, the correct option is (*c*)

Q38. The differential equation for $y = A\cos\alpha x + B\sin\alpha x$, where A and B are arbitrary constants is:

(*a*) $\frac{d^2y}{dx^2} - \alpha^2 y = 0$ (*b*) $\frac{d^2y}{dx^2} + \alpha^2 y = 0$

(*c*) $\frac{d^2y}{dx^2} + \alpha y = 0$ (*d*) $\frac{d^2y}{dx^2} - \alpha y = 0$

Sol. Given equation is : $y = A\cos\alpha x + B\sin\alpha x$

Differentiating both sides w.r.t. x, we have

$$\frac{dy}{dx} = -A\sin\alpha x \,.\, \alpha + B\cos\alpha x \,.\, \alpha$$
$$= -A\alpha\sin\alpha x + B\alpha\cos\alpha x$$

Again differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = -A\alpha^2\cos\alpha x - B\alpha^2\sin\alpha x$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\alpha^2(A\cos\alpha x + B\sin\alpha x)$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\alpha^2 y \Rightarrow \frac{d^2y}{dx^2} + \alpha^2 y = 0$$

Hence, the correct option is (*b*).

Q39. Solution of differential equation $x\,dy - y\,dx = 0$ represents:

(*a*) a rectangular hyperbola

(*b*) parabola whose vertex is at origin

(*c*) straight line passing through origin

(*d*) a circle whose centre is at origin.

Sol. The given differential equation is

$$xdy - ydx = 0$$

Sol. The given differential equation is

$$xdy - ydx = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x} \Rightarrow \frac{dy}{y} = \frac{dx}{x}$$

Integrating both sides, we get

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\Rightarrow \quad \log y = \log x + \log c \Rightarrow \log y = \log xc$$

$\Rightarrow$ $y = xc$ which is a straight line passing through the origin.

Hence, the correct answer is (*c*).

Q40. Integrating factor of the differential equation

$\cos x \cdot \frac{dy}{dx} + y \sin x = 1$ is

(*a*) $\cos x$ (*b*) $\tan x$ (*c*) $\sec x$ (*d*) $\sin x$

Sol. The given differential equation is

$$\cos x \cdot \frac{dy}{dx} + y \sin x = 1$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{\sin x}{\cos x} y = \frac{1}{\cos x} \Rightarrow \frac{dy}{dx} + \tan x \, y = \sec x$$

Here, P = $\tan x$ and Q = $\sec x$

$\therefore$ Integrating factor = $e^{\int P dx} = e^{\int \tan x dx} = e^{\log \sec x} = \sec x$

Hence, the correct option is (*c*).

Q41. Solution of differential equation

$\tan y \sec^2 x \, dx + \tan x \sec^2 y \, dy = 0$ is

(*a*) $\tan x + \tan y = k$ (*b*) $\tan x - \tan y = k$

(*c*) $\frac{\tan x}{\tan y} = k$ (*d*) $\tan x \cdot \tan y = k$

Sol. The given differential equation is

$$\tan y \sec^2 x \, dx + \tan x \sec^2 y \, dy = 0$$

$$\Rightarrow \tan x \sec^2 y \, dy = -\tan y \sec^2 x \, dx$$

$$\Rightarrow \quad \frac{\sec^2 y}{\tan y} \cdot dy = \frac{-\sec^2 x}{\tan x} \cdot dx$$

Integrating both sides, we get

$$\Rightarrow \quad \int \frac{\sec^2 y}{\tan y} dy = \int \frac{-\sec^2 x}{\tan x} dx$$

$$\Rightarrow \quad \log |\tan y| = -\log |\tan x| + \log c$$

$$\Rightarrow \quad \log |\tan y| + \log |\tan x| = \log c$$

Q42. Family $y = Ax + A^3$ of curves is represented by the differential equation of degree

(*a*) 1 (*b*) 2 (*c*) 3 (*d*) 4

Sol. Given equation is $y = Ax + A^3$

Differentiating both sides, we get

$$\frac{dy}{dx} = \text{A which has degree 1.}$$

Hence, the correct answer is (*a*).

Q43. Integrating factor of $x\frac{dy}{dx} - y = x \;-\; x$ is

(*a*) x (*b*) $\log x$

(*c*) $\frac{1}{x}$ (*d*) $-x$

Sol. The given differential equation is

$$x\frac{dy}{dx} - y = x^4 - 3x \Rightarrow \frac{dy}{dx} - \frac{y}{x} = x^3 - 3$$

Here, $P = -\frac{1}{x}$ and $Q = x^3 - 3$

So, integrating factor $= e^{\int P dx} = e^{\int -\frac{1}{x}dx} = e^{-\log x} = e^{\log\frac{1}{x}} = \frac{1}{x}$

Hence, the correct option is (*c*).

Q44. Solution of $\frac{dy}{dx} - y = 1, y(0) = 1$ is given by

(*a*) $xy = -e^x$ (*b*) $xy = -e^{-x}$

(*c*) $xy = -1$ (*d*) $y = 2e^x - 1$

Sol. The given differential equation is

$$\frac{dy}{dx} - y = 1$$

Here, $P = -1, Q = 1$

$\therefore$ Integrating factor, I.F. $= e^{\int P dx} = e^{\int -1 dx} = e^{-x}$

So, the solution is

$$y \times \text{I.F.} = \int Q.\text{I.F.}\, dx + c$$

$$\Rightarrow \quad y \times e^{-x} = \int 1.e^{-x}\, dx + c$$

$$\Rightarrow \quad y\,.\,e^{-x} = -e^{-x} + c$$

Put $x = 0, y = 1$

$$\Rightarrow \quad 1\,.\,e^0 = -e^0 + c$$

$$\Rightarrow \quad 1 = -1 + c \qquad \therefore c = 2$$

So the equation is $y \cdot e^{-x} = -e^{-x} + 2$

$\Rightarrow \qquad y = -1 + 2e^{x} = 2e^{x} - 1$

Hence, the correct option is (*d*).

Q45. The number of solutions of $\dfrac{dy}{dx} = \dfrac{y+1}{x-1}$ when $y(1) = 2$ is

(*a*) none (*b*) one

(*c*) two (*d*) infinite

Sol. The given differential equation is $\dfrac{dy}{dx} = \dfrac{y+1}{x-1}$

$$\Rightarrow \qquad \frac{dy}{y+1} = \frac{dx}{x-1}$$

Integrating both sides, we get

$$\int \frac{dy}{y+1} = \int \frac{dx}{x-1}$$

$$\Rightarrow \qquad \log(y+1) = \log(x-1) + \log c$$

$$\Rightarrow \quad \log(y+1) - \log(x-1) = \log c$$

$$\Rightarrow \qquad \log\left|\frac{y+1}{x-1}\right| = \log c \quad \Rightarrow \quad \frac{y+1}{x-1} = c$$

Put $x = 1$ and $y = 2$

$$\Rightarrow \qquad \frac{2+1}{1-1} = c \qquad \therefore c = \infty$$

$$\therefore \qquad \frac{y+1}{x-1} = \frac{1}{0} \quad \Rightarrow \quad x - 1 = 0 \quad \Rightarrow \quad x = 1$$

Hence, the correct option is (*b*).

Q46. Which of the following is a second order differential equation?

(*a*) $(y')^2 + x = y^2$ (*b*) $y'y'' + y = \sin x$

(*c*) $y''' + (y'')^2 + y = 0$ (*d*) $y' = y^2$

Sol. Second order differential equation is $y'y'' + y = \sin x$

Hence, the correct option is (*b*).

Q47. Integrating factor of the differential equation

$$(1 - x^2)\frac{dy}{dx} - xy = 1 \text{ is}$$

(*a*) $-x$ (*b*) $\dfrac{x}{1+x^2}$

(*c*) $\sqrt{1-x^2}$ (*d*) $\dfrac{1}{2}\log(1-x^2)$

Sol. The given differential equation is

$$(1 - x^2)\frac{dy}{dx} - xy = 1$$

$$\Rightarrow \qquad \frac{dy}{dx} - \frac{x}{1-x^2} \cdot y = \frac{1}{1-x^2}$$

Here, $P = -\dfrac{x}{1-x^2}$ and $Q = \dfrac{1}{1-x^2}$

$\therefore$ Integrating factor

$$\text{I.F.} = e^{\int P dx} = e^{\int \frac{-x}{1-x^2} dx} = e^{\frac{1}{2}\log(1-x^2)} = \sqrt{1-x^2}$$

Hence, the correct option is (*c*).

Q48. $\tan^{-1}x + \tan^{-1}y = c$ is the general solution of the differential equation:

(*a*) $\dfrac{dy}{dx} = \dfrac{1+y^2}{1+x^2}$ (*b*) $\dfrac{dy}{dx} = \dfrac{1+x^2}{1+y^2}$

(*c*) $(1+x^2)\,dy + (1+y^2)\,dx = 0$

(*d*) $(1+x^2)\,dx + (1+y^2)\,dy = 0$

Sol. Given equation is $\tan^{-1}x + \tan^{-1}y = c$

Differentiating w.r.t. x, we have

$$\frac{1}{1+x^2} + \frac{1}{1+y^2}\cdot\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \left(\frac{1}{1+y^2}\right)\frac{dy}{dx} = -\left(\frac{1}{1+x^2}\right) \Rightarrow \frac{dy}{dx} = -\left(\frac{1+y^2}{1+x^2}\right)$$

$$\Rightarrow \quad (1+x^2)\,dy = -(1+y^2)\,dx$$

$$\Rightarrow \quad (1+x^2)\,dy + (1+y^2)\,dx = 0$$

Hence the correct option is (*c*).

Q49. The differential equation $y\dfrac{dy}{dx} + x = c$ represents:

(*a*) Family of hyperbolas (*b*) Family of parabolas

(*c*) Family of ellipses (*d*) Family of circles

Sol. Given differential equation is

$$y\frac{dy}{dx} + x = c$$

$$\Rightarrow \quad y\frac{dy}{dx} = c - x \quad \Rightarrow y\,dy = (c-x)\,dx$$

$\therefore$ Integrating both sides, we get

$$\int y\,dy = \int (c-x)\,dx$$

$$\Rightarrow \quad \frac{y^2}{2} = cx - \frac{x^2}{2} + k \Rightarrow \frac{x^2}{2} + \frac{y^2}{2} - cx = k$$

$\Rightarrow$ $x^2 + y^2 - 2cx = 2k$ which is a family of circles.

Hence, the correct option is (*d*).

Q50. The general solution of $e^x \cos y\,dx - e^x \sin y\,dy = 0$ is:

(*a*) $e^x \cos y = k$ (*b*) $e^x \sin y = k$

(*c*) $e^x = k \cos y$ (*d*) $e^x = k \sin y$

Sol. The given differential equation is

$$e^x \cos y\, dx - e^x \sin y\, dy = 0$$

$\Rightarrow$ $e^x(\cos y\, dx - \sin y\, dy) = 0$

$\Rightarrow$ $\cos y\, dx - \sin y\, dy = 0$ $\quad [\because e^x \neq 0]$

$\Rightarrow$ $\sin y\, dy = \cos y\, dx \Rightarrow \dfrac{\sin y}{\cos y} dy = dx$

Integrating both sides, we have

$$\int \frac{\sin y}{\cos y} dy = \int dx$$

$\Rightarrow$ $-\log|\cos y| = x + \log k \Rightarrow \log \dfrac{1}{\cos y} - \log k = x$

$\Rightarrow$ $\log\left(\dfrac{1}{k \cos y}\right) = x \Rightarrow \dfrac{1}{k \cos y} = e^x$

$\Rightarrow$ $\dfrac{1}{k} = e^x \cos y \Rightarrow e^x \cos y = c$ $\quad \left[c = \dfrac{1}{k}\right]$

Hence, the correct option is (*a*).

Q51. The degree of the differential equation:

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 + 6y^5 = 0 \text{ is}$$

(*a*) 1 (*b*) 2 (*c*) 3 (*d*) 5

Sol. The degree of the given differential equation is 1 as the power of the highest order is 1.

Hence, the correct option is (*a*).

Q52. The solution of the differential equation

$$\frac{dy}{dx} + y = e^{-x},\ y(0) = 0 \text{ is}$$

(*a*) $y = e^x(x-1)$ (*b*) $y = xe^{-x}$

(*c*) $y = x e^{-x} + 1$ (*d*) $y = (x+1)e^{-x}$

Sol. The given differential equation is

$$\frac{dy}{dx} + y = e^{-x}$$

Since, it is a linear differential equation

$\therefore$ $P = 1$ and $Q = e^{-x}$

$\therefore$ I.F $= e^{\int 1.dx} = e^x$

So, the solution is

$$y \times \text{I.F.} = \int Q.\text{I.F.}\, dx + c \Rightarrow y.e^x = \int e^{-x}.e^x dx + c$$

$\Rightarrow$ $y.e^x = \int 1.dx + c \Rightarrow y.e^x = x + c$

Put $x = 0, y = 0$, we have $0 = 0 + c \quad \therefore c = 0$

So, the solution is $y\,e^x = x \quad \Rightarrow \quad y = x \,.\, e^{-x}$
Hence, the correct option is (*b*).

Q53. Integrating factor of the differential equation

$$\frac{dy}{dx} + y\tan x - \sec x = 0 \text{ is}$$

(*a*) $\cos x$ (*b*) $\sec x$
(*c*) $e^{\cos x}$ (*d*) $e^{\sec x}$

Sol. Given differential equation is

$$\frac{dy}{dx} + y\tan x - \sec x = 0 \Rightarrow \frac{dy}{dx} + y\tan x = \sec x$$

Here, $P = \tan x$ and $Q = \sec x$

$\therefore$ I.F. $= e^{\int P dx} = e^{\int \tan x\, dx} = e^{\log \sec x} = \sec x$

Hence, the correct option is (*b*).

Q54. The solution of the differential equation

$$\frac{dy}{dx} = \frac{1+y^2}{1+x^2} \text{ is}$$

(*a*) $y = \tan^{-1} x$ (*b*) $y - x = k(1+xy)$
(*c*) $x = \tan^{-1} y$ (*d*) $\tan(xy) = k$

Sol. The given differential equation is

$$\frac{dy}{dx} = \frac{1+y^2}{1+x^2} \quad \Rightarrow \quad \frac{dy}{1+y^2} = \frac{dx}{1+x^2}$$

Integrating both sides, we get

$$\int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2}$$

$\Rightarrow \quad \tan^{-1} y = \tan^{-1} x + c \Rightarrow \tan^{-1} y - \tan^{-1} x = c$

$\Rightarrow \quad \tan^{-1}\left(\dfrac{y-x}{1+xy}\right) = c$

$\Rightarrow \quad \dfrac{y-x}{1+xy} = \tan c \quad \Rightarrow \quad \dfrac{y-x}{1+xy} = k \quad [k = \tan c]$

$\Rightarrow \quad y - x = k(1+xy)$

Hence, the correct option is (*b*).

Q55. The integrating factor of the differential equation

$$\frac{dy}{dx} + y = \frac{1+y}{x} \text{ is:}$$

(*a*) $\dfrac{x}{e^x}$ (*b*) $\dfrac{e^x}{x}$ (*c*) xe^x (*d*) e^x

Sol. The given differential equation is

$$\frac{dy}{dx} + y = \frac{1+y}{x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1+y}{x} - y$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{x} + y\frac{(1-x)}{x} \Rightarrow \frac{dy}{dx} - \left(\frac{1-x}{x}\right)y = \frac{1}{x}$$

Here, $P = -\left(\frac{1-x}{x}\right)$ and $Q = \frac{1}{x}$

$\therefore$ Integrating factor I.F. $= e^{\int P dx} = e^{\int \frac{x-1}{x} dx} = e^{\int \left(1 - \frac{1}{x}\right) dx}$

$= e^{(x - \log x)} = e^x . e^{-\log x}$

$= e^x . e^{\log \frac{1}{x}} = e^x . \frac{1}{x}$

Hence, the correct option is (*b*).

Q56. $y = ae^{mx} + be^{-mx}$ satisfies which of the following differential equations?

(*a*) $\frac{dy}{dx} + my = 0$ (*b*) $\frac{dy}{dx} - my = 0$

(*c*) $\frac{d^2y}{dx^2} - m^2y = 0$ (*d*) $\frac{d^2y}{dx^2} + m^2y = 0$

Sol. The given equation is $y = ae^{mx} + be^{-mx}$

On differentiation, we get $\frac{dy}{dx} = a . me^{mx} - b . me^{-mx}$

Again differentiating w.r.t., we have

$$\frac{d^2y}{dx^2} = am^2 e^{mx} + bm^2 e^{-mx}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = m^2 (ae^{mx} + be^{-mx}) \Rightarrow \frac{d^2y}{dx^2} = m^2y \Rightarrow \frac{d^2y}{dx^2} - m^2y = 0$$

Hence, the correct option is (*c*).

Q57. The solution of the differential equation $\cos x \sin y \, dx + \sin x \cos y \, dy = 0$ is

(*a*) $\frac{\sin x}{\sin y} = c$ (*b*) $\sin x \sin y = c$

(*c*) $\sin x + \sin y = c$ (*d*) $\cos x \cos y = c$

Sol. The given differential equation is

$\cos x \sin y \, dx + \sin x \cos y \, dy = 0$

$$\Rightarrow \quad \sin x \cos y \, dy = -\cos x \sin y \, dx$$

$$\Rightarrow \quad \frac{\cos y}{\sin y} dy = -\frac{\cos x}{\sin x} dx \Rightarrow \cot y \, dy = -\cot x \, dx$$

Integrating both sides, we have

$$\Rightarrow \quad \int \cot y \, dy = -\int \cot x \, dx$$

$\Rightarrow \quad \log|\sin y| = -\log|\sin x| + \log c$

$\Rightarrow \quad \log|\sin y| + \log|\sin x| = \log c$

$\Rightarrow \quad \log|\sin y \cdot \sin x| = \log c \Rightarrow \sin x \sin y = c$

Hence, the correct option is (*b*).

Q58. The solution of $x\dfrac{dy}{dx} + y = e^x$ is:

(*a*) $y = \dfrac{e^x}{x} + \dfrac{k}{x}$ (*b*) $y = xe^x + cx$

(*c*) $y = x \cdot e^x + k$ (*d*) $x = \dfrac{e^y}{y} + \dfrac{k}{y}$

Sol. The given differential equation is $x\dfrac{dy}{dx} + y = e^x$

$\Rightarrow \quad \dfrac{dy}{dx} + \dfrac{y}{x} = \dfrac{e^x}{x}$

Here $P = \dfrac{1}{x}$ and $Q = \dfrac{e^x}{x}$

$\therefore$ Integrating factor I.F. $= e^{\int \frac{1}{x} dx} = e^{\log|x|} = x$

So, the solution is

$y \times \text{I.F.} = \int Q \times \text{I.F.}\, dx + k \Rightarrow y \times x = \int \dfrac{e^x}{x} \times x\, dx + k$

$\Rightarrow \quad y \times x = \int e^x\, dx + k \quad \Rightarrow y \times x = e^x + k$

$\therefore \quad y = \dfrac{e^x}{x} + \dfrac{k}{x}$

Hence, the correct option is (*a*).

Q59. The differential equation of the family of curves $x^2 + y^2 - 2ay = 0$, where a is arbitrary constant, is:

(*a*) $(x^2 - y^2)\dfrac{dy}{dx} = 2xy$ (*b*) $2(x^2 + y^2)\dfrac{dy}{dx} = xy$

(*c*) $2(x^2 - y^2)\dfrac{dy}{dx} = xy$ (*d*) $(x^2 + y^2)\dfrac{dy}{dx} = 2xy$

Sol. The given equation is

$$x^2 + y^2 - 2ay = 0 \quad \text{..(1)}$$

Differentiating w.r.t. x, we have

$2x + 2y \cdot \dfrac{dy}{dx} - 2a\dfrac{dy}{dx} = 0$

$\Rightarrow \quad x + y\dfrac{dy}{dx} - a\dfrac{dy}{dx} = 0 \quad \Rightarrow \quad x + (y - a)\dfrac{dy}{dx} = 0$

$\Rightarrow \quad (y - a)\dfrac{dy}{dx} = -x \quad \Rightarrow y - a = \dfrac{-x}{dy/dx}$

$$\Rightarrow \quad a = y + \frac{x}{dy/dx} \Rightarrow a = \frac{y \cdot \frac{dy}{dx} + x}{\frac{dy}{dx}}$$

Putting the value of a in eq. (1) we get

$$x^2 + y^2 - 2y\left[\frac{y\frac{dy}{dx} + x}{\frac{dy}{dx}}\right] = 0$$

$$\Rightarrow (x^2 + y^2)\frac{dy}{dx} - 2y\left(y\frac{dy}{dx} + x\right) = 0$$

$$\Rightarrow (x^2 + y^2)\frac{dy}{dx} - 2y^2\frac{dy}{dx} - 2xy = 0$$

$$\Rightarrow \quad (x^2 + y^2 - 2y^2)\frac{dy}{dx} = 2xy \Rightarrow (x^2 - y^2)\frac{dy}{dx} = 2xy$$

$\therefore$ Hence the correct option is (*a*).

Q60. Family $y = Ax + A^3$ of curves will correspond to a differential equation of order

(*a*) 3 (*b*) 2 (*c*) 1 (*d*) not defined

Sol. The given equation is

$$y = Ax + A^3$$

Differentiating both sides, we get $\frac{dy}{dx} = A$

Again differentiating both sides, we have $\frac{d^2y}{dx^2} = 0$

So the order of the differential equation is 2.

Hence, the correct option is (*b*).

Q61. The general solution of $\frac{dy}{dx} = 2xe^{x^2-y}$ is :

(*a*) $e^{x^2-y} = c$ (*b*) $e^{-y} + e^{x^2} = c$

(*c*) $e^y = e^{x^2} + c$ (*d*) $e^{x^2+y} = c$

Sol. The given differential equation is

$$\frac{dy}{dx} = 2x \cdot e^{x^2-y}$$

$$\Rightarrow \quad \frac{dy}{dx} = 2x \cdot e^{x^2} \cdot e^{-y} \Rightarrow \frac{dy}{e^{-y}} = 2x \cdot e^{x^2} dx$$

Integrating both sides, we have

$$\int \frac{dy}{e^{-y}} = \int 2x \cdot e^{x^2} dx \Rightarrow \int e^y dy = \int 2x \cdot e^{x^2} dx$$

Put in RHS $x^2 = t \therefore 2x\,dx = dt$

$\therefore \quad \int e^y dy = \int e^t dt$

$\Rightarrow \quad e^y = e^t + c \Rightarrow e^y = e^{x^2} + c$

Hence, the correct option is (*c*).

Q62. The curve for which the slope of the tangent at any point is equal to the ratio of the abscissa to the ordinate of the point is:

(*a*) an ellipse (*b*) parabola

(*c*) circle (*d*) rectangular hyperbola

Sol. Since, the slope of the tangent to the curve $= x : y$

$$\therefore \quad \frac{dy}{dx} = \frac{x}{y} \Rightarrow ydy = xdx$$

Integrating both sides, we get $\int y\,dy = \int x\,dx$

$$\Rightarrow \quad \frac{y^2}{2} = \frac{x^2}{2} + c \Rightarrow y^2 = x^2 + 2c$$

$\Rightarrow \quad y^2 - x^2 = 2c = k$ which is rectangular hyperbola.

Hence, the correct option is (*d*).

Q63. The general solution of the differential equation $\frac{dy}{dx} = e^{\frac{x^2}{2}} + xy$ is:

(*a*) $y = c\,.\,e^{\frac{-x^2}{2}}$ (*b*) $y = c\,.\,e^{\frac{x^2}{2}}$

(*c*) $y = (x + c)\,.\,e^{\frac{x^2}{2}}$ (*d*) $y = (c - x)\,e^{\frac{x^2}{2}}$

Sol. The given differential equation is

$$\frac{dy}{dx} = e^{\frac{x^2}{2}} + xy \Rightarrow \frac{dy}{dx} - xy = e^{\frac{x^2}{2}}$$

Since it is linear differential equation where $P = -x$ and $Q = e^{\frac{x^2}{2}}$

$\therefore$ Integrating factor I.F. $= e^{\int P dx} = e^{\int -x\,dx} = e^{-\frac{x^2}{2}}$

So, the solution is

$$y \times \text{I.F.} = \int Q \times \text{I.F.}\, dx + c$$

$$\Rightarrow \quad y \times e^{-\frac{x^2}{2}} = \int e^{\frac{x^2}{2}} e^{-\frac{x^2}{2}} dx + c$$

$$\Rightarrow \quad y \times e^{-\frac{x^2}{2}} = \int e^0\, dx + c$$

$$\Rightarrow \quad y \times e^{-\frac{x^2}{2}} = \int 1\,.\,dx + c \Rightarrow y \times e^{-\frac{x^2}{2}} = x + c$$

$\therefore \quad y = (x+c)\, e^{\frac{x^2}{2}}$

Hence the correct option is (*c*).

Q64. The solution of the equation $(2y-1)\,dx - (2x+3)\,dy = 0$ is

(*a*) $\dfrac{2x-1}{2y+3} = k$ (*b*) $\dfrac{2y+1}{2x-3} = k$

(*c*) $\dfrac{2x+3}{2y-1} = k$ (*d*) $\dfrac{2x-1}{2y-1} = k$

Sol. The given differential equation is

$$(2y-1)\,dx - (2x+3)\,dy = 0 \Rightarrow (2x+3)\,dy = (2y-1)\,dx$$

$$\Rightarrow \quad \frac{dy}{2y-1} = \frac{dx}{2x+3}$$

Integrating both sides, we get

$$\int \frac{dy}{2y-1} = \int \frac{dx}{2x+3}$$

$$\Rightarrow \quad \frac{1}{2}\log|2y-1| = \frac{1}{2}\log|2x+3| + \log c$$

$$\Rightarrow \quad \log|2y-1| = \log|2x+3| + 2\log c$$

$$\Rightarrow \quad \log|2y-1| - \log|2x+3| = \log c^2$$

$$\Rightarrow \quad \log\left|\frac{2y-1}{2x+3}\right| = \log c^2$$

$$\Rightarrow \quad \frac{2y-1}{2x+3} = c^2 \Rightarrow \frac{2x+3}{2y-1} = \frac{1}{c^2}$$

$$\Rightarrow \quad \frac{2x+3}{2y-1} = k, \text{ where } k = \frac{1}{c^2}$$

Hence, the correct option is (*c*).

Q65. The differential equation for which $y = a\cos x + b\sin x$ is a solution, is:

(*a*) $\dfrac{d^2y}{dx^2} + y = 0$ (*b*) $\dfrac{d^2y}{dx^2} - y = 0$

(*c*) $\dfrac{d^2y}{dx^2} + (a+b)\,y = 0$ (*d*) $\dfrac{d^2y}{dx^2} + (a-b)\,y = 0$

Sol. The given equation is

$$y = a\cos x + b\sin x$$

$$\frac{dy}{dx} = -a\sin x + b\cos x$$

$$\frac{d^2y}{dx^2} = -a\cos x - b\sin x$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -(a\cos x + b\sin x) \Rightarrow \frac{d^2y}{dx^2} = -y \quad \Rightarrow \frac{d\ y}{dx} + y = 0$$

Hence, the correct option is (*a*).

Q66. The solution of $\frac{dy}{dx} + y = e^{-x}, y(0) = 0$ is:

(*a*) $y = e^{-x}(x-1)$ (*b*) $y = x \cdot e^{x}$

(*c*) $y = xe^{-x} + 1$ (*d*) $y = x \cdot e^{-x}$

Sol. The given differential equation is $\frac{dy}{dx} + y = e^{-x}$

Since, it is a linear differential equation then $P = 1$ and $Q = e^{-x}$

Integrating factor I.F. $= e^{\int P dx} = e^{\int 1 . dx} = e^{x}$

$\therefore$ Solution is

$$y \times \text{I.F.} = \int Q \times \text{I.F.}\, dx + c$$

$$\Rightarrow \quad y \times e^{x} = \int e^{-x} \times e^{x} dx + c \quad \Rightarrow y \times e^{x} = \int e^{0} dx + c$$

$$\Rightarrow \quad y \times e^{x} = \int 1 . dx + c \quad \Rightarrow y \times e^{x} = x + c$$

Put $y = 0$ and $x = 0$

$\therefore \quad 0 = 0 + c \quad \therefore \quad c = 0$

$\therefore$ equation is $\quad y \times e^{x} = x$

So $\quad y = x \cdot e^{-x}$

Hence, the correct option is (*d*).

Q67. The order and degree of the differential equation

$$\left[\frac{d^3y}{dx^3}\right]^2 - 3\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^4 = y^4 \text{ are}$$

(*a*) 1, 4 (*b*) 3, 4

(*c*) 2, 4 (*d*) 3, 2

Sol. The given differential equation is

$$\left[\frac{d^3y}{dx^3}\right]^2 - 3\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^4 = y^4$$

Here the highest derivative is $\frac{d^3y}{dx^3}$.

$\therefore$ the order of the differential equation is 3 and since, the power of highest order is 2

$\therefore$ its degree is 2

Hence, the correct option is (d).

Q68. The order and degree of the differential equation $\left[1+\left(\frac{dy}{dx}\right)^2\right]=\frac{d^2y}{dx^2}$ are:

(a) $2, \frac{3}{2}$ (b) 2, 3

(c) 2, 1 (d) 3, 4

Sol. The given differential equation is

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]=\frac{d^2y}{dx^2}$$

Here, the highest derivative is 2,

$\therefore$ order = 2

and the power of the highest derivative is 1

$\therefore$ degree = 1

Hence, the correct option is (c).

Q69. The differential equation of the family of curves $y^2 = 4a(x + a)$ is:

(a) $y^2 = 4\frac{dy}{dx}\left(x+\frac{dy}{dx}\right)$ (b) $2y\cdot\frac{dy}{dx} = 4a$

(c) $y\cdot\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2 = 0$ (d) $2x\cdot\frac{dy}{dx}+y\left(\frac{dy}{dx}\right)^2 - y$

Sol. The given equation of family of curves is

$$y^2 = 4a(x+a)$$

$$\Rightarrow \quad y^2 = 4ax + 4a^2 \quad \text{...(1)}$$

Differentiating both sides, w.r.t. x, we get

$$2y\cdot\frac{dy}{dx} = 4a$$

$$\Rightarrow \quad y\cdot\frac{dy}{dx} = 2a \Rightarrow \frac{y}{2}\frac{dy}{dx} = a$$

Now, putting the value of a in eq. (1) we get

$$y^2 = 4x\left(\frac{y}{2}\frac{dy}{dx}\right)+4\left(\frac{y}{2}\cdot\frac{dy}{dx}\right)^2$$

$$\Rightarrow \quad y^2 = 2xy\frac{dy}{dx}+y^2\left(\frac{dy}{dx}\right)^2 \quad \Rightarrow y = 2x\frac{dy}{dx}+y\left(\frac{dy}{dx}\right)^2$$

$$\Rightarrow \quad 2x.\frac{dy}{dx} + y.\left(\frac{dy}{dx}\right)^2 - y = 0$$

Hence, the correct option is (*d*).

Q70. Which of the following is the general solution of

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0?$$

(*a*) $y = (Ax + B).e^x$ (*b*) $y = (Ax + B)e^{-x}$

(*c*) $y = Ae^x + Be^{-x}$ (*d*) $y = A\cos x + B\sin x$

Sol. The given differential equation is

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$$

Since the above equation is of second order and first degree

$\therefore \quad D^2y - 2Dy + y = 0$, where $D = \frac{d}{dx}$

$\Rightarrow \quad (D^2 - 2D + 1)y = 0$

$\therefore$ auxiliary equation is

$$m^2 - 2m + 1 = 0 \quad \Rightarrow \quad (m-1)^2 = 0 \quad \Rightarrow \quad m = 1, 1$$

If the roots of Auxiliary equation are real and equal say (m) then $CF = (c_1 x + c_2).e^{mx}$

$\therefore \quad CF = (Ax + B)e^x$

So $\quad y = (Ax + B).e^x$

Hence, the correct option is (*a*).

Q71. General solution of $\frac{dy}{dx} + y\tan x = \sec x$ is:

(*a*) $y\sec x = \tan x + c$ (*b*) $y\tan x = \sec x + c$

(*c*) $\tan x = y\tan x + c$ (*d*) $x\sec x = \tan y + c$

Sol. The given differential equation is $\frac{dy}{dx} + y\tan x = \sec x$

Since, it is a linear differential equation

$\therefore \quad P = \tan x$ and $Q = \sec x$

Integrating factor I.F. $= e^{\int P dx} = e^{\int \tan x\, dx} = e^{\log \sec x} = \sec x$

$\therefore$ Solution is

$$y \times \text{I.F.} = \int Q \times \text{I.F.}\, dx + c$$

$$\Rightarrow \quad y \times \sec x = \int \sec x . \sec x\, dx + c$$

$$\Rightarrow \quad y\sec x = \int \sec^2 x\, dx + c \quad \Rightarrow y\sec x = \tan x + c$$

Hence, the correct option is (*a*).

Q72. Solution of differential equation $\frac{dy}{dx} + \frac{y}{x} = \sin x$ is:

(a) $x(y + \cos x) = \sin x + c$ (b) $x(y - \cos x) = \sin x + c$

(c) $xy \cos x = \sin x + c$ (d) $x(y + \cos x) = \cos x + c$

Sol. The given differential equation is $\frac{dy}{dx} + \frac{y}{x} = \sin x$

Since, it is a linear differential equation

$\therefore \quad P = \frac{1}{x}$ and $Q = \sin x$

Integrating factor I.F. $= e^{\int \frac{1}{x} dx} = e^{\log x} = x$

$\therefore$ Solution is $\quad y \times \text{I.F.} = \int Q \times \text{I.F.}\, dx + c$

$$y \times x = \int \sin x \,.\, x \, dx + c \Rightarrow y \times x = \int x \sin x \, dx + c$$

$$\Rightarrow \quad yx = x \,.\int \sin x \, dx - \int \left(D(x) \int \sin x \, dx\right) dx + c$$

$$\Rightarrow \quad yx = x(-\cos x) - \int -\cos x \, dx$$

$$\Rightarrow \quad yx = -x \cos x + \int \cos x \, dx \Rightarrow yx = -x \cos x + \sin x + c$$

$$\Rightarrow \quad yx + x \cos x = \sin x + c$$

$$\Rightarrow \quad x(y + \cos x) = \sin x + c$$

Hence, the correct option is (a).

Q73. The general solution of the differential equation $(e^x + 1)\, y \, dy = (y + 1)\, e^x \, dx$ is:

(a) $(y + 1) = k(e^x + 1)$

(b) $y + 1 = e^x + 1 + k$

(c) $y = \log [k(y + 1)(e^x + 1)]$

(d) $y = \log \left\{\frac{e^x + 1}{y + 1}\right\} + k$

Sol. The given differential equation is

$$(e^x + 1)\, y \, dy = (y + 1)\, e^x \, dx$$

$$\Rightarrow \quad \frac{y}{y+1} dy = \frac{e^x}{e^x + 1} dx$$

Integrating both sides, we get

$$\int \frac{y}{y+1} dy = \int \frac{e^x}{e^x + 1} dx$$

$$\Rightarrow \quad \int \frac{y + 1 - 1}{y + 1} dy = \int \frac{e^x}{e^x + 1} dx$$

$$\Rightarrow \quad \int 1 \,.\, dy - \int \frac{1}{y+1} dy = \int \frac{e^x}{e^x + 1} dx$$

$\Rightarrow \quad y - \log|y+1| = \log|e^x+1| + \log k$

$\Rightarrow \quad y = \log|y+1| + \log|e^x+1| + \log k$

$\Rightarrow \quad y = \log|k(y+1)(e^x+1)|$

Hence, the correct option is (*c*).

Q74. The solution of differential equation $\frac{dy}{dx} = e^{x-y} + x^2e^{-y}$ is:

(*a*) $y = e^{x-y} - x^2e^{-y} + c$ (*b*) $e^y - e^x = \frac{x^3}{3} + c$

(*c*) $e^x + e^y = \frac{x^3}{3} + c$ (*d*) $e^x - e^y = \frac{x^3}{3} + c$

Sol. The given differential equation is

$$\frac{dy}{dx} = e^{x-y} + x^2e^{-y}$$

$$\Rightarrow \quad \frac{dy}{dx} = e^x.e^{-y} + x^2.e^{-y} \Rightarrow \frac{dy}{dx} = e^{-y}(e^x + x^2)$$

$$\Rightarrow \quad \frac{dy}{e^{-y}} = (e^x + x^2)\,dx \Rightarrow e^y.dy = (e^x + x^2)\,dx$$

Integrating both sides, we have

$$\int e^y dy = \int (e^x + x^2)\,dx$$

$$\Rightarrow \quad e^y = e^x + \frac{x^3}{3} + c \Rightarrow e^y - e^x = \frac{x^3}{3} + c$$

Hence, the correct option is (*b*).

Q75. The solution of the differential equation

$$\frac{dy}{dx} + \frac{2xy}{1+x^2} = \frac{1}{(1+x^2)^2} \text{ is:}$$

(*a*) $y(1+x^2) = c + \tan^{-1}x$ (*b*) $\frac{y}{1+x^2} = c + \tan^{-1}x$

(*c*) $y\log(1+x^2) = c + \tan^{-1}x$ (*d*) $y(1+x^2) = c + \sin^{-1}x$

Sol. The given differential equation is

$$\frac{dy}{dx} + \frac{2xy}{1+x^2} = \frac{1}{(1+x^2)^2}$$

Since, it is a linear differential equation

$$P = \frac{2x}{1+x^2} \text{ and } Q = \frac{1}{(1+x^2)^2}$$

Integrating factor I.F. $= e^{\int P dx} = e^{\int \frac{2x}{1+x^2}dx} = e^{\log(1+x^2)} = (1+x^2)$

$\therefore$ Solution is $y \times \text{I.F.} = \int Q \times \text{I.F.}\,dx + c$

$$\Rightarrow \quad y(1+x^2) = \int \frac{1}{(1+x^2)^2} \times (1+x^2)\, dx + c$$

$$\Rightarrow \quad y(1+x^2) = \int \frac{1}{(1+x^2)}\, dx + c \Rightarrow y(1+x^2) = \tan^{-1} x + c$$

Hence, the correct option is (*a*).

Q76. Fill in the blanks of the following (*i* to *xi*):

(*i*) The degree of the differential equation $\frac{d^2y}{dx^2} + e^{dy/dx} = 0$ is ____________ .

(*ii*) The degree of the differential equation $\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = x$ is ____________ .

(*iii*) The number of arbitrary constants in the general solution of a differential equation of order three is ____________ .

(*iv*) $\frac{dy}{dx} + \frac{y}{x \log x} = \frac{1}{x}$ is an equation of the type ____________ .

(*v*) General solution of the differential equation of the type $\frac{dx}{dy} + P_1 x = Q_1$ is given by ____________ .

(*vi*) The solution of the differential equation $x\frac{dy}{dx} + 2y = x^2$ is ____________ .

(*vii*) The solution of $(1+x^2)\frac{dy}{dx} + 2xy - 4x^2 = 0$ is ____________ .

(*viii*) The solution of the differential equation $ydx + (x + xy)\, dy = 0$ is ____________ .

(*ix*) General solution of $\frac{dy}{dx} + y = \sin x$ is ____________ .

(*x*) The solution of differential equation $\cot y\, dx = x\, dy$ is ____________ .

(*xi*) The integrating factor of $\frac{dy}{dx} + y = \frac{1+y}{x}$ is ____________ .

Sol. (*i*) The degree of the differential equation $\frac{d^2y}{dx^2} + e^{dy/dx} = 0$ is **not defined.**

(*ii*) The given differential equation is $\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = x$

Squaring both sides, we gett

$$1 + \left(\frac{dy}{dx}\right)^2 = x^2$$

So, the degree of the equation is 2.

(*iii*) The number of arbitrary constants in the solution is 3.

(*iv*) The given differential equation $\frac{dy}{dx}+\frac{y}{x\log x}=\frac{1}{x}$ is of the type $\frac{dy}{dx}+Py=Q$.

(*v*) General solution of the differential equation of the type $\frac{dx}{dy}+P_1x=Q_1$ is given by $x\times\text{I.F.}=\int Q\times\text{I.F.}\,dy+c$

$$\Rightarrow \quad x\,.\,e^{\int P_1\,dy}=\int Q_1\,.\,e^{\int P_1\,dy}dy+c.$$

(*vi*) The given differential equation is $x\frac{dy}{dx}+2y=x^2$

$$\Rightarrow \qquad \frac{dy}{dx}+\frac{2}{x}y=x.$$

Since, it is linear differential equation

$\therefore \quad P=\frac{2}{x}$ and $Q=x$

Integrating factor I.F. $=e^{\int Pdx}=e^{\int\frac{2}{x}dx}=e^{2\log x}=e^{\log x^2}=x^2$

$\therefore$ Solution is

$$y\times\text{I.F.}=\int Q\times\text{I.F.}\,dx+c$$

$$\Rightarrow \quad y\,.\,x^2=\int x\,.\,x^2\,dx+c \quad\Rightarrow y\,.\,x^2=\int x^3\,dx+c$$

$$\Rightarrow \quad y\,.\,x^2=\frac{1}{4}x^4+c \quad\Rightarrow y=\frac{1}{4}x^2+c\,.\,x^{-2}$$

Hence, the solution is $y=\frac{1}{4}x^2+c\,.\,x^{-2}$.

(*vii*) The given differential equation is

$$(1+x^2)\frac{dy}{dx}+2xy-4x^2=0$$

$$\Rightarrow \quad \frac{dy}{dx}+\frac{2xy}{1+x^2}=\frac{4x^2}{1+x^2}$$

Since it is a linear differential equation

$\therefore \quad P=\frac{2x}{1+x^2}$ and $Q=\frac{4x^2}{1+x^2}$

Integrating factor I.F. $=e^{\int Pdx}=e^{\int\frac{2x}{1+x^2}dx}=e^{\log(1+x^2)}=(1+x^2)$

$\therefore$ Solution is $\quad y\times\text{I.F}=\int Q\times\text{I.F.}\,dx+c$

$$\Rightarrow \quad y \times (1+x^2) = \int \frac{4x}{1+x^2} \times (1+x^2)\,dx + c$$

$$\Rightarrow \quad y \times (1+x^2) = \int 4x^2\,dx + c \;\Rightarrow y \times (1+x^2) = \frac{4}{3}x^3 + c$$

$$\Rightarrow \quad y = \frac{4}{3}\frac{x^3}{(1+x^2)} + c\,(1+x^2)^{-1}$$

Hence, the required solution is $y = \frac{4}{3}\frac{x^3}{(1+x^2)} + c\,(1+x^2)^{-1}$.

(*viii*) The given differential equation is

$$ydx + (x+xy)\,dy = 0$$

$$\Rightarrow \quad (x+xy)\,dy = -y\,dx \;\Rightarrow x\,(1+y)\,dy = -y\,dx$$

$$\Rightarrow \quad \frac{1+y}{y}dy = -\frac{1}{x}dx$$

Integrating both sides, we get

$$\int \frac{1+y}{y}dy = -\int \frac{1}{x}dx$$

$$\Rightarrow \quad \int \left(\frac{1}{y}+1\right)dy = -\int \frac{1}{x}dx$$

$$\Rightarrow \quad \log y + y = -\log x + \log c$$

$$\Rightarrow \quad \log x + \log y + \log e^y = \log c$$

$$\Rightarrow \quad \log (xy\,.\,e^y) = \log c$$

$$\therefore \quad xy = c\,e^{-y}$$

Hence, the required solution is $xy = c\,e^{-y}$.

(*ix*) The given differential equation is $\frac{dy}{dx} + y = \sin x$

Since, it it a linear differential equation

$\therefore$ P = 1 and Q = $\sin x$

Integrating factor I.F. $= e^{\int P dx} = e^{\int 1.dx} = e^x$

$\therefore$ Solution is $y \times \text{I.F.} = \int \text{Q} \times \text{I.F.}\,dx + c$

$$\Rightarrow \quad y\,.\,e^x = \int \sin x\,.\,e^x dx + c \quad ...(1)$$

Let $\text{I} = \int \underset{\text{I}}{\sin x}\,.\,\underset{\text{II}}{e^x}\,dx$

$$\text{I} = \sin x\,.\int e^x dx - \int \left(D(\sin x)\,.\int e^x dx\right)dx$$

$$\text{I} = \sin x\,.\,e^x - \int \underset{\text{I}}{\cos x}\,.\,\underset{\text{II}}{e^x}\,dx$$

$$\text{I} = \sin x\,.\,e^x - \left[\cos x\,.\int e^x dx - \int \left(D\,(\cos x)\int e^x dx\right)dx\right]$$

$$I = \sin x \cdot e^x - \left[\cos x \cdot e^x - \int -\sin x \cdot e^x dx\right]$$

$$I = \sin x \cdot e^x - \cos x \cdot e^x - \int \sin x \cdot e^x dx$$

$$I = \sin x \cdot e^x - \cos x \cdot e^x - I$$

$$\Rightarrow I + I = e^x (\sin x - \cos x)$$

$$\Rightarrow \quad 2I = e^x (\sin x - \cos x)$$

$$\therefore \quad I = \frac{e^x}{2}(\sin x - \cos x)$$

From eq. (1) we get

$$y \cdot e^x = \frac{e^x}{2}(\sin x - \cos x) + c$$

$$y = \left(\frac{\sin x - \cos x}{2}\right) + c \cdot e^{-x}$$

Hence, the required solution is

$$y = \left(\frac{\sin x - \cos x}{2}\right) + c \cdot e^{-x}$$

(x) The given differential equation is $\cot y \, dx = x \, dy$

$$\Rightarrow \frac{dy}{\cot y} = \frac{dx}{x} \Rightarrow \tan y \, dy = \frac{dx}{x}$$

Integrating both sides, we get

$$\int \tan y \, dy = \int \frac{dx}{x} \Rightarrow \log \sec y = \log x + \log c$$

$$\Rightarrow \quad \log \sec y - \log x = \log c$$

$$\Rightarrow \quad \log \left|\frac{\sec y}{x}\right| = \log C$$

$$\therefore \quad \frac{\sec y}{x} = C \Rightarrow \frac{x}{\sec y} = \frac{1}{C} \Rightarrow \frac{x}{\sec y} = C \qquad \left[\frac{1}{c} = C\right]$$

$$\therefore \quad x = C \sec y$$

Hence, the required solution is $x = C \sec y$.

(xi) The given differential equation is

$$\frac{dy}{dx} + y = \frac{1+y}{x}$$

$$\Rightarrow \quad \frac{dy}{dx} + y = \frac{1}{x} + \frac{y}{x}$$

$$\Rightarrow \quad \frac{dy}{dx} + y - \frac{y}{x} = \frac{1}{x} \Rightarrow \frac{dy}{dx} + \left(1 - \frac{1}{x}\right)y = \frac{1}{x}$$

Here $P = \left(1 - \frac{1}{x}\right)$

$\therefore$ I.F. $= e^{\int P dx} = e^{\int\left(1-\frac{1}{x}\right)dx} = e^{(x - \log x)}$

$= e^x . e^{-\log x} = e^x . e^{\log\frac{1}{x}} = e^x . \frac{1}{x}$

Hence, the required I.F. $= e^x . \frac{1}{x}$.

Q77. State True or False for the following:

(*i*) Integrating factor of the differential equation of the form $\frac{dx}{dy} + P_1 x = Q_1$ is given by $e^{\int P_1 dy}$.

(*ii*) Solution of the differential equation of the type $\frac{dx}{dy} + P_1 x = Q_1$ is given by x . I.F. $= \int$(I.F.) $Q_1 dy$.

(*iii*) Correct substitution for the solution of the differential equation of the type $\frac{dy}{dx} = f(x, y)$, where $f(x, y)$ is a homogeneous function of zero degree is $y = vx$.

(*iv*) Correct substitution for the solution of the differential equation of the type $\frac{dx}{dy} = g(x, y)$, where g (x, y) is a homogeneous function of the degree zero is $x = vy$.

(*v*) Number of arbitrary constants in the particular solution of a differential equation of order two is two.

(*vi*) The differential equation representing the family of circles $x^2 + (y - a)^2 = a^2$ will be of order two.

(*vii*) The solution of $\frac{dy}{dx} = \left(\frac{y}{x}\right)^{1/3}$ is $y^{2/3} - x^{2/3} = c$.

(*viii*) Differential equation representing the family of curves $y = e^x (A \cos x + B \sin x)$ is $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$.

(*ix*) The solution of differential equation $\frac{dy}{dx} = \frac{x + 2y}{x}$ is $x + y = kx^2$.

(*x*) Solution of $\frac{xdy}{dx} = y + x \tan\frac{y}{x}$ is $\sin\left(\frac{y}{x}\right) = cx$.

(*xi*) The differential equation of all non-horizontal lines in a plane is $\frac{d^2x}{dy^2}=0$.

Sol. (*i*) True

I.F. of the given differential equation

$$\frac{dx}{dy}+P_1x=Q \text{ is } e^{\int P_1 dy}$$

(*ii*) True (*iii*) True (*iv*) True (*v*) False

Since particular solution of a differential equation has no arbitrary constant.

(*vi*) False

We know that the order of the differential equation is equal to the number of arbitrary constants.

(*vii*) True

The given differential equation is

$$\frac{dy}{dx}=\left(\frac{y}{x}\right)^{1/3}$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{y^{1/3}}{x^{1/3}} \quad \Rightarrow \quad \frac{dy}{y^{1/3}}=\frac{dx}{x^{1/3}}$$

Integrating both sides, we get

$$\int\frac{dy}{y^{1/3}}=\int\frac{dx}{x^{1/3}} \quad \Rightarrow \int y^{-1/3}\,dy=\int x^{-1/3}\,dx$$

$$\Rightarrow \frac{1}{-\frac{1}{3}+1}y^{-\frac{1}{3}+1}=\frac{1}{-\frac{1}{3}+1}.x^{-\frac{1}{3}+1}+c$$

$$\Rightarrow \quad \frac{3}{2}y^{2/3}=\frac{3}{2}x^{2/3}+c$$

$$\Rightarrow \quad y^{2/3}=x^{2/3}+\frac{2}{3}c \quad \Rightarrow \quad y^{2/3}-x^{2/3}=k\left[k=\frac{2}{3}c\right]$$

(*viii*) True

Given equation is

$$y=e^x(A\cos x+B\sin x)$$

Differentiating both sides, we get

$$\frac{dy}{dx}=e^x(-A\sin x+B\cos x)+(A\cos x+B\sin x)\,e^x$$

$$\frac{dy}{dx}=e^x(-A\sin x+B\cos x)+y$$

Again differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = e^x(-A\cos x - B\sin x) + (-A\sin x + B\cos x).e^x + \frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = -e^x(A\cos x + B\sin x) + \frac{dy}{dx} - y + \frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = -y - y + 2\frac{dy}{dx} \quad \therefore \quad \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$$

(*ix*) True

The given differential equation is

$$\frac{dy}{dx} = \frac{x+2y}{x}$$

$$\Rightarrow \quad \frac{dy}{dx} = 1 + 2\frac{y}{x} \quad \Rightarrow \quad \frac{dy}{dx} - \frac{2y}{x} = 1$$

Here, $P = \frac{-2}{x}$ and $Q = 1$

Integrating factor I.F. $= e^{\int \frac{-2}{x}dx} = e^{-2\log x} = e^{\log x^{-2}} = \frac{1}{x^2}$

$\therefore$ Solution is $y \times \text{I.F.} = \int Q \times \text{I.F.}\, dx + c$

$$\Rightarrow \quad y \times \frac{1}{x^2} = \int 1 \times \frac{1}{x^2} dx + c$$

$$\Rightarrow \quad \frac{y}{x^2} = \int \frac{1}{x^2} dx + c \quad \Rightarrow \quad \frac{y}{x^2} = -\frac{1}{x} + c$$

$$\Rightarrow \quad y = -x + cx^2 \quad \Rightarrow y + x = cx^2$$

(*x*) True

The given differential equation is

$$x\frac{dy}{dx} = y + x\tan\left(\frac{y}{x}\right)$$

$$x\frac{dy}{dx} = -x\tan\left(\frac{y}{x}\right) = y$$

$$\Rightarrow \quad \frac{dy}{dx} - \tan\left(\frac{y}{x}\right) = \frac{y}{x} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right)$$

Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\Rightarrow \quad v + x.\frac{dv}{dx} = \frac{vx}{x} + \tan\left(\frac{vx}{x}\right)$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = v + \tan v \quad \Rightarrow \quad x\frac{dv}{dx} = \tan v$$

$$\Rightarrow \quad \frac{dv}{\tan v} = \frac{dx}{x} \quad \Rightarrow \cot v \, dv = \frac{dx}{x}$$

Integrating both sides, we get

$$\int \cot v \, dv = \int \frac{dx}{x} \Rightarrow \log \sin v = \log x + \log c$$

$$\Rightarrow \quad \log \sin v - \log x = \log c \quad \Rightarrow \quad \log \sin \frac{y}{x} = \log xc$$

$$\therefore \quad \sin \frac{y}{x} = xc$$

(xi) True

Let $y = mx + c$ be the non-horizontal line in a plane

$\therefore \frac{dy}{dx} = m$ and $\frac{d^2y}{dx^2} = 0.$

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10 Vector Algebra

10.3 EXERCISE

SHORT ANSWER TYPE QUESTIONS

Q1. Find the unit vector in the direction of sum of vectors $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = 2\hat{j} + \hat{k}$.

Sol. Given that

$$\vec{a} = 2\hat{i} - \hat{j} + \hat{k} \text{ and } \vec{b} = 2\hat{j} + \hat{k}$$

$$\vec{a} + \vec{b} = (2\hat{i} - \hat{j} + \hat{k}) + (2\hat{j} + \hat{k}) = 2\hat{i} + \hat{j} + 2\hat{k}$$

$\therefore$ Unit vector in the direction of $\vec{a} + \vec{b} = \dfrac{\vec{a} + \vec{b}}{|\vec{a} + \vec{b}|}$

$$= \frac{2\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{(2)^2 + (1)^2 + (2)^2}} = \frac{2\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{4 + 1 + 4}}$$

$$= \frac{2\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{9}} = \frac{2\hat{i} + \hat{j} + 2\hat{k}}{3} = \frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}$$

Hence, the required unit vector is $\frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}$.

Q2. If $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} - 2\hat{k}$, find the unit vector in the direction of (*i*) $6\vec{b}$ (*ii*) $2\vec{a} - \vec{b}$

Sol. Given that $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} - 2\hat{k}$

(*i*) $6\vec{b} = 6(2\hat{i} + \hat{j} - 2\hat{k}) = 12\hat{i} + 6\hat{j} - 12\hat{k}$

$\therefore$ Unit vector in the direction of $6\vec{b} = \dfrac{6\vec{b}}{|6\vec{b}|}$

$$= \frac{12\hat{i} + 6\hat{j} - 12\hat{k}}{\sqrt{(12)^2 + (6)^2 + (-12)^2}} = \frac{12\hat{i} + 6\hat{j} - 12\hat{k}}{\sqrt{144 + 36 + 144}}$$

$$= \frac{12\hat{i} + 6\hat{j} - 12\hat{k}}{\sqrt{324}} = \frac{12\hat{i} + 6\hat{j} - 12\hat{k}}{18}$$

$$= \frac{6}{18}(2\hat{i} + \hat{j} - 2\hat{k}) = \frac{1}{3}(2\hat{i} + \hat{j} - 2\hat{k})$$

Hence, the required unit vector is $\frac{1}{3}(2\hat{i} + \hat{j} - 2\hat{k})$.

(*ii*) $2\vec{a} - \vec{b} = 2(\hat{i} + \hat{j} + 2\hat{k}) - (2\hat{i} + \hat{j} - 2\hat{k})$

$$= 2\hat{i} + 2\hat{j} + 4\hat{k} - 2\hat{i} - \hat{j} + 2\hat{k} = \hat{j} + 6\hat{k}$$

$\therefore$ Unit vector in the direction of $2\vec{a}-\vec{b}$

$$= \frac{2\vec{a}-\vec{b}}{|2\vec{a}-\vec{b}|} = \frac{\hat{j}+6\hat{k}}{\sqrt{(1)^2+(6)^2}} = \frac{\hat{j}+6\hat{k}}{\sqrt{1+36}}$$

$$= \frac{\hat{j}+6\hat{k}}{\sqrt{37}} = \frac{1}{\sqrt{37}}[\hat{j}+6\hat{k}]$$

Hence, the required unit vector is $\frac{1}{\sqrt{37}}[\hat{j}+6\hat{k}]$.

Q3. Find a unit vector in the direction of $\overrightarrow{PQ}$, where P and Q have coordinates (5, 0, 8) and (3, 3, 2) respectively.

Sol. Given coordinates are P(5, 0, 8) and Q(3, 3, 2)

$$\therefore \quad \overrightarrow{PQ} = (3-5)\hat{i}+(3-0)\hat{j}+(2-8)\hat{k} = -2\hat{i}+3\hat{j}-6\hat{k}$$

$\therefore$ Unit vector in the direction of $\overrightarrow{PQ} = \frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|}$

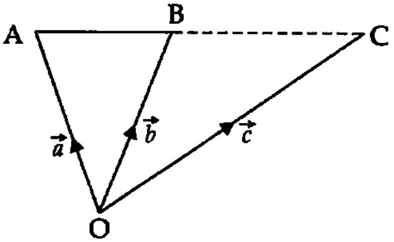
$$= \frac{-2\hat{i}+3\hat{j}-6\hat{k}}{\sqrt{(-2)^2+(3)^2+(-6)^2}} = \frac{-2\hat{i}+3\hat{j}-6\hat{k}}{\sqrt{4+9+36}} = \frac{-2\hat{i}+3\hat{j}-6\hat{k}}{\sqrt{49}}$$

$$= \frac{-2\hat{i}+3\hat{j}-6\hat{k}}{7} = \frac{1}{7}(-2\hat{i}+3\hat{j}-6\hat{k})$$

Hence, the required unit vector is $\frac{1}{7}(-2\hat{i}+3\hat{j}-6\hat{k})$.

Q4. If $\vec{a}$ and $\vec{b}$ are the position vectors of A and B respectively, find the position vector of a point C in BA produced such that BC = 1.5 BA.

Sol. Given that



$$\text{BC} = 1.5\,\text{BA}$$

$$\Rightarrow \quad \frac{\text{BC}}{\text{BA}} = 1.5 = \frac{3}{2}$$

$$\Rightarrow \quad \frac{\vec{c}-\vec{b}}{\vec{a}-\vec{b}} = \frac{3}{2}$$

$$\Rightarrow 2\vec{c}-2\vec{b} = 3\vec{a}-3\vec{b} \Rightarrow 2\vec{c} = 3\vec{a}-3\vec{b}+2\vec{b} \Rightarrow 2\vec{c} = 3\vec{a}-\vec{b}$$

$$\therefore \quad \vec{c} = \frac{3\vec{a}-\vec{b}}{2}$$

Hence, the required vector is $\vec{c} = \frac{3\vec{a}-\vec{b}}{2}$.

Q5. Using vectors, find the value of k, such that the points $(k, -10, 3)$, $(1, -1, 3)$ and $(3, 5, 3)$ are collinear.

Sol. Let the given points are $A(k,-10,3)$, $B(1,-1,3)$ and $C(3,5,3)$

$$\overrightarrow{AB} = (1-k)\hat{i} + (-1+10)\hat{j} + (3-3)\hat{k}$$

$$\overrightarrow{AB} = (1-k)\hat{i} + 9\hat{j} + 0\hat{k}$$

$$\therefore \quad |\overrightarrow{AB}| = \sqrt{(1-k)^2 + (9)^2} = \sqrt{(1-k)^2 + 81}$$

$$\overrightarrow{BC} = (3-1)\hat{i} + (5+1)\hat{j} + (3-3)\hat{k} = 2\hat{i} + 6\hat{j} + 0\hat{k}$$

$$\therefore \quad |\overrightarrow{BC}| = \sqrt{(2)^2 + (6)^2} = \sqrt{4+36} = \sqrt{40} = 2\sqrt{10}$$

$$\overrightarrow{AC} = (3-k)\hat{i} + (5+10)\hat{j} + (3-3)\hat{k} = (3-k)\hat{i} + 15\hat{j} + 0\hat{k}$$

$$\therefore \quad |\overrightarrow{AC}| = \sqrt{(3-k)^2 + (15)^2} = \sqrt{(3-k)^2 + 225}$$

If A, B and C are collinear, then

$$|\overrightarrow{AB}| + |\overrightarrow{BC}| = |\overrightarrow{AC}|$$

$$\sqrt{(1-k)^2 + 81} + \sqrt{40} = \sqrt{(3-k)^2 + 225}$$

Squaring both sides, we have

$$\left[\sqrt{(1-k)^2 + 81} + \sqrt{40}\right]^2 = \left[\sqrt{(3-k)^2 + 225}\right]^2$$

$$\Rightarrow \quad (1-k)^2 + 81 + 40 + 2\sqrt{40}\sqrt{(1-k)^2 + 81} = (3-k)^2 + 225$$

$$\Rightarrow \quad 1 + k^2 - 2k + 121 + 2\sqrt{40}\sqrt{1 + k^2 - 2k + 81} = 9 + k^2 - 6k + 225$$

$$\Rightarrow \quad 122 - 2k + 2\sqrt{40}\sqrt{k^2 - 2k + 82} = 234 - 6k$$

Dividing by 2, we get

$$\Rightarrow \quad 61 - k + \sqrt{40}\sqrt{k^2 - 2k + 82} = 117 - 3k$$

$$\Rightarrow \quad \sqrt{40}\sqrt{k^2 - 2k + 82} = 117 - 61 - 3k + k$$

$$\Rightarrow \sqrt{40}\sqrt{k^2 - 2k + 82} = 56 - 2k \Rightarrow 2\sqrt{10}\sqrt{k^2 - 2k + 82} = 56 - 2k$$

$$\Rightarrow \quad \sqrt{10}\sqrt{k^2 - 2k + 82} = 28 - k \qquad \text{(Dividing by 2)}$$

Squaring both sides, we get

$$\Rightarrow \quad 10(k^2 - 2k + 82) = 784 + k^2 - 56k$$

$$\Rightarrow \quad 10k^2 - 20k + 820 = 784 + k^2 - 56k$$

$$\Rightarrow \quad 10k^2 - k^2 - 20k + 56k + 820 - 784 = 0$$

$$\Rightarrow \quad 9k^2 + 36k + 36 = 0 \quad \Rightarrow k^2 + 4k + 4 = 0 \quad \Rightarrow (k+2)^2 = 0$$

$\Rightarrow \quad k+2=0 \quad \Rightarrow k=-2$

Hence, the required value is $k=-2$

Q6. A vector $\vec{r}$ is inclined at equal angles to the three axes. If the magnitude of $\vec{r}$ is $2\sqrt{3}$ units, find $\vec{r}$.

Sol. Since, the vector $\vec{r}$ makes equal angles with the axes, their direction cosines should be same

$$\therefore \quad l=m=n$$

We know that $\quad l^2+m^2+n^2=1 \quad \Rightarrow l^2+l^2+l^2=1$

$$\Rightarrow \quad 3l^2=1 \quad \Rightarrow l^2=\frac{1}{3} \quad \Rightarrow \quad l=\pm\frac{1}{\sqrt{3}}$$

$$\therefore \quad \hat{r} = \pm\frac{1}{\sqrt{3}}\hat{i} \pm \frac{1}{\sqrt{3}}\hat{j} \pm \frac{1}{\sqrt{3}}\hat{k} \quad \Rightarrow \hat{r} = \pm\frac{1}{\sqrt{3}}(\hat{i}+\hat{j}+\hat{k})$$

We know that $\quad \vec{r} = (\hat{r})|\vec{r}|$

$$= \pm\frac{1}{\sqrt{3}}(\hat{i}+\hat{j}+\hat{k})\,2\sqrt{3} = \pm 2(\hat{i}+\hat{j}+\hat{k})$$

Hence, the required value of $\vec{r}$ is $\pm 2(\hat{i}+\hat{j}+\hat{k})$.

Q7. A vector $\vec{r}$ has magnitude 14 and direction ratios 2, 3 and -6. Find the direction cosines and components of $\vec{r}$, given that $\vec{r}$ makes an acute angle with x-axis.

Sol. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three vectors such that $\vec{a}=2k, \vec{b}=3k$ and $\vec{c}=-6k$

If l, m and n are the direction cosines of vector $\vec{r}$, then

$$l=\frac{\vec{a}}{|\vec{r}|}=\frac{2k}{14}=\frac{k}{7}$$

$$m=\frac{\vec{b}}{|\vec{r}|}=\frac{3k}{14} \quad \text{and} \quad n=\frac{\vec{c}}{|\vec{r}|}=\frac{-6k}{14}=\frac{-3k}{7}$$

We know that $l^2+m^2+n^2=1$

$$\therefore \quad \frac{k^2}{49}+\frac{9k^2}{196}+\frac{9k^2}{49}=1$$

$$\Rightarrow \quad \frac{4k^2+9k^2+36k^2}{196}=1 \quad \Rightarrow 49k^2=196 \quad \Rightarrow k^2=4$$

$$\therefore \quad k=\pm 2 \quad \text{and} \quad l=\frac{k}{7}=\frac{2}{7}$$

$$m=\frac{3k}{14}=\frac{3\times 2}{14}=\frac{3}{7} \quad \text{and} \quad n=\frac{-3k}{7}\frac{-3\times 2}{7}=\frac{-6}{7}$$

$$\therefore \quad \hat{r} = \pm\left(\frac{2}{7}\hat{i}+\frac{3}{7}\hat{j}-\frac{6}{7}\hat{k}\right)$$

$$\hat{r} = \hat{r}|\vec{r}|$$

$$\Rightarrow \quad \vec{r} = \pm\left(\frac{2}{7}\hat{i}+\frac{3}{7}\hat{j}-\frac{6}{7}\hat{k}\right)\cdot 14 = \pm(4\hat{i}+6\hat{j}-12\hat{k})$$

Hence, the required direction cosines are $\frac{2}{7}, \frac{3}{7}, \frac{-6}{7}$ and the components of $\vec{r}$ are $4\hat{i}, 6\hat{j}$ and $-12\hat{k}$.

Q8. Find a vector of magnitude 6, which is perpendicular to both the vectors $2\hat{i}-\hat{j}+2\hat{k}$ and $4\hat{i}-\hat{j}+3\hat{k}$.

Sol. Let $\vec{a} = 2\hat{i}-\hat{j}+2\hat{k}$ and $\vec{b} = 4\hat{i}-\hat{j}+3\hat{k}$

We know that unit vector perpendicular to $\vec{a}$ and $\vec{b} = \frac{(\vec{a}\times\vec{b})}{|\vec{a}\times\vec{b}|}$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 2 \\ 4 & -1 & 3 \end{vmatrix}$$

$$= \hat{i}(-3+2)-\hat{j}(6-8)+\hat{k}(-2+4) = -\hat{i}+2\hat{j}+2\hat{k}$$

$$\therefore \quad |\vec{a}\times\vec{b}| = \sqrt{(-1)^2+(2)^2+(2)^2} = \sqrt{1+4+4} = \sqrt{9} = 3$$

$$\text{so,} \quad \frac{(\vec{a}\times\vec{b})}{|\vec{a}\times\vec{b}|} = \frac{-\hat{i}+2\hat{j}+2\hat{k}}{3} = \frac{1}{3}(-\hat{i}+2\hat{j}+2\hat{k})$$

Now the vector of magnitude $6 = \frac{1}{3}(-\hat{i}+2\hat{j}+2\hat{k})\cdot 6$

$$= 2(-\hat{i}+2\hat{j}+2\hat{k}) = -2\hat{i}+4\hat{j}+4\hat{k}$$

Hence, the required vector is $-2\hat{i}+4\hat{j}+4\hat{k}$.

Q9. Find the angle between the vectors $2\hat{i}-\hat{j}+\hat{k}$ and $3\hat{i}+4\hat{j}-\hat{k}$.

Sol. Let $\vec{a} = 2\hat{i}-\hat{j}+\hat{k}$ and $\vec{b} = 3\hat{i}+4\hat{j}-\hat{k}$

and let θ be the angle between $\vec{a}$ and $\vec{b}$.

$$\therefore \quad \cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{(2\hat{i}-\hat{j}+\hat{k})\cdot(3\hat{i}+4\hat{j}-\hat{k})}{\sqrt{4+1+1}\cdot\sqrt{9+16+1}}$$

$$= \frac{6-4-1}{\sqrt{6}\cdot\sqrt{26}} \Rightarrow \frac{1}{2\sqrt{3}\cdot\sqrt{13}} = \frac{1}{2\sqrt{39}}$$

$$\therefore \quad \theta = \cos^{-1}\frac{1}{2\sqrt{39}} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{156}\right)$$

Hence, the required value of θ is $\cos^{-1}\left(\frac{1}{156}\right)$.

Q10. If $\vec{a}+\vec{b}+\vec{c} = 0$ show that $\vec{a}\times\vec{b} = \vec{b}\times\vec{c} = \vec{c}\times\vec{a}$. Interpret the result geometrically.

Sol. Given that $\vec{a}+\vec{b}+\vec{c}=0$

So, $\vec{a}\times(\vec{a}+\vec{b}+\vec{c}) = \vec{a}\times 0$

$\Rightarrow \vec{a}\times\vec{a}+\vec{a}\times\vec{b}+\vec{a}\times\vec{c}=0$

$\Rightarrow \vec{0}+\vec{a}\times\vec{b}+\vec{a}\times\vec{c}=0$ $\quad(\vec{a}\times\vec{a}=0)$

$\Rightarrow \vec{a}\times\vec{b}-\vec{c}\times\vec{a}=0$ $\quad(\vec{a}\times\vec{c}=-\vec{c}\times\vec{a})$

$\Rightarrow \vec{a}\times\vec{b}=\vec{c}\times\vec{a}$ ...(*i*)

Now $\vec{a}+\vec{b}+\vec{c}=0$

$\Rightarrow \vec{b}\times(\vec{a}+\vec{b}+\vec{c})=\vec{b}\times 0$

$\Rightarrow \vec{b}\times\vec{a}+\vec{b}\times\vec{b}+\vec{b}\times\vec{c}=0$

$\Rightarrow \vec{b}\times\vec{a}+\vec{0}+\vec{b}\times\vec{c}=0$ $\quad(\because\ \vec{b}\times\vec{b}=0)$

$\Rightarrow -(\vec{a}\times b)+\vec{b}\times\vec{c}=0$

$\therefore \vec{b}\times\vec{c}=\vec{a}\times\vec{b}$ ...(*ii*)

From eq. (*i*) and (*ii*) we get

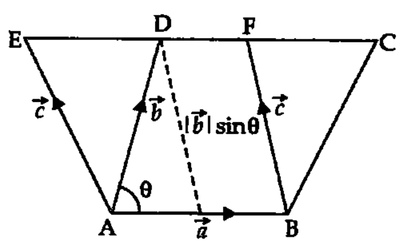
$\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}$. Hence proved.

Geometrical Interpretation

According to figure, we have

Area of parallelogram ABCD is

$\Rightarrow |\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta$



Since, the parallelograms on the same base and between the same parallel lines are equal in area

$\therefore |\vec{a}\times\vec{b}|=|\vec{b}\times\vec{c}|=|\vec{c}\times\vec{a}|$

$\Rightarrow \vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}.$

Q11. Find the sine of the angle between the vectors $\vec{a}=3\hat{i}+\hat{j}+2\hat{k}$ and $\vec{b}=2\hat{i}-2\hat{j}+4\hat{k}$.

Sol. Given that $\vec{a}=3\hat{i}+\hat{j}+2\hat{k}$ and $\vec{b}=2\hat{i}-2\hat{j}+4\hat{k}$

We know that $|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta$

$$\therefore \quad \vec{a}\times\vec{b} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & 1 & 2\\ 2 & -2 & 4\end{vmatrix}$$

$$= \hat{i}(4+4)-\hat{j}(12-4)+\hat{k}(-6-2)$$

$$= 8\hat{i}-8\hat{j}-8\hat{k}$$

$$|\vec{a}\times\vec{b}| = \sqrt{(8)^2+(-8)^2+(-8)^2}$$

$$= \sqrt{64+64+64} = \sqrt{192} = \sqrt{64 \times 3} = 8\sqrt{3}$$

$$|\vec{a}| = \sqrt{(3)^2 + (1)^2 + (2)^2} = \sqrt{9+1+4} = \sqrt{14}$$

$$|\vec{b}| = \sqrt{(2)^2 + (-2)^2 + (4)^2} = \sqrt{4+4+16}$$

$$= \sqrt{24} = 2\sqrt{6}$$

$$\therefore \qquad \sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|} = \frac{8\sqrt{3}}{\sqrt{14} \cdot 2\sqrt{6}}$$

$$\Rightarrow \qquad = \frac{4\sqrt{3}}{\sqrt{84}} = \frac{4\sqrt{3}}{2\sqrt{21}} = \frac{2}{\sqrt{7}}$$

Hence, $\sin\theta = \frac{2}{\sqrt{7}}$.

Q12. If A, B, C, D are the points with position vectors $\hat{i} + \hat{j} - \hat{k}$, $2\hat{i} - \hat{j} + 3\hat{k}$, $2\hat{i} - 3\hat{k}$, $3\hat{i} - 2\hat{j} + \hat{k}$, respectively, find the projection of $\overline{AB}$ along $\overline{CD}$.

Sol. Here, Position vector of A = $\hat{i} + \hat{j} - \hat{k}$

Position vector of B = $2\hat{i} - \hat{j} + 3\hat{k}$

Position vector of C = $2\hat{i} - 3\hat{k}$

Position vector of D = $3\hat{i} - 2\hat{j} + \hat{k}$

$$\overrightarrow{AB} = \text{P.V of B} - \text{P.V of A}$$

$$= (2\hat{i} - \hat{j} + 3\hat{k}) - (\hat{i} + \hat{j} - \hat{k}) = \hat{i} - 2\hat{j} + 4\hat{k}$$

$$\overrightarrow{CD} = \text{P.V. of D} - \text{P.V. of C}$$

$$= (3\hat{i} - 2\hat{j} + \hat{k}) - (2\hat{i} - 3\hat{k}) = \hat{i} - 2\hat{j} + 4\hat{k}$$

$$\text{Projection of } \overrightarrow{AB} \text{ on } \overrightarrow{CD} = \frac{\overrightarrow{AB} \cdot \overrightarrow{CD}}{|\overrightarrow{CD}|}$$

$$= \frac{(\hat{i} - 2\hat{j} + 4\hat{k}) \cdot (\hat{i} - 2\hat{j} + 4\hat{k})}{\sqrt{(1)^2 + (-2)^2 + (4)^2}}$$

$$= \frac{1+4+16}{\sqrt{1+4+16}} = \frac{21}{\sqrt{21}} = \sqrt{21}$$

Hence, the required projection = $\sqrt{21}$.

Q13. Using vectors, find the area of the triangle ABC with vertices A(1, 2, 3), B(2, –1, 4) and C(4, 5, –1).

Sol. Given that A(1, 2, 3), B(2, –1, 4) and C(4, 5, –1)

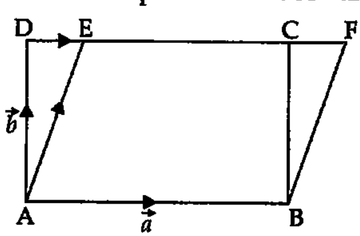
$$\overrightarrow{AB} = (2-1)\hat{i} + (-1-2)\hat{j} + (4-3)\hat{k}$$
$$\overrightarrow{AB} = \hat{i} - 3\hat{j} + \hat{k}$$
$$\overrightarrow{AC} = (4-1)\hat{i} + (5-2)\hat{j} + (-1-3)\hat{k} = 3\hat{i} + 3\hat{j} - 4\hat{k}$$

$$\text{Area of } \Delta ABC = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 1 \\ 3 & 3 & -4 \end{vmatrix}$$
$$= \frac{1}{2}[\hat{i}(12-3) - \hat{j}(-4-3) + \hat{k}(3+9)]$$
$$= \frac{1}{2}\left|9\hat{i} + 7\hat{j} + 12\hat{k}\right| = \frac{1}{2}\sqrt{(9)^2 + (7)^2 + (12)^2}$$
$$= \frac{1}{2}\sqrt{81 + 49 + 144} = \frac{1}{2}\sqrt{274}$$

Hence, the required area is $\frac{\sqrt{274}}{2}$.

Q14. Using vectors, prove that the parallelogram on the same base and between the same parallels, are equal in area.

Sol. Let ABCD and ABFE be two parallelograms on the same base AB and between same parallel lines AB and DF.



Let $\quad \overrightarrow{AB} = \vec{a}$ and $\overrightarrow{AD} = \vec{b}$

$\therefore$ Area of parallelogram ABCD $= |\vec{a} \times \vec{b}|$

Now Area of parallelogram ABFE $= \left|\overrightarrow{AB} \times \overrightarrow{AE}\right|$

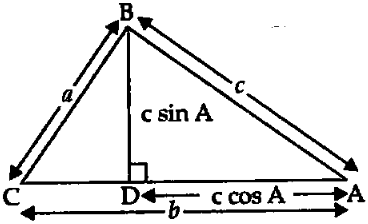
$$= \left|\vec{a} \times (\overrightarrow{AD} + \overrightarrow{DE})\right| = \left|\vec{a} \times (\vec{b} \times K\vec{a})\right|$$
$$= \left|(\vec{a} \times \vec{b}) + K(\vec{a} \times \vec{a})\right| = |\vec{a} \times \vec{b}| + 0 \qquad [\because \vec{a} \times \vec{a} = 0]$$
$$= |\vec{a} \times \vec{b}|$$

Hence proved.

LONG ANSWER TYPE QUESTIONS

Q15. Prove that in any triangle ABC, $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$, where a, b, c are the magnitudes of the sides opposite to the vertices

A, B, C respectively.



Sol. Here, in the given figure, the components of c are c cos A and c sin A.

$\therefore \quad \overrightarrow{CD} = b - c\cos A$

In ΔBDC,

$$a^2 = CD^2 + BD^2$$

$$\Rightarrow \quad a^2 = (b - c\cos A)^2 + (c\sin A)^2$$

$$\Rightarrow \quad a^2 = b^2 + c^2\cos^2 A - 2bc\cos A + c^2\sin^2 A$$

$$\Rightarrow \quad a^2 = b^2 + c^2(\cos^2 A + \sin^2 A) - 2bc\cos A$$

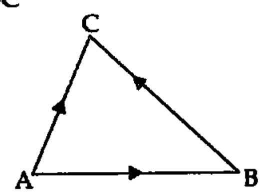
$$\Rightarrow \quad a^2 = b^2 + c^2 - 2bc\cos A \quad \Rightarrow 2bc\cos A = b^2 + c^2 - a^2$$

$$\therefore \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Hence Proved.

Q16. If $\vec{a}, \vec{b}$ and $\vec{c}$ determine the vertices of a triangle, show that $\frac{1}{2}[\vec{b}\times\vec{c} + \vec{c}\times\vec{a} + \vec{a}\times\vec{b}]$ gives the vector area of the triangle. Hence deduce the condition that the three points $\vec{a}, \vec{b}$ and $\vec{c}$ are collinear. Also, find the unit vector normal to the plane of the triangle.

Sol. Since, $\vec{a}, \vec{b}$ and $\vec{c}$ are the vertices of ΔABC



$\therefore \quad \overrightarrow{AB} = \vec{b} - \vec{a}, \overrightarrow{BC} = \vec{c} - \vec{b}$

and $\quad \overrightarrow{AC} = \vec{c} - \vec{a}$

$\therefore \quad$ Area of $\Delta ABC = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$

$$= \frac{1}{2}\left|(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})\right|$$

$$= \frac{1}{2}\left|\vec{b}\times\vec{c} - \vec{b}\times\vec{a} - \vec{a}\times\vec{c} + \vec{a}\times\vec{a}\right|$$

$$= \frac{1}{2}\left|\vec{b}\times\vec{c} + \vec{a}\times\vec{b} + \vec{c}\times\vec{a}\right| \quad \left[\begin{array}{l} \because \ \vec{a}\times\vec{b} = -\vec{b}\times\vec{a} \\ \vec{c}\times\vec{a} = -\vec{a}\times\vec{c} \\ \vec{a}\times\vec{a} = \vec{0} \end{array}\right]$$

For three vectors are collinear, area of $\Delta ABC = 0$

$$\therefore \quad \frac{1}{2}\left|\vec{b}\times\vec{c} + \vec{a}\times\vec{b} + \vec{c}\times\vec{a}\right| = 0$$

$$\left|\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a}\right| = 0$$

which is the condition of collinearity of $\vec{a}, \vec{b}$ and $\vec{c}$.

Let $\hat{n}$ be the unit vector normal to the plane of the ΔABC

$$\therefore \quad \hat{n} = \frac{\overrightarrow{AB} \times \overrightarrow{AC}}{|\overrightarrow{AB} \times \overrightarrow{AC}|}$$

$$\Rightarrow \quad = \frac{\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}}{|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|}$$

Q17. Show that area of the parallelogram whose diagonals are given by $\vec{a}$ and $\vec{b}$ is $\frac{|\vec{a} \times \vec{b}|}{2}$. Also, find the area of the parallelogram, whose diagonals are $2\hat{i} - \hat{j} + \hat{k}$ and $\hat{i} + 3\hat{j} - \hat{k}$.

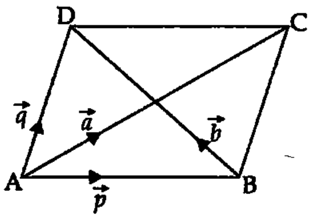
Sol. Let ABCD be a parallelogram such that,

$$\overrightarrow{AB} = \vec{p}, \overrightarrow{AD} = \vec{q} = \overrightarrow{BC}$$

$\therefore$ by law of triangle, we get

$$\overrightarrow{AC} = \vec{a} = \vec{p} + \vec{q} \quad \text{...}(i)$$

and $$\overrightarrow{BD} = \vec{b} = -\vec{p} + \vec{q} \quad \text{...}(ii)$$



Adding eq. (i) and (ii) we get,

$$\vec{a} + \vec{b} = 2\vec{q} \quad \Rightarrow \quad \vec{q} = \left(\frac{\vec{a} + \vec{b}}{2}\right)$$

Subtracting eq. (*ii*) from eq. (*i*) we get

$$\vec{a} - \vec{b} = 2\vec{p} \quad \Rightarrow \quad \vec{p} = \left(\frac{\vec{a} - \vec{b}}{2}\right)$$

$$\therefore \vec{p} \times \vec{q} = \frac{1}{4}(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) = \frac{1}{4}(\vec{a} \times \vec{a} - \vec{a} \times \vec{b} + \vec{b} \times \vec{a} - \vec{b} \times \vec{b})$$

$$= \frac{1}{4}(-\vec{a} \times \vec{b} + \vec{b} \times \vec{a}) \qquad \left[\because \begin{array}{l} \vec{a} \times \vec{a} = 0 \\ \vec{b} \times \vec{b} = 0 \end{array}\right]$$

$$= \frac{1}{4}(\vec{a} \times \vec{b} + \vec{a} \times \vec{b}) = \frac{1}{4} \cdot 2(\vec{a} \times \vec{b}) = \frac{|\vec{a} \times \vec{b}|}{2}$$

So, the area of the parallelogram ABCD $= |\vec{p} \times \vec{q}| = \frac{1}{2}|\vec{a} \times \vec{b}|$

Now area of parallelogram whose diagonals are $2\hat{i} - \hat{j} + \hat{k}$ and $\hat{i} + 3\hat{j} - \hat{k} = \frac{1}{2}|(2\hat{i} - \hat{j} + \hat{k}) \times (\hat{i} + 3\hat{j} - \hat{k})|$

$$= -\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 1 \\ 1 & 3 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left|\hat{i}(1-3) - \hat{j}(-2-1) + \hat{k}(6+1)\right| = \frac{1}{2}\left|-2\hat{i} + 3\hat{j} + 7\hat{k}\right|$$

$$= \frac{1}{2}\sqrt{(-2)^2 + (3)^2 + (7)^2} = \frac{1}{2}\sqrt{4+9+49}$$

$$= \frac{1}{2}\sqrt{62} \text{ sq. units}$$

Hence, the required area is $\frac{1}{2}\sqrt{62}$ sq. units.

Q18. If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{j} - \hat{k}$, find a vector $\vec{c}$ such that $\vec{a} \times \vec{c} = \vec{b}$ and $\vec{a} \cdot \vec{c} = 3$.

Sol. Let $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

Also given that $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{j} - \hat{k}$

Since, $\vec{a} \times \vec{c} = \vec{b}$

$$\therefore \quad \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ c_1 & c_2 & c_3 \end{vmatrix} = \hat{j} - \hat{k}$$

$$= \hat{i}(c_3 - c_2) - \hat{j}(c_3 - c_1) + \hat{k}(c_2 - c_1) = \hat{j} - \hat{k}$$

On comparing the like terms, we get

$$c_3 - c_2 = 0 \quad \text{...}(i)$$

$$c_1 - c_3 = 1 \quad \text{...}(ii)$$

and $$c_2 - c_1 = -1 \quad \text{...}(iii)$$

Now for $\vec{a} \cdot \vec{c} = 3$

$$(\hat{i} + \hat{j} + \hat{k}) \cdot (c_1\hat{i} + c_2\hat{j} + c_3\hat{k}) = 3$$

$$\therefore \quad c_1 + c_2 + c_3 = 3 \quad \text{...}(iv)$$

Adding eq. (*ii*) and eq. (*iii*) we get,

$$c_2 - c_3 = 0 \quad \text{...}(v)$$

From (*iv*) and (*v*) we get

$$c_1 + 2c_2 = 3 \quad \text{...}(vi)$$

From (*iii*) and (*vi*) we get

$$\begin{array}{r} c_1 + 2c_2 = 3 \\ -c_1 + c_2 = -1 \\ \hline \text{Adding} \quad 3c_2 = 2 \end{array}$$

$$\therefore \quad c_2 = \frac{2}{3}$$

$$c_3 - c_2 = 0 \Rightarrow c_3 - \frac{2}{3} = 0$$

$$\therefore \qquad c_3 = \frac{2}{3}$$

Now $\qquad c_2 - c_1 = -1 \Rightarrow \frac{2}{3} - c_1 = -1$

$$\Rightarrow \qquad c_1 = 1 + \frac{2}{3} = \frac{5}{3}$$

$$\therefore \qquad \vec{c} = \frac{5}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$$

Hence, $\qquad \vec{c} = \frac{1}{3}(5\hat{i} + 2\hat{j} + 2\hat{k})$.

OBJECTIVE TYPE QUESTIONS

Choose the correct answer from the given four options in each of the Exercises from 19 to 33 (MCQ).

Q19. The vector in the direction of the vector $\hat{i} - 2\hat{j} + 2\hat{k}$ that has magnitude 9 is

(*a*) $\hat{i} - 2\hat{j} + 2\hat{k}$ (*b*) $\frac{\hat{i} - 2\hat{j} + 2\hat{k}}{3}$

(*c*) $3(\hat{i} - 2\hat{j} + 2\hat{k})$ (*d*) $9(\hat{i} - 2\hat{j} + 2\hat{k})$

Sol. Let $\vec{a} = \hat{i} - 2\hat{j} + 2\hat{k}$

Unit vector in the direction of $\vec{a} = \frac{\vec{a}}{|\vec{a}|}$

$$= \frac{\hat{i} - 2\hat{j} + 2\hat{k}}{\sqrt{(1)^2 + (-2)^2 + (2)^2}} = \frac{\hat{i} - 2\hat{j} + 2\hat{k}}{\sqrt{1+4+4}} = \frac{\hat{i} - 2\hat{j} + 2\hat{k}}{3}$$

$\therefore$ Vector of magnitude $9 = \frac{9(\hat{i} - 2\hat{j} + 2\hat{k})}{3} = 3(\hat{i} - 2\hat{j} + 2\hat{k})$

Hence, the correct option is (*c*).

Q20. The position vector of the point which divides the join of points $2\vec{a} - 3\vec{b}$ and $\vec{a} + \vec{b}$ in the ratio $3:1$ is

(*a*) $\frac{3\vec{a} - 2\vec{b}}{2}$ (*b*) $\frac{7\vec{a} - 8\vec{b}}{4}$ (*c*) $\frac{3\vec{a}}{4}$ (*d*) $\frac{5\vec{a}}{4}$

Sol. The given vectors are $2\vec{a} - 3\vec{b}$ and $\vec{a} + \vec{b}$ and the ratio is $3:1$.

$\therefore$ The position vector of the required point c which divides the join of the given vectors $\vec{a}$ and $\vec{b}$ is

$$\vec{c} = \frac{m_1x_2 + m_2x_1}{m_1 + m_2}$$

$$= \frac{1\cdot(2\vec{a}-3\vec{b})+3(\vec{a}+\vec{b})}{3+1} = \frac{2\vec{a}-3\vec{b}+3\vec{a}+3\vec{b}}{4}$$

$$= \frac{5\vec{a}}{4} = \frac{5}{4}\vec{a}$$

Hence, the correct option is (*d*).

Q21. The vector having initial and terminal points as (2, 5, 0) and (–3, 7, 4), respectively is

(*a*) $-\hat{i}+12\hat{j}+4\hat{k}$ (*b*) $5\hat{i}+2\hat{j}-4\hat{k}$

(*c*) $-5\hat{i}+2\hat{j}+4\hat{k}$ (*d*) $\hat{i}+\hat{j}+\hat{k}$

Sol. Let A and B be two points whose coordinates are given as (2, 5, 0) and (–3, 7, 4)

$$\therefore \quad \overrightarrow{AB} = (-3-2)\hat{i}+(7-5)\hat{j}+(4-0)\hat{k}$$

$$\Rightarrow \quad \overrightarrow{AB} = -5\hat{i}+2\hat{j}+4\hat{k}$$

Hence, the correct option is (*c*).

Q22. The angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitudes $\sqrt{3}$ and 4 respectively and $\vec{a}\cdot\vec{b}=2\sqrt{3}$ is

(*a*) $\frac{\pi}{6}$ (*b*) $\frac{\pi}{3}$ (*c*) $\frac{\pi}{2}$ (*d*) $\frac{5\pi}{2}$

Sol. Here, given that $|\vec{a}|=\sqrt{3}, |\vec{b}|=4$ and $\vec{a}\cdot\vec{b}=2\sqrt{3}$

$\therefore$ From scalar product, we know that

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$\Rightarrow \quad 2\sqrt{3} = \sqrt{3}\cdot 4\cdot\cos\theta$$

$$\Rightarrow \quad \cos\theta = \frac{2\sqrt{3}}{\sqrt{3}\cdot 4} = \frac{1}{2}$$

$$\therefore \quad \theta = \frac{\pi}{3}$$

Hence, the correct option is (*b*).

Q23. Find the value of λ such that the vectors $\vec{a}=2\hat{i}+\lambda\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}+3\hat{k}$ are orthogonal

(*a*) 0 (*b*) 1 (*c*) $\frac{3}{2}$ (*d*) $\frac{-5}{2}$

Sol. Given that $\vec{a}=2\hat{i}+\lambda\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}+3\hat{k}$

Since $\vec{a}$ and $\vec{b}$ are orthogonal

$$\therefore \quad \vec{a}\cdot\vec{b} = 0$$

$$\Rightarrow \quad (2\hat{i}+\lambda\hat{j}+\hat{k})\cdot(\hat{i}+2\hat{j}+3\hat{k}) = 0$$

$$\Rightarrow \quad 2 + 2\lambda + 3 = 0$$

$$\Rightarrow \quad 5 + 2\lambda = 0 \Rightarrow \lambda = \frac{-5}{2}$$

Hence, the correct option is (*d*).

Q24. The value of λ for which the vectors $3\hat{i} - 6\hat{j} + \hat{k}$ and $2\hat{i} - 4\hat{j} + \lambda\hat{k}$ are parallel is

(*a*) $\frac{2}{3}$ (*b*) $\frac{3}{2}$ (*c*) $\frac{5}{2}$ (*d*) $\frac{2}{5}$

Sol. Let

$$\vec{a} = 3\hat{i} - 6\hat{j} + \hat{k}$$

$$\vec{b} = 2\hat{i} - 4\hat{j} + \lambda\hat{k}$$

Since the given vectors are parallel,

$\therefore$ Angle between them is 0°

so

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos 0$$

$$\Rightarrow (3\hat{i} - 6\hat{j} + \hat{k}) \cdot (2\hat{i} - 4\hat{j} + \lambda\hat{k}) = |3\hat{i} - 6\hat{j} + \hat{k}||2\hat{i} - 4\hat{j} + \lambda\hat{k}|$$

$$6 + 24 + \lambda = \sqrt{9 + 36 + 1} \cdot \sqrt{4 + 16 + \lambda^2}$$

$$30 + \lambda = \sqrt{46} \cdot \sqrt{20 + \lambda^2}$$

Squaring both sides, we get

$$900 + \lambda^2 + 60\lambda = 46(20 + \lambda^2)$$

$$\Rightarrow \quad 900 + \lambda^2 + 60\lambda = 920 + 46\lambda^2$$

$$\Rightarrow \quad \lambda^2 - 46\lambda^2 + 60\lambda + 900 - 920 = 0$$

$$\Rightarrow \quad -45\lambda^2 + 60\lambda - 20 = 0$$

$$\Rightarrow \quad 9\lambda^2 - 12\lambda + 4 = 0$$

$$\Rightarrow \quad (3\lambda - 2)^2 = 0$$

$$\Rightarrow \quad 3\lambda - 2 = 0$$

$$\Rightarrow \quad 3\lambda = 2$$

$$\therefore \quad \lambda = 2/3$$

Alternate method:

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

If $\vec{a} \parallel \vec{b}$

$$\therefore \quad \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$$

$$\Rightarrow \quad \frac{3}{2} = \frac{-6}{-4} = \frac{1}{\lambda} \Rightarrow \frac{1}{\lambda} = \frac{3}{2} \Rightarrow \lambda = \frac{2}{3}$$

Hence, the correct option is (*a*).

Q25. The vectors from origin to the points A and B are

$\vec{a} = 2\hat{i} - 3\hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} + 3\hat{j} + \hat{k}$ respectively, then the area of ΔOAB is equal to

(*a*) 340 (*b*) $\sqrt{25}$ (*c*) $\sqrt{229}$ (*d*) $\frac{1}{2}\sqrt{229}$

Sol. Let O be the origin

$$\therefore \quad \overrightarrow{OA} = 2\hat{i} - 3\hat{j} + 2\hat{k}$$

$$\text{and} \quad \overrightarrow{OB} = 2\hat{i} + 3\hat{j} + \hat{k}$$

$$\therefore \quad \text{Area of } \Delta OAB = \frac{1}{2}\left|\overrightarrow{OA} \times \overrightarrow{OB}\right| = \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -3 & 2 \\ 2 & 3 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left|\hat{i}(-3-6) - \hat{j}(2-4) + \hat{k}(6+6)\right|$$

$$= \frac{1}{2}\left|-9\hat{i} + 2\hat{j} + 12\hat{k}\right|$$

$$= \frac{1}{2}\sqrt{(-9)^2 + (2)^2 + (12)^2}$$

$$= \frac{1}{2}\sqrt{81 + 4 + 144} = \frac{1}{2}\sqrt{229}$$

Hence the correct option is (*d*).

Q26. For any vector $\vec{a}$, the value of $(\vec{a} \times \hat{i})^2 + (\vec{a} \times \hat{j})^2 + (\vec{a} \times \hat{k})^2$ is

(*a*) $\vec{a}^2$ (*b*) $3\vec{a}^2$ (*c*) $4\vec{a}^2$ (*d*) $2\vec{a}^2$

Sol. Let $\quad \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$$\therefore \quad \vec{a}^2 = a_1^2 + a_2^2 + a_3^2$$

$$\text{Now,} \quad \vec{a} \times \hat{i} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \times \hat{i}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(0-a_3) + \hat{k}(0-a_2) = a_3\hat{j} - a_2\hat{k}$$

$$\therefore \quad (\vec{a} \times \hat{i})^2 = (a_3\hat{j} - a_2\hat{k}) \cdot (a_3\hat{j} - a_2\hat{k}) = a_3^2 + a_2^2$$

$$\text{Similarly} \quad (\vec{a} \times \hat{j})^2 = a_1^2 + a_3^2$$

$$\text{and} \quad (\vec{a} \times \hat{k})^2 = a_1^2 + a_2^2$$

$$\therefore (\vec{a} \times \hat{i})^2 + (\vec{a} \times \hat{j})^2 + (\vec{a} \times \hat{k})^2 = a_3^2 + a_2^2 + a_1^2 + a_3^2 + a_1^2 + a_2^2$$

$$= 2(a_1^2 + a_2^2 + a_3^2) = 2\vec{a}^2$$

Hence, the correct option is (*d*).

Q27. If $|\vec{a}|=10, |\vec{b}|=2$ and $\vec{a}\cdot\vec{b}=12$, then value of $|\vec{a}\times\vec{b}|$ is

(*a*) 5 (*b*) 10 (*c*) 14 (*d*) 16

Sol. Given that $|\vec{a}|=10, |\vec{b}|=2$ and $\vec{a}\cdot\vec{b}=12$

$$\therefore \quad \vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$\Rightarrow \quad 12 = 10\cdot 2\cdot\cos\theta$$

$$\Rightarrow \quad \cos\theta = \frac{12}{20} = \frac{3}{5}$$

$$\therefore \quad \sin\theta = \sqrt{1-\cos^2\theta}$$

$$\Rightarrow \quad \sin\theta = \sqrt{1-\left(\frac{3}{5}\right)^2} \quad \Rightarrow \quad \sin\theta = \sqrt{1-\frac{9}{25}}$$

$$\Rightarrow \quad \sin\theta = \sqrt{\frac{16}{25}} \quad \Rightarrow \quad \sin\theta = \frac{4}{5}$$

Now $$|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta$$

$$= 10\cdot 2\cdot\frac{4}{5} = 16$$

Hence, the correct option is (*d*).

Q28. The vectors $\lambda\hat{i}+\hat{j}+2\hat{k}, \hat{i}+\lambda\hat{j}-\hat{k}$ and $2\hat{i}-\hat{j}+\lambda\hat{k}$ are coplanar if

(*a*) $\lambda=-2$ (*b*) $\lambda=0$ (*c*) $\lambda=1$ (*d*) $\lambda=-1$

Sol. Let
$$\vec{a} = \lambda\hat{i}+\hat{j}+2\hat{k}$$
$$\vec{b} = \hat{i}+\lambda\hat{j}-\hat{k}$$
$$\vec{c} = 2\hat{i}-\hat{j}+\lambda\hat{k}$$

If $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar, then

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = 0$$

$$\therefore \quad \begin{vmatrix} \lambda & 1 & 2 \\ 1 & \lambda & -1 \\ 2 & -1 & \lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda(\lambda^2-1)-1(\lambda+2)+2(-1-2\lambda) = 0$$

$$\Rightarrow \quad \lambda^3-\lambda-\lambda-2-2-4\lambda = 0$$

$$\Rightarrow \quad \lambda^3-6\lambda-4 = 0$$

$$\Rightarrow \quad (\lambda+2)(\lambda^2-2\lambda-2) = 0$$

$$\Rightarrow \quad \lambda=-2 \quad \text{or} \quad \lambda^2-2\lambda-2 = 0$$

$$\Rightarrow \qquad \lambda = \frac{2 \pm \sqrt{4+8}}{2}$$

$$\Rightarrow \qquad \lambda = \frac{2 \pm 2\sqrt{3}}{2}$$

$$\therefore \qquad \lambda = -2 \quad \text{or} \quad \lambda = 1 \pm \sqrt{3}$$

Hence, the correct option is (*a*).

Q29. If $\vec{a}, \vec{b}, \vec{c}$ are unit vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, then the value of $\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}$ is

(*a*) 1 (*b*) 3 (*c*) $-\frac{3}{2}$ (*d*) None of these

Sol. Given that $|\vec{a}| = |\vec{b}| = |\vec{c}| = 1$

and $\vec{a} + \vec{b} + \vec{c} = \vec{0}$

$$\therefore \qquad (\vec{a} + \vec{b} + \vec{c}) \cdot (\vec{a} + \vec{b} + \vec{c}) = \vec{0} \cdot \vec{0} = 0$$

$$|\vec{a}|^2 + \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{a} + |\vec{b}|^2 + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} + \vec{c} \cdot \vec{b} + |\vec{c}|^2 = 0$$

$$\Rightarrow \qquad |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2\vec{a} \cdot \vec{b} + 2\vec{b} \cdot \vec{c} + 2\vec{c} \cdot \vec{a} = 0$$

$$\Rightarrow \qquad 1 + 1 + 1 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot a) = 0$$

$$\Rightarrow \qquad 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = -3$$

$$\therefore \qquad \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} = \frac{-3}{2}$$

Hence, the correct option is (*c*).

Q30. The projection vector of $\vec{a}$ on $\vec{b}$ is :

(*a*) $\left(\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}\right)\vec{b}$ (*b*) $\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$ (*c*) $\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$ (*d*) $\left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2}\right)\vec{b}$

Sol. The projection vector of $\vec{a}$ on $\vec{b} = \left(\frac{\vec{a} . \vec{b}}{|\vec{b}|}\right) \cdot \vec{b}$

Hence, the correct option is (*a*).

Q31. If $\vec{a}, \vec{b}, \vec{c}$ are three vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ and $|\vec{a}| = 2$, $|\vec{b}| = 3, |\vec{c}| = 5$, then the value of $\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}$ is

(*a*) 0 (*b*) 1 (*c*) -19 (*d*) 38

Sol. Given that $|\vec{a}| = 2$, $|\vec{b}| = 3, |\vec{c}| = 5$,

and $\qquad \vec{a} + \vec{b} + \vec{c} = \vec{0}$

$$(\vec{a} + \vec{b} + \vec{c}) \cdot (\vec{a} + \vec{b} + \vec{c}) = \vec{0} \cdot \vec{0} = 0$$

$$\Rightarrow |\vec{a}|^2 + \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{a} + |\vec{b}|^2 + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} + \vec{c} \cdot \vec{b} + |\vec{c}|^2 = 0$$

$$\Rightarrow \qquad |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2\vec{a} \cdot \vec{b} + 2\vec{b} \cdot \vec{c} + 2\vec{c} \cdot \vec{a} = 0$$

$$\Rightarrow \quad (2)^2 + (3)^2 + (5)^2 + 2(\vec{a}.\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}) = 0$$
$$\Rightarrow \quad 4 + 9 + 25 + 2\ (\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}) = 0$$
$$\Rightarrow \quad 38 + 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}) = 0$$
$$\Rightarrow \quad 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}) = -38$$
$$\therefore \quad \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a} = -19$$

Hence, the correct option is (*c*).

Q32. If $|\vec{a}| = 4$ and $-3 \le \lambda \le 2$, then the range of $|\lambda\vec{a}|$ is

(*a*) [0, 8] (*b*) [– 12, 8] (*c*) [0, 12] (*d*) [8, 12]

Sol. Given that $|\vec{a}| = 4, -3 \le \lambda \le 2$

Now $|\lambda\vec{a}| = \lambda|\vec{a}| = \lambda \cdot 4 = 4\lambda$

Here $\quad -3 \le \lambda \le 2$

$\Rightarrow \quad -3.4 \le 4\lambda \le 2.4 \quad \Rightarrow \quad -12 \le 4\lambda \le 8$

$\therefore \quad 4\lambda = [-12, 8]$

Hence, the correct option is (*b*).

Q33. The number of vectors of unit length perpendicular to the vectors $\vec{a} = 2\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = \hat{j} + \hat{k}$ is

(*a*) one (*b*) two (*c*) three (*d*) infinite

Sol. The number of vectors of unit length perpendicular to vectors $\vec{a}$ and $\vec{b}$ is $\vec{c}$ (let)

$$\therefore \quad \vec{c} = \pm(\vec{a} \times \vec{b})$$

So, there will be two vectors of unit length perpendicular to vectors $\vec{a}$ and $\vec{b}$.

Hence, the correct option is (*b*).

Fill in the blanks in each of the Exercises from 34 to 40.

Q34. The vector $\vec{a} + \vec{b}$ bisects the angle between the non-collinear vectors $\vec{a}$ and $\vec{b}$ if ________________.

Sol. If vector $\vec{a} + \vec{b}$ bisects the angle between non-collinear vectors $\vec{a}$ and $\vec{b}$ then the angle between $\vec{a} + \vec{b}$ and $\vec{a}$ is equal to the angle between $\vec{a} + \vec{b}$ and $\vec{b}$.

So, $$\cos\theta = \frac{\vec{a}\cdot(\vec{a}+\vec{b})}{|\vec{a}||\vec{a}+\vec{b}|} = \frac{\vec{a}\cdot(\vec{a}+\vec{b})}{|\vec{a}|\sqrt{a^2+b^2}} \quad \text{...}(i)$$

Also, $$\cos\theta = \frac{\vec{b}\cdot(\vec{a}+\vec{b})}{|\vec{b}|\cdot|\vec{a}+\vec{b}|} \quad [\because \ \theta \text{ is same}]$$
$$= \frac{\vec{b}\cdot(\vec{a}+\vec{b})}{|\vec{b}|\sqrt{a^2+b^2}} \quad \text{...}(ii)$$

From eq. (*i*) and eq. (*ii*) we get,

$$\frac{\vec{a}\cdot(\vec{a}+\vec{b})}{|\vec{a}|\sqrt{a^2+b^2}} = \frac{\vec{b}\cdot(\vec{a}+\vec{b})}{|\vec{b}|\sqrt{a^2+b^2}}$$

$$\Rightarrow \quad \frac{\vec{a}}{|\vec{a}|} = \frac{\vec{b}}{|\vec{b}|}$$

$$\Rightarrow \quad \hat{a} = \hat{b} \quad \Rightarrow \quad \vec{a} = \vec{b}$$

Hence, the required filler is $\vec{a} = \vec{b}$.

Q35. If $\vec{r}\cdot\vec{a}=0, \vec{r}\cdot\vec{b}=0$ and $\vec{r}\cdot\vec{c}=0$ for some non-zero vector $\vec{r}$, then the value of $\vec{a}\cdot(\vec{b}\times\vec{c})$ is ____________.

Sol. If $\vec{r}$ is a non-zero vector, then $\vec{a}$, $\vec{b}$ and $\vec{c}$ can be in the same plane.

Since angles between $\vec{a}$, and $\vec{c}$ are zero i.e. $\theta = 0$

$\therefore \quad \vec{a}\cdot(\vec{b}\times\vec{c}) = 0$

Hence the required value is 0.

Q36. The vectors $\vec{a} = 3\hat{i} - 2\hat{j} + 2\hat{k}$ and $\vec{b} = -\hat{i} - 2\hat{k}$ are the adjacent sides of a parallelogram. The acute angle between its diagonals is ____________.

Sol. Given that $\quad \vec{a} = 3\hat{i} - 2\hat{j} + 2\hat{k}$

and $\quad \vec{b} = -\hat{i} - 2\hat{k}$

$\therefore \quad \vec{a}+\vec{b} = 2\hat{i} - 2\hat{j}$ and $\vec{a}-\vec{b} = 4\hat{i} - 2\hat{j} + 4\hat{k}$

Let θ be the angle between the two diagonal vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ then

$$\cos\theta = \frac{(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})}{|\vec{a}+\vec{b}||\vec{a}-\vec{b}|} = \frac{(2\hat{i}-2\hat{j})\cdot(4\hat{i}-2\hat{j}+4\hat{k})}{\sqrt{(2)^2+(-2)^2}\cdot\sqrt{(4)^2+(-2)^2+(4)^2}}$$

$$= \frac{8+4}{2\sqrt{2}\cdot 6} = \frac{12}{2\sqrt{2}\cdot 6} = \frac{1}{\sqrt{2}}$$

$\therefore \quad \theta = \dfrac{\pi}{4}$

Hence the value of required filler is $\dfrac{\pi}{4}$.

Q37. The values of k, for which $|k\vec{a}| < |\vec{a}|$ and $k\vec{a} + \frac{1}{2}\vec{a}$ is parallel to $\vec{a}$ holds true are ____________.

Sol. Given that $|k\vec{a}| < |\vec{a}|$ and $k\vec{a} + \frac{1}{2}\vec{a}$ is parallel to $\vec{a}$

$\therefore \quad |k\vec{a}| < |\vec{a}| \Rightarrow |k||\vec{a}| < |\vec{a}| \Rightarrow |k| < 1 \Rightarrow -1 < k < 1$

Now since $k\vec{a}+\frac{1}{2}\vec{a}$ is parallel to $\vec{a}$

Here we see that at $k=-\frac{1}{2}$, $k\vec{a}+\frac{1}{2}\vec{a}$ become null vector and then it will not be parallel to $\vec{a}$.

$\therefore$ $k\vec{a}+\frac{1}{2}\vec{a}$ is parallel to $\vec{a}$ when $k\in(-1,1)$ and $k\neq\frac{1}{2}$.

Hence, the required value of $k\in(-1, 1)$ and $k\neq\frac{1}{2}$.

Q38. The value of the expression $|\vec{a}\times\vec{b}|^2+(\vec{a}\cdot\vec{b})^2$ is ________.

Sol.

$$\begin{aligned}|\vec{a}\times\vec{b}|^2+(\vec{a}\cdot\vec{b})^2 &= \left(|\vec{a}||\vec{b}|\sin\theta\right)^2+\left(|\vec{a}||\vec{b}|\cos\theta\right)^2\\ &= |\vec{a}|^2|\vec{b}|^2\sin^2\theta+|\vec{a}|^2|\vec{b}|^2\cos^2\theta\\ &= |\vec{a}|^2|\vec{b}|^2\cdot(\sin^2\theta+\cos^2\theta)\\ &= |\vec{a}|^2|\vec{b}|^2\cdot 1 = |\vec{a}|^2|\vec{b}|^2\end{aligned}$$

Hence, the value of the filler is $|\vec{a}|^2|\vec{b}|^2$.

Q39. If $|\vec{a}\times\vec{b}|^2+|\vec{a}\cdot\vec{b}|^2=144$ and $|\vec{a}|=4$, then $|\vec{b}|$ is equal to ________.

Sol.

$$\begin{aligned}&& |\vec{a}\times\vec{b}|^2+(\vec{a}\cdot\vec{b})^2 &= 144\\ \Rightarrow && \left(|\vec{a}||\vec{b}|\sin\theta\right)^2+\left(|\vec{a}||\vec{b}|\cos\theta\right)^2 &= 144\\ \Rightarrow && |\vec{a}|^2|\vec{b}|^2\sin^2\theta+|\vec{a}|^2|\vec{b}|^2\cos^2\theta &= 144\\ \Rightarrow && |\vec{a}|^2|\vec{b}|^2(\sin^2\theta+\cos^2\theta) &= 144\\ \Rightarrow && |\vec{a}|^2|\vec{b}|^2 &= 144\\ \Rightarrow && |\vec{a}||\vec{b}| &= 12\\ \Rightarrow && 4\cdot|\vec{b}| &= 12\\ \therefore && |\vec{b}| &= 3\end{aligned}$$

Hence, the value of the filler is 3.

Q40. If $\vec{a}$ is any non-zero vector, then $(\vec{a}\cdot\hat{i})\hat{i}+(\vec{a}\cdot\hat{j})\hat{j}+(\vec{a}\cdot\hat{k})\hat{k}$ equals ________.

Sol. Let

$$\begin{aligned}&& \vec{a} &= a_1\hat{i}+a_2\hat{j}+a_3\hat{k}\\ \therefore && \vec{a}\cdot\hat{i} &= (a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\cdot\hat{i}\\ &&&= a_1\end{aligned}$$

Similarly, $\vec{a}\cdot\hat{j}=a_2$ and $\vec{a}\cdot\hat{k}=a_3$

$\therefore (\vec{a}\cdot\hat{i})\cdot\hat{i} + (\vec{a}\cdot\hat{j})\hat{j} + (\vec{a}\cdot\hat{k})\cdot\hat{k} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k} = \vec{a}$

Hence, the value of the filler is $\vec{a}$.

State True or False in each of the following Exercises.

Q41. If $|\vec{a}| = |\vec{b}|$, then necessarily it implies $\vec{a} = \pm\vec{b}$.

Sol. If $|\vec{a}| = |\vec{b}|$ then $\vec{a} = \pm\vec{b}$ which is true.

Hence, the statement is True.

Q42. Position vector of a point P is a vector whose initial point is origin.

Sol. True

Q43. If $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$, then the vectors $\vec{a}$ and $\vec{b}$ are orthogonal.

Sol. Given that $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$

Squaring both sides, we get

$$|\vec{a}+\vec{b}|^2 = |\vec{a}-\vec{b}|^2$$

$$\Rightarrow \quad |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}\cdot\vec{b} = |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a}\cdot\vec{b}$$

$$\Rightarrow \quad 2\vec{a}\cdot\vec{b} = -2\vec{a}\cdot\vec{b} \Rightarrow \vec{a}\cdot\vec{b} = -\vec{a}\cdot\vec{b}$$

$$\Rightarrow \quad 2\vec{a}\cdot\vec{b} = 0 \Rightarrow \vec{a}\cdot\vec{b} = 0$$

which implies that $\vec{a}$ and $\vec{b}$ are orthogonal.

Hence the given statement is True.

Q44. The formula $(\vec{a}+\vec{b})^2 = \vec{a}^2 + \vec{b}^2 + 2\vec{a}\times\vec{b}$ is valid for non-zero vectors $\vec{a}$ and $\vec{b}$.

Sol.

$$(\vec{a}+\vec{b})^2 = (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})$$

$$= |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}\cdot\vec{b}$$

Hence, the given statement is False.

Q45. If $\vec{a}$ and $\vec{b}$ are adjacent sides of a rhombus, then $\vec{a}\cdot\vec{b} = 0$.

Sol. If $\vec{a}\cdot\vec{b} = 0$ then $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos 90°$

So the angle between the adjacent sides of the rhombus should be 90° which is not possible.

Hence, the given statement is False.

❑❑❑

11 Three Dimensional Geometry

11.3 EXERCISE

SHORT ANSWER TYPE QUESTIONS

Q1. Find the position vector of a point A in space such that $\overrightarrow{OA}$ is inclined at 60° to OX and at 45° to OY and $|\overrightarrow{OA}| = 10$ units.

Sol. Let $\alpha = 60°$, $\beta = 45°$ and the angle inclined to OZ axis be γ

We know that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$\Rightarrow \cos^2 60° + \cos^2 45° + \cos^2\gamma = 1$$

$$\Rightarrow \left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \cos^2\gamma = 1 \quad \Rightarrow \frac{1}{4} + \frac{1}{2} + \cos^2\gamma = 1$$

$$\Rightarrow \frac{3}{4} + \cos^2\gamma = 1 \quad \Rightarrow \cos^2\gamma = 1 - \frac{3}{4} = \frac{1}{4}$$

$$\therefore \quad \cos\gamma = \pm\frac{1}{2} \quad \Rightarrow \cos\gamma = \frac{1}{2}$$

(Rejecting $\cos\gamma = -\frac{1}{2}$, since $\gamma < 90°$)

$$\therefore \quad \overrightarrow{OA} = |\overrightarrow{OA}|\left(\frac{1}{2}\hat{i} + \frac{1}{\sqrt{2}}\hat{j} + \frac{1}{2}\hat{k}\right) = 10\left(\frac{1}{2}\hat{i} + \frac{1}{\sqrt{2}}\hat{j} + \frac{1}{2}\hat{k}\right)$$

$$= 5\hat{i} + 5\sqrt{2}\hat{j} + 5\hat{k}$$

Hence, the position vector of A is $(5\hat{i} + 5\sqrt{2}\hat{j} + 5\hat{k})$.

Q2. Find the vector equation of the line which is parallel to the vector $3\hat{i} - 2\hat{j} + 6\hat{k}$ and which passes through the point $(1, -2, 3)$.

Sol. We know that the equation of line is

$$\vec{r} = \vec{a} + \vec{b}\lambda$$

Here, $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + 6\hat{k}$

$\therefore$ Equation of line is $\vec{r} = (\hat{i} - 2\hat{j} + 3\hat{k}) + \lambda(3\hat{i} - 2\hat{j} + 6\hat{k})$

$\Rightarrow (x\hat{i} + y\hat{j} + z\hat{k}) = (\hat{i} - 2\hat{j} + 3\hat{k}) + \lambda(3\hat{i} - 2\hat{j} + 6\hat{k})$

$\Rightarrow (x\hat{i} + y\hat{j} + z\hat{k}) - (\hat{i} - 2\hat{j} + 3\hat{k}) = \lambda(3\hat{i} - 2\hat{j} + 6\hat{k})$

$\Rightarrow (x-1)\hat{i} + (y+2)\hat{j} + (z-3)\hat{k} = \lambda(3\hat{i} - 2\hat{j} + 6\hat{k})$

Hence, the required equation is

$(x-1)\hat{i} + (y+2)\hat{j} + (z-3)\hat{k} = \lambda(3\hat{i} - 2\hat{j} + 6\hat{k})$

Q3. Show that the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-4}{5}=\frac{y-1}{2}=z$ intersect. Also, find their point of intersection.

Sol. The given equations are

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4} \text{ and } \frac{x-4}{5}=\frac{y-1}{2}=z$$

Let $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}=\lambda$

$\therefore x=2\lambda+1,\ y=3\lambda+2$ and $z=4\lambda+3$

and $\frac{x-4}{5}=\frac{y-1}{2}=\frac{z}{1}=\mu$

$\therefore x=5\mu+4,\ y=2\mu+1$ and $z=\mu$

If the two lines intersect each other at one point,

then $2\lambda+1=5\mu+4 \Rightarrow 2\lambda-5\mu=3$ ...(*i*)

$3\lambda+2=2\mu+1 \Rightarrow 3\lambda-2\mu=-1$ ...(*ii*)

and $4\lambda+3=\mu \Rightarrow 4\lambda-\mu=-3$ ...(*iii*)

Solving eqns. (*i*) and (*ii*) we get

$2\lambda-5\mu=3$ [multiply by 3]

$3\lambda-2\mu=-1$ [multiply by 2]

$$\begin{array}{rrcr} \Rightarrow 6\lambda & -15\mu & = & 9 \\ 6\lambda & -4\mu & = & -2 \\ (-) & (+) & & (+) \\ \hline & -11\mu & = & 11 \end{array} \quad \therefore \mu=-1$$

Putting the value of μ in eq. (*i*) we get,

$$2\lambda-5(-1)=3$$
$$\Rightarrow \quad 2\lambda+5=3$$
$$\Rightarrow \quad 2\lambda=-2 \quad \therefore \lambda=-1$$

Now putting the value of λ and μ in eq. (*iii*) then

$$4(-1)-(-1)=-3$$
$$-4+1=-3$$
$$-3=-3 \text{ (satisfied)}$$

$\therefore$ Coordinates of the point of intersection are

$$x=5(-1)+4=-5+4=-1$$
$$y=2(-1)+1=-2+1=-1$$
$$z=-1$$

Hence, the given lines intersect each other at $(-1,-1,-1)$.

Alternately: If two lines intersect each other at a point, then the shortest distance between them is equal to 0.

For this we will use SD $=\frac{(\vec{a}_2-\vec{a}_1)(\vec{b}_1\times\vec{b}_2)}{|\vec{b}_1\times\vec{b}_2|}=0.$

Q4. Find the angle between the lines

$$\vec{r} = 3\hat{i} - 2\hat{j} + 6\hat{k} + \lambda(2\hat{i} + \hat{j} + 2\hat{k}) \text{ and}$$
$$\vec{r} = (2\hat{j} - 5\hat{k}) + \mu(6\hat{i} + 3\hat{j} + 2\hat{k})$$

Sol. Here, $\vec{b}_1 = 2\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b}_2 = 6\hat{i} + 3\hat{j} + 2\hat{k}$

$$\therefore \quad \cos\theta = \frac{\vec{b}_1 . \vec{b}_2}{|\vec{b}_1||\vec{b}_2|} = \frac{(2\hat{i} + \hat{j} + 2\hat{k}).(6\hat{i} + 3\hat{j} + 2\hat{k})}{\sqrt{(2)^2 + (1)^2 + (2)^2} . \sqrt{(6)^2 + (3)^2 + (2)^2}}$$

$$= \frac{12 + 3 + 4}{\sqrt{4 + 1 + 4} . \sqrt{36 + 9 + 4}} = \frac{19}{\sqrt{9} . \sqrt{49}} = \frac{19}{3 \cdot 7} = \frac{19}{21}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{19}{21}\right)$$

Hence, the required angle is $\cos^{-1}\left(\frac{19}{21}\right)$.

Q5. Prove that the line through A(0, – 1, – 1) and B(4, 5, 1) intersects the line through C(3, 9, 4) and D(– 4, 4, 4).

Sol. Given points are A(0, – 1, – 1) and B(4, 5, 1)

C(3, 9, 4) and D(– 4, 4, 4)

Cartesian form of equation AB is

$$\frac{x - 0}{4 - 0} = \frac{y + 1}{5 + 1} = \frac{z + 1}{1 + 1} \Rightarrow \frac{x}{4} = \frac{y + 1}{6} = \frac{z + 1}{2}$$

and its vector form is $\vec{r} = (-\hat{j} - \hat{k}) + \lambda(4\hat{i} + 6\hat{j} + 2\hat{k})$

Similarly, equation of CD is

$$\frac{x - 3}{-4 - 3} = \frac{y - 9}{4 - 9} = \frac{z - 4}{4 - 4} \Rightarrow \frac{x - 3}{-7} = \frac{y - 9}{-5} = \frac{z - 4}{0}$$

and its vector form is $\vec{r} = (3\hat{i} + 9\hat{j} + 4\hat{k}) + \mu(-7\hat{i} - 5\hat{j})$

Now, here $\vec{a}_1 = -\hat{j} - \hat{k}, \ \vec{b}_1 = 4\hat{i} + 6\hat{j} + 2\hat{k}$

$\vec{a}_2 = 3\hat{i} + 9\hat{j} + 4\hat{k}, \ \vec{b}_2 = -7\hat{i} - 5\hat{j}$

Shortest distance between AB and CD

$$\text{S.D.} = \left|\frac{(\vec{a}_2 - \vec{a}_1)\cdot(\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|}\right|$$

$$\vec{a}_2 - \vec{a}_1 = (3\hat{i} + 9\hat{j} + 4\hat{k}) - (-\hat{j} - \hat{k}) = 3\hat{i} + 10\hat{j} + 5\hat{k}.$$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 6 & 2 \\ -7 & -5 & 0 \end{vmatrix}$$

$$= \hat{i}(0 + 10) - \hat{j}(0 + 14) + \hat{k}(-20 + 42)$$
$$= 10\hat{i} - 14\hat{j} + 22\hat{k}$$

$$|\vec{b}_1 \times \vec{b}_2| = \sqrt{(10)^2 + (-14)^2 + (22)^2}$$
$$= \sqrt{100 + 196 + 484} = \sqrt{780}$$
$$\therefore \quad \text{S.D} = \frac{(3\hat{i} + 10\hat{j} + 5\hat{k})\cdot(10\hat{i} - 14\hat{j} + 22\hat{k})}{\sqrt{780}}$$
$$= \frac{30 - 140 + 110}{\sqrt{780}} = 0$$

Hence, the two lines intersect each other.

Q6. Prove that the lines $x = py + q$, $z = ry + s$ and $x = p'y + q'$, $z = r'y + s'$ are perpendicular, if $pp' + rr' + 1 = 0$

Sol. Given that: $x = py + q \Rightarrow y = \dfrac{x-q}{p}$

and $z = ry + s \Rightarrow y = \dfrac{z-s}{r}$

$\therefore$ the equation becomes

$\dfrac{x-q}{p} = \dfrac{y}{1} = \dfrac{z-s}{r}$ in which d'ratios are $a_1 = p, b_1 = 1, c_1 = r$

Similarly $x = p'y + q' \Rightarrow y = \dfrac{x-q'}{p'}$

and $z = r'y + s' \Rightarrow y = \dfrac{z-s'}{r'}$

$\therefore$ the equation becomes

$\dfrac{x-q'}{p'} = \dfrac{y}{1} = \dfrac{z-s'}{r'}$ in which $a_2 = p', b_2 = 1, c_2 = r'$

If the lines are perpendicular to each other, then

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$
$$pp' + 1.1 + rr' = 0$$

Hence, $pp' + rr' + 1 = 0$ is the required condition.

Q7. Find the equation of a plane which bisects perpendicularly the line joining the points A(2, 3, 4), B(4, 5, 8) at right angles.

Sol. Given that A(2, 3, 4) and B(4, 5, 8)

Coordinates of mid-point C are $\left(\dfrac{2+4}{2}, \dfrac{3+5}{2}, \dfrac{4+8}{2}\right) = (3, 4, 6)$

Now direction ratios of the normal to the plane

$= \text{direction ratios of AB}$

$= 4 - 2, 5 - 3, 8 - 4 = (2, 2, 4)$

Equation of the plane is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$
$$\Rightarrow \quad 2(x-3) + 2(y-4) + 4(z-6) = 0$$
$$\Rightarrow \quad 2x - 6 + 2y - 8 + 4z - 24 = 0$$
$$\Rightarrow \quad 2x + 2y + 4z = 38 \Rightarrow x + y + 2z = 19$$

Hence, the required equation of plane is

$x + y + 2z = 19$ or $\vec{r}(\hat{i} + \hat{j} + 2\hat{k}) = 19.$

Q8. Find the equation of a plane which is at a distance $3\sqrt{3}$ units from origin and the normal to which is equally inclined to coordinate axis.

Sol. Since, the normal to the plane is equally inclined to the axes

$\therefore \cos\alpha = \cos\beta = \cos\gamma$

$\Rightarrow \cos^2\alpha + \cos^2\alpha + \cos^2\alpha = 1$

$$\Rightarrow \quad 3\cos^2\alpha = 1 \quad \Rightarrow \quad \cos\alpha = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad \cos\alpha = \cos\beta = \cos\gamma = \frac{1}{\sqrt{3}}$$

So, the normal is $\quad \vec{N} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}$

$\therefore$ Equation of the plane is $\vec{r}.\vec{N} = d$

$$\Rightarrow \quad \vec{r}.\frac{\vec{N}}{|\vec{N}|} = d$$

$$\Rightarrow \quad \frac{\vec{r}.\left(\frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}\right)}{1} = 3\sqrt{3}$$

$$\Rightarrow \quad \vec{r}.\left(\frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}\right) = 3\sqrt{3}$$

$$\Rightarrow (x\hat{i} + y\hat{j} + z\hat{k}).\frac{1}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k}) = 3\sqrt{3}$$

$$\Rightarrow \quad x + y + z = 3\sqrt{3}.\sqrt{3} \quad \Rightarrow x + y + z = 9$$

Hence, the required equation of plane is $x + y + z = 9$.

Q9. If the line drawn from the point $(-2, -1, -3)$ meets a plane at right angle at the point $(1, -3, 3)$, find the equation of the plane.

Sol. Direction ratios of the normal to the plane are

$(1 + 2, -3 + 1, 3 + 3) \Rightarrow (3, -2, 6)$

Equation of plane passing through one point (x_1, y_1, z_1) is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

$$\Rightarrow \quad 3(x - 1) - 2(y + 3) + 6(z - 3) = 0$$

$$\Rightarrow \quad 3x - 3 - 2y - 6 + 6z - 18 = 0$$

$$\Rightarrow \quad 3x - 2y + 6z - 27 = 0 \quad \Rightarrow 3x - 2y + 6z = 27$$

Hence, the required equation is $3x - 2y + 6z = 27$.

Q10. Find the equation of the plane passing through the points $(2, 1, 0)$, $(3, -2, -2)$ and $(3, 1, 7)$.

Sol. Since, the equation of the plane passing through the points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is

$$\Rightarrow \begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-2 & y-1 & z-0 \\ 3-2 & -2-1 & -2-0 \\ 3-2 & 1-1 & 7-0 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} x-2 & y-1 & z \\ 1 & -3 & -2 \\ 1 & 0 & 7 \end{vmatrix} = 0$$

$$\Rightarrow (x-2)\begin{vmatrix} -3 & -2 \\ 0 & 7 \end{vmatrix} - (y-1)\begin{vmatrix} 1 & -2 \\ 1 & 7 \end{vmatrix} + z\begin{vmatrix} 1 & -3 \\ 1 & 0 \end{vmatrix} = 0$$

$$\Rightarrow \quad (x-2)(-21) - (y-1)(7+2) + z(3) = 0$$
$$\Rightarrow \quad -21(x-2) - 9(y-1) + 3z = 0$$
$$\Rightarrow \quad -21x + 42 - 9y + 9 + 3z = 0$$
$$\Rightarrow \quad -21x - 9y + 3z + 51 = 0 \Rightarrow 7x + 3y - z - 17 = 0$$

Hence, the required equation is $7x + 3y - z - 17 = 0$.

Q11. Find the equations of two lines through the origin which intersect the line $\dfrac{x-3}{2} = \dfrac{y-3}{1} = \dfrac{z}{1}$ at angles of $\dfrac{\pi}{3}$ each.

Sol. Any point on the given line is

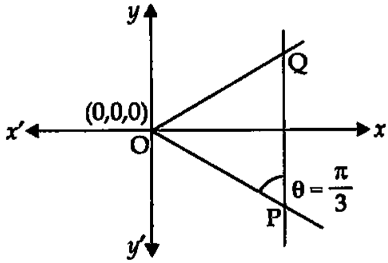
$$\frac{x-3}{2} = \frac{y-3}{1} = \frac{z}{1} = \lambda$$

$\Rightarrow \quad x = 2\lambda + 3, y = \lambda + 3$

and $\quad z = \lambda$

Let it be the coordinates of P

$\therefore$ Direction ratios of OP are



$(2\lambda + 3 - 0), (\lambda + 3 - 0)$ and $(\lambda - 0) \Rightarrow 2\lambda + 3, \lambda + 3, \lambda$

But the direction ratios of the line PQ are 2, 1, 1

$$\therefore \quad \cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\,.\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$\cos\frac{\pi}{3} = \frac{2(2\lambda+3) + 1(\lambda+3) + 1.\lambda}{\sqrt{(2)^2 + (1)^2 + (1)^2}\,.\sqrt{(2\lambda+3)^2 + (\lambda+3)^2 + \lambda^2}}$$

$$\Rightarrow \quad \frac{1}{2} = \frac{4\lambda + 6 + \lambda + 3 + \lambda}{\sqrt{6}\,.\sqrt{4\lambda^2 + 9 + 12\lambda + \lambda^2 + 9 + 6\lambda + \lambda^2}}$$

$$\Rightarrow \quad \frac{\sqrt{6}}{2} = \frac{6\lambda + 9}{\sqrt{6\lambda^2 + 18\lambda + 18}} = \frac{6\lambda + 9}{\sqrt{6}\sqrt{\lambda^2 + 3\lambda + 3}}$$

$$\Rightarrow \quad \frac{6}{2} = \frac{3(2\lambda+3)}{\sqrt{\lambda^2+3\lambda+3}} \Rightarrow 3 = \frac{3(2\lambda+3)}{\sqrt{\lambda^2+3\lambda+3}}$$

$$\Rightarrow \quad 1 = \frac{2\lambda+3}{\sqrt{\lambda^2+3\lambda+3}} \Rightarrow \sqrt{\lambda^2+3\lambda+3} = 2\lambda+3$$

$\Rightarrow \quad \lambda^2+3\lambda+3 = 4\lambda^2+9+12\lambda$ (Squaring both sides)

$\Rightarrow \quad 3\lambda^2+9\lambda+6 = 0 \Rightarrow \lambda^2+3\lambda+2=0$

$\Rightarrow \quad (\lambda+1)(\lambda+2) = 0$

$\therefore \quad \lambda = -1, \lambda = -2$

$\therefore$ Direction ratios are $[2(-1)+3, -1+3, -1]$ i.e., $1, 2, -1$ when $\lambda = -1$ and $[2(-2)+3, -2+3, -2]$ i.e., $-1, 1, -2$ when $\lambda = -2$.

Hence, the required equations are

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{-1} \text{ and } \frac{x}{-1} = \frac{y}{1} = \frac{z}{-2}.$$

Q12. Find the angle between the lines whose direction cosines are given by the equations $l+m+n=0$ and $l^2+m^2-n^2=0$

Sol. The given equations are

$$l+m+n = 0 \quad \text{...(i)}$$

$$l^2+m^2-n^2 = 0 \quad \text{...(ii)}$$

From equation (*i*) $n = -(l+m)$

Putting the value of n in eq. (*ii*) we get

$$l^2+m^2+[-(l+m)^2] = 0$$

$\Rightarrow \quad l^2+m^2-l^2-m^2-2lm = 0$

$\Rightarrow \quad -2lm = 0$

$\Rightarrow \quad lm = 0 \Rightarrow (-m-n)m = 0 \; [\because l = -m-n]$

$\Rightarrow \quad (m+n)m = 0 \Rightarrow m = 0$ or $m = -n$

$\Rightarrow \quad l = 0$ or $l = -n$

$\therefore$ Direction cosines of the two lines are

$0, -n, n$ and $-n, 0, n \Rightarrow 0, -1, 1$ and $-1, 0, 1$

$$\therefore \quad \cos\theta = \frac{(0\hat{i}-\hat{j}+\hat{k}).(-\hat{i}+0\hat{j}+\hat{k})}{\sqrt{(-1)^2+(1)^2}\sqrt{(-1)^2+(1)^2}} = \frac{1}{\sqrt{2}.\sqrt{2}} = \frac{1}{2}$$

$$\therefore \quad \theta = \frac{\pi}{3}$$

Hence, the required angle is $\frac{\pi}{3}$.

Q13. If a variable line in two adjacent positions has direction cosines l, m, n and $l+\delta l, m+\delta m, n+\delta n$, show that the small angle $\delta\theta$ between the two positions is given by $\delta\theta^2 = \delta l^2 + \delta m^2 + \delta n^2$.

Sol. Given that l, m, n and $l+\delta l, m+\delta m, n+\delta n$, are the direction cosines of a variable line in two positions

$$\therefore \quad l^2+m^2+n^2 = 1 \quad \text{...(i)}$$

and $(l+\delta l)^2+(m+\delta m)^2+(n+\delta n)^2=1$ ...(ii)

$\Rightarrow l^2+\delta l^2+2l.\delta l+m^2+\delta m^2+2m.\delta m+n^2+\delta n^2+2n.\delta n=1$

$\Rightarrow (l^2+m^2+n^2)+(\delta l^2+\delta m^2+\delta n^2)+2(l.\delta l+m.\delta m+n.\delta n)=1$

$\Rightarrow 1+(\delta l^2+\delta m^2+\delta n^2)+2(l.\delta l+m.\delta m+n.\delta n)=1$

$\Rightarrow l.\delta l+m.\delta m+n.\delta n=-\frac{1}{2}(\delta l^2+\delta m^2+\delta n^2)$

Let $\vec{a}$ and $\vec{b}$ be the unit vectors along a line with d'cosines l, m, n and $(l+\delta l), (m+\delta m), (n+\delta n)$.

$\therefore\ \vec{a}=l\hat{i}+m\hat{j}+n\hat{k}$ and $\vec{b}=(l+\delta l)\hat{i}+(m+\delta m)\hat{j}+(n+\delta n)\hat{k}$

$$\cos\delta\theta=\frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$$

$$\cos\delta\theta=\frac{(l\hat{i}+m\hat{j}+n\hat{k}).\left[(l+\delta l)\hat{i}+(m+\delta m)\hat{j}+(n+\delta n)\hat{k}\right]}{1.1}$$

$\left[\because |\vec{a}|=|\vec{b}|=1\right]$

$\Rightarrow\ \cos\delta\theta=l(l+\delta l)+m(m+\delta m)+n(n+\delta n)$

$\Rightarrow\ \cos\delta\theta=l^2+l.\delta l+m^2+m.\delta m+n^2+n.\delta n$

$\Rightarrow\ \cos\delta\theta=(l^2+m^2+n^2)+(l.\delta l+m.\delta m+n.\delta n)$

$\Rightarrow\ \cos\delta\theta=1-\frac{1}{2}(\delta l^2+\delta m^2+\delta n^2)$

$\Rightarrow 1-\cos\delta\theta=\frac{1}{2}(\delta l^2+\delta m^2+\delta n^2)$

$\Rightarrow 2\sin^2\frac{\delta\theta}{2}=\frac{1}{2}(\delta l^2+\delta m^2+\delta n^2)$

$\Rightarrow 4\sin^2\frac{\delta\theta}{2}=\delta l^2+\delta m^2+\delta n^2$

$\left[\because \frac{\delta\theta}{2}\text{ is very small so, } \sin\frac{\delta\theta}{2}=\frac{\delta\theta}{2}\right]$

$\Rightarrow\ 4\left(\frac{\delta\theta}{2}\right)^2=\delta l^2+\delta m^2+\delta n^2$

$\Rightarrow\ (\delta\theta)^2=\delta l^2+\delta m^2+\delta n^2$ Hence proved.

Q14. O is the origin and A is (a, b, c). Find the direction cosines of the line OA and the equation of plane through A at right angle to OA.

Sol. We have $A(a, b, c)$ and $O(0, 0, 0)$

$\therefore$ direction ratios of OA $= a-0, b-0, c-0$

$= a, b, c$

$\therefore$ direction cosines of line OA

$$=\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}$$

Now direction ratios of the normal to the plane are (a, b, c).

$\therefore$ Equation of the plane passing through the point $A(a, b, c)$ is

$a(x-a)+b(y-b)+c(z-c)=0$

$\Rightarrow \quad ax - a^2 + by - b^2 + cz - c^2 = 0$

$\Rightarrow \quad ax + by + cz = a^2 + b^2 + c^2$

Hence, the required equation is $ax + by + cz = a^2 + b^2 + c^2$.

Q15. Two systems of rectangular axis have the same origin. If a plane cuts them at distances a, b, c and a', b', c' respectively from the origin, prove that $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{a'^2}+\frac{1}{b'^2}+\frac{1}{c'^2}$.

Sol. Let OX, OY, OZ and ox, oy, oz be two rectangular systems

$\therefore$ Equations of two planes are

$\frac{X}{a}+\frac{Y}{b}+\frac{Z}{c}=1 \quad ...(i)$ and $\frac{x}{a'}+\frac{y}{b'}+\frac{z}{c'}=1 \quad ...(ii)$

Length of perpendicular from origin to plane (i) is

$$= \left|\frac{\frac{0}{a}+\frac{0}{b}+\frac{0}{c}-1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}\right| = \frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}$$

Length of perpendicular from origin to plane (ii)

$$= \left|\frac{\frac{0}{a'}+\frac{0}{b'}+\frac{0}{c'}-1}{\sqrt{\frac{1}{a'^2}+\frac{1}{b'^2}+\frac{1}{c'^2}}}\right| = \frac{1}{\sqrt{\frac{1}{a'^2}+\frac{1}{b'^2}+\frac{1}{c'^2}}}$$

As per the condition of the question

$$\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}} = \frac{1}{\sqrt{\frac{1}{a'^2}+\frac{1}{b'^2}+\frac{1}{c'^2}}}$$

Hence, $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{a'^2}+\frac{1}{b'^2}+\frac{1}{c'^2}$

LONG ANSWER TYPE QUESTIONS

Q16. Find the foot of perpendicular from the point $(2, 3, -8)$ to the line $\frac{4-x}{2}=\frac{y}{6}=\frac{1-z}{3}$. Also, find the perpendicular distance from the given point to the line.

Sol. Given that: $\frac{4-x}{2}=\frac{y}{6}=\frac{1-z}{3}$ is the equation of line

$$\Rightarrow \quad \frac{x-4}{-2}=\frac{y}{6}=\frac{z-1}{-3}=\lambda$$

$\therefore$ Coordinates of any point Q on the line are

$x = -2\lambda + 4$, $y = 6\lambda$ and $z = -3\lambda + 1$

and the given point is P(2, 3, – 8)
Direction ratios of PQ are $-2\lambda+4-2, 6\lambda-3, -3\lambda+1+8$
i.e., $-2\lambda+2, 6\lambda-3, -3\lambda+9$
and the D'ratios of the given line are – 2, 6, – 3.
If PQ $\perp$ line
then $-2(-2\lambda+2)+6(6\lambda-3)-3(-3\lambda+9)=0$

$\Rightarrow \quad 4\lambda-4+36\lambda-18+9\lambda-27=0$

$\Rightarrow \quad 49\lambda-49=0 \Rightarrow \lambda=1$

$\therefore$ The foot of the perpendicular is $-2(1)+4, 6(1), -3(1)+1$
i.e., 2, 6, – 2

Now, distance PQ $= \sqrt{(2-2)^2+(3-6)^2+(-8+2)^2}$

$= \sqrt{9+36} = \sqrt{45} = 3\sqrt{5}$

Hence, the required coordinates of the foot of perpendicular are 2, 6, – 2 and the required distance is $3\sqrt{5}$ units.

Q17. Find the distance of a point (2, 4, – 1) from the line

$$\frac{x+5}{1}=\frac{y+3}{4}=\frac{z-6}{-9}.$$

Sol. The given equation of line is

$$\frac{x+5}{1}=\frac{y+3}{4}=\frac{z-6}{-9}=\lambda \text{ and any point P(2, 4, – 1)}$$

Let Q be any point on the given line
$\therefore$ Coordinates of Q are $x=\lambda-5, y=4\lambda-3, z=-9\lambda+6$
D'ratios of PQ are $\lambda-5-2, 4\lambda-3-4, -9\lambda+6+1$
i.e., $\lambda-7, 4\lambda-7, -9\lambda+7$
and the d'ratios of the line are 1, 4, – 9
If PQ $\perp$ line then

$1(\lambda-7)+4(4\lambda-7)-9(-9\lambda+7)=0$

$\lambda-7+16\lambda-28+81\lambda-63=0$

$\Rightarrow \quad 98\lambda-98=0 \quad \therefore \quad \lambda=1$

So, the coordinates of Q are $1-5, 4\times1-3, -9\times1+6$
i.e., – 4, 1, – 3

$\therefore \quad PQ = \sqrt{(-4-2)^2+(1-4)^2+(-3+1)^2}$

$= \sqrt{(-6)^2+(-3)^2+(-2)^2} = \sqrt{36+9+4} = \sqrt{49} = 7$

Hence, the required distance is 7 units.

Q18. Find the length and foot of perpendicular from the point $\left(1, \frac{3}{2}, 2\right)$ to the plane $2x-2y+4z+5=0$.

Sol. Given plane is $2x-2y+4z+5=0$ and given point is $\left(1, \frac{3}{2}, 2\right)$

D'ratios of the normal to the plane are 2, – 2, 4

So, the equation of the line passing through $\left(1, \frac{3}{2}, 2\right)$ and whose d'ratios are equal to the d'ratios of the normal to the plane i.e., 2, – 2, 4 is $\frac{x-1}{2} = \frac{y-\frac{3}{2}}{-2} = \frac{z-2}{4} = \lambda$

$\therefore$ Any point in the plane is $2\lambda + 1,\ -2\lambda + \frac{3}{2},\ 4\lambda + 2$

Since, the point lies in the plane, then

$$2(2\lambda + 1) - 2\left(-2\lambda + \frac{3}{2}\right) + 4(4\lambda + 2) + 5 = 0$$

$$\Rightarrow 4\lambda + 2 + 4\lambda - 3 + 16\lambda + 8 + 5 = 0$$

$$\Rightarrow 24\lambda + 12 = 0 \quad \therefore \lambda = -\frac{1}{2}$$

So, the coordinates of the point in the plane are

$$2\left(-\frac{1}{2}\right) + 1,\ -2\left(-\frac{1}{2}\right) + \frac{3}{2},\ 4\left(-\frac{1}{2}\right) + 2 \text{ i.e., } 0, \frac{5}{2}, 0$$

Hence, the foot of the perpendicular is $\left(0, \frac{5}{2}, 0\right)$ and the

$$\text{required length} = \sqrt{(1-0)^2 + \left(\frac{3}{2} - \frac{5}{2}\right)^2 + (2-0)^2}$$

$$= \sqrt{1 + 1 + 4} = \sqrt{6} \text{ units}$$

Q19. Find the equations of the line passing through the point (3, 0, 1) and parallel to the planes $x + 2y = 0$ and $3y - z = 0$.

Sol. Given point is (3, 0, 1) and the equation of planes are

$$x + 2y = 0 \quad \ldots(i)$$

and $$3y - z = 0 \quad \ldots(ii)$$

Equation of any line l passing through (3, 0, 1) is

$$l: \frac{x-3}{a} = \frac{y-0}{b} = \frac{z-1}{c}$$

Direction ratios of the normal to plane (*i*) and plane (*ii*) are (1, 2, 0) and (0, 3, – 1)

Since the line is parallel to both the planes.

$\therefore \quad 1.a + 2.b + 0.c = 0 \quad \Rightarrow \quad a + 2b + 0c = 0$

and $\quad 0.a + 3.b - 1.c = 0 \quad \Rightarrow \quad 0.a + 3b - c = 0$

So $$\frac{a}{-2-0} = \frac{-b}{-1-0} = \frac{c}{3-0} = \lambda$$

$\therefore a = -2\lambda, b = \lambda, c = 3\lambda$

So, the equation of line is

$$\frac{x-3}{-2\lambda} = \frac{y}{\lambda} = \frac{z-1}{3\lambda}$$

Hence, the required equation is

$$\frac{x-3}{-2}=\frac{y}{1}=\frac{z-1}{3}$$

or in vector form is

$$(x-3)\hat{i}+y\hat{j}+(z-1)\hat{k} = \lambda(-2\hat{i}+\hat{j}+3\hat{k})$$

Q20. Find the equation of the plane through the points (2, 1, – 1) and (– 1, 3, 4), and perpendicular to the plane $x-2y+4z=10$.

Sol. Equation of the plane passing through two points (x_1, y_1, z_1) and (x_2, y_2, z_2) with its normal's d'ratios is

$$a(x-x_1)+b(y-y_1)+c(z-z_1)=0 \qquad ...(i)$$

If the plane is passing through the given points (2, 1, – 1) and (– 1, 3, 4) then

$$a(x_2-x_1)+b(y_2-y_1)+c(z_2-z_1)=0$$

$$\Rightarrow \quad a(-1-2)+b(3-1)+c(4+1)=0$$

$$\Rightarrow \quad -3a+2b+5c=0 \qquad ...(ii)$$

Since the required plane is perpendicular to the given plane $x-2y+4z=10$, then

$$1.a-2.b+4.c=10 \qquad ...(iii)$$

Solving (*ii*) and (*iii*) we get,

$$\frac{a}{8+10}=\frac{-b}{-12-5}=\frac{c}{6-2}=\lambda$$

$a=18\lambda,\ b=17\lambda,\ c=4\lambda$

Hence, the required plane is

$$18\lambda(x-2)+17\lambda(y-1)+4\lambda(z+1)=0$$

$$\Rightarrow \quad 18x-36+17y-17+4z+4=0$$

$$\Rightarrow \quad 18x+17y+4z-49=0$$

Q21. Find the shortest distance between the lines given by

$$\vec{r} = (8+3\lambda)\hat{i}-(9+16\lambda)\hat{j}+(10+7\lambda)\hat{k}$$

and $\quad \vec{r} = 15\hat{i}+29\hat{j}+5\hat{k}+\mu(3\hat{i}+8\hat{j}-5\hat{k})$.

Sol. Given equations of lines are

$$\vec{r} = (8+3\lambda)\hat{i}-(9+16\lambda)\hat{j}+(10+7\lambda)\hat{k} \qquad ...(i)$$

and $\quad \vec{r} = 15\hat{i}+29\hat{j}+5\hat{k}+\mu(3\hat{i}+8\hat{j}-5\hat{k}) \qquad ...(ii)$

Equation (*i*) can be re-written as

$$\vec{r} = 8\hat{i}-9\hat{j}+10\hat{k}+\lambda(3\hat{i}-16\hat{j}+7\hat{k}) \qquad ...(iii)$$

Here, $\quad \vec{a}_1 = 8\hat{i}-9\hat{j}+10\hat{k}$ and $\vec{a}_2=15\hat{i}+29\hat{j}+5\hat{k}$

$\qquad \vec{b}_1 = 3\hat{i}-16\hat{j}+7\hat{k}$ and $\vec{b}_2=3\hat{i}+8\hat{j}-5\hat{k}$

$$\vec{a}_2-\vec{a}_1 = 7\hat{i}+38\hat{j}-5\hat{k}$$

$$\vec{b}_1\times\vec{b}_2 = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & -16 & 7\\ 3 & 8 & -5\end{vmatrix}$$

$$= \hat{i}(80-56) - \hat{j}(-15-21) + \hat{k}(24+48)$$
$$= 24\hat{i} + 36\hat{j} + 72\hat{k}$$

$\therefore$ Shortest distance, SD $= \left|\dfrac{(\vec{a}_2 - \vec{a}_1).(\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|}\right|$

$$= \left|\frac{(7\hat{i} + 38\hat{j} - 5\hat{k}).(24\hat{i} + 36\hat{j} + 72\hat{k})}{\sqrt{(24)^2 + (36)^2 + (72)^2}}\right|$$

$$= \left|\frac{168 + 1368 - 360}{\sqrt{576 + 1296 + 5184}}\right| = \left|\frac{168 + 1008}{\sqrt{7056}}\right| = \frac{1176}{84} = 14 \text{ units}$$

Hence, the required distance is 14 units.

Q22. Find the equation of the plane which is perpendicular to the plane $5x+3y+6z+8=0$ and which contains the line of intersection of the planes $x+2y+3z-4=0$ and $2x+y-z+5=0$.

Sol. The given planes are

$P_1 : \quad 5x + 3y + 6z + 8 = 0$

$P_2 : \quad x + 2y + 3z - 4 = 0$

$P_3 : \quad 2x + y - z + 5 = 0$

Equation of the plane passing through the line of intersection of P_2 and P_3 is

$$(x + 2y + 3z - 4) + \lambda(2x + y - z + 5) = 0$$
$$\Rightarrow \quad (1 + 2\lambda)x + (2 + \lambda)y + (3 - \lambda)z - 4 + 5\lambda = 0 \quad \text{...}(i)$$

Plane (*i*) is perpendicular to P_1, then

$$5(1 + 2\lambda) + 3(2 + \lambda) + 6(3 - \lambda) = 0$$
$$\Rightarrow \quad 5 + 10\lambda + 6 + 3\lambda + 18 - 6\lambda = 0$$
$$\Rightarrow \quad 7\lambda + 29 = 0$$
$$\therefore \quad \lambda = \frac{-29}{7}$$

Putting the value of λ in eq. (*i*), we get

$$\left[1 + 2\left(\frac{-29}{7}\right)\right]x + \left[2 - \frac{29}{7}\right]y + \left[3 + \frac{29}{7}\right]z - 4 + 5\left(\frac{-29}{7}\right) = 0$$

$$\Rightarrow \frac{-15}{7}x - \frac{15}{7}y + \frac{50}{7}z - 4 - \frac{145}{7} = 0$$

$$\Rightarrow \quad -15x - 15y + 50z - 28 - 145 = 0$$

$$\Rightarrow \quad -15x - 15y + 50z - 173 = 0 \quad \Rightarrow \quad 51x + 15y - 50z + 173 = 0$$

Q23. The plane $ax + by = 0$ is rotated about its line of intersection with plane $z = 0$ through an angle α. Prove that the equation of the plane in its new position is $ax + by \pm \left(\sqrt{a^2 + b^2} \tan \alpha\right)z = 0$.

Sol. Given planes are:

$$ax + by = 0 \quad \text{...(i)}$$
$$z = 0 \quad \text{...(ii)}$$

Equation of any plane passing through the line of intersection of plane (*i*) and (*ii*) is

$$(ax + by) + kz = 0 \quad \Rightarrow \quad ax + by + kz = 0 \quad \text{...(iii)}$$

Dividing both sides by $\sqrt{a^2 + b^2 + k^2}$, we get

$$\frac{a}{\sqrt{a^2 + b^2 + k^2}}x + \frac{b}{\sqrt{a^2 + b^2 + k^2}}y + \frac{k}{\sqrt{a^2 + b^2 + k^2}}z = 0$$

∴ Direction cosines of the normal to the plane are

$$\frac{a}{\sqrt{a^2 + b^2 + k^2}}, \frac{b}{\sqrt{a^2 + b^2 + k^2}}, \frac{k}{\sqrt{a^2 + b^2 + k^2}}$$

and the direction cosines of the plane (*i*) are

$$\frac{a}{\sqrt{a^2 + b^2}}, \frac{b}{\sqrt{a^2 + b^2}}, 0$$

Since, α is the angle between the planes (*i*) and (*iii*), we get

$$\cos\alpha = \frac{a.a + b.b + k.0}{\sqrt{a^2 + b^2 + k^2}.\sqrt{a^2 + b^2}}$$

$$\Rightarrow \quad \cos\alpha = \frac{a^2 + b^2}{\sqrt{a^2 + b^2 + k^2}.\sqrt{a^2 + b^2}}$$

$$\Rightarrow \quad \cos\alpha = \frac{\sqrt{a^2 + b^2}}{\sqrt{a^2 + b^2 + k^2}} \quad \Rightarrow \cos^2\alpha = \frac{a^2 + b^2}{a^2 + b^2 + k^2}$$

$$\Rightarrow \quad (a^2 + b^2 + k^2)\cos^2\alpha = a^2 + b^2$$

$$\Rightarrow a^2\cos^2\alpha + b^2\cos^2\alpha + k^2\cos^2\alpha = a^2 + b^2$$

$$\Rightarrow \quad k^2\cos^2\alpha = a^2 - a^2\cos^2\alpha + b^2 - b^2\cos^2\alpha$$

$$\Rightarrow \quad k^2\cos^2\alpha = \alpha^2(1 - \cos^2\alpha) + b^2(1 - \cos^2\alpha)$$

$$\Rightarrow \quad k^2\cos^2\alpha = a^2\sin^2\alpha + b^2\sin^2\alpha$$

$$\Rightarrow \quad k^2\cos^2\alpha = (a^2 + b^2)\sin^2\alpha$$

$$\Rightarrow \quad k^2 = (a^2 + b^2)\frac{\sin^2\alpha}{\cos^2\alpha} \quad \Rightarrow k = \pm\sqrt{a^2 + b^2}.\tan\alpha$$

Putting the value of *k* in eq. (*iii*) we get

$ax + by \pm \left(\sqrt{a^2 + b^2}.\tan\alpha\right)z = 0$ which is the required equation of plane.

Hence proved.

Q24. Find the equation of the plane through the intersection of the planes $\vec{r}.(\hat{i} + 3\hat{j}) - 6 = 0$ and $\vec{r}.(3\hat{i} - \hat{j} - 4\hat{k}) = 0$, whose perpendicular distance from origin is unity.

Sol. Given planes are;

$$\vec{r}.(\hat{i}+3\hat{j})-6=0 \quad \Rightarrow \quad x+3y-6=0 \qquad ...(i)$$

and $$\vec{r}.(3\hat{i}-\hat{j}-4\hat{k})=0 \quad \Rightarrow \quad 3x-y-4z=0 \qquad ...(ii)$$

Equation of the plane passing through the line of intersection of plane (*i*) and (*ii*) is

$$(x+3y-6)+k(3x-y-4z)=0 \qquad ...(iii)$$

$$(1+3k)x+(3-k)y-4kz-6=0$$

Perpendicular distance from origin

$$=\left|\frac{-6}{\sqrt{(1+3k)^2+(3-k)^2+(-4k)^2}}\right|=1$$

$$\Rightarrow \frac{36}{1+9k^2+6k+9+k^2-6k+16k^2}=1 \text{ [Squaring both sides]}$$

$$\Rightarrow \frac{36}{26k^2+10}=1 \quad \Rightarrow 26k^2+10=36$$

$$\Rightarrow 26k^2=26 \quad \Rightarrow k^2=1 \quad \therefore k=\pm 1$$

Putting the value of *k* in eq. (*iii*) we get,

$$(x+3y-6)\pm(3x-y-4z)=0$$

$\Rightarrow x+3y-6+3x-y-4z=0$ and $x+3y-6-3x+y+4z=0$

$\Rightarrow 4x+2y-4z-6=0$ and $-2x+4y+4z-6=0$

Hence, the required equations are:

$4x+2y-4z-6=0$ and $-2x+4y+4z-6=0$.

Q25. Show that the points $(\hat{i}-\hat{j}+3\hat{k})$ and $3(\hat{i}+\hat{j}+\hat{k})$ are equidistant from the plane $\vec{r}.(5\hat{i}+2\hat{j}-7\hat{k})+9=0$ and lies on opposite side of it.

Sol. Given points are $P(\hat{i}-\hat{j}+3\hat{k})$ and $Q(3\hat{i}+3\hat{j}+3\hat{k})$ and the plane $\vec{r}.(5\hat{i}+2\hat{j}-7\hat{k})+9=0$

Perpendicular distance of $P(\hat{i}-\hat{j}+3\hat{k})$ from the plane

$$\vec{r}.(5\hat{i}+2\hat{j}-7\hat{k})+9=\left|\frac{(\hat{i}-\hat{j}+3\hat{k}).(5\hat{i}+2\hat{j}-7\hat{k})+9}{\sqrt{(5)^2+(2)^2+(-7)^2}}\right|$$

$$=\left|\frac{5-2-21+9}{\sqrt{25+4+49}}\right|=\left|\frac{-9}{\sqrt{78}}\right|$$

and perpendicular distance of $Q(3\hat{i}+3\hat{j}+3\hat{k})$ from the plane

$$=\left|\frac{(3\hat{i}+3\hat{j}+3\hat{k}).(5\hat{i}+2\hat{j}-7\hat{k})+9}{\sqrt{25+4+49}}\right|$$

$$=\left|\frac{15+6-21+9}{\sqrt{78}}\right|=\left|\frac{9}{\sqrt{78}}\right|$$

Hence, the two points are equidistant from the given plane. Opposite sign shows that they lie on either side of the plane.

Q26. $\overrightarrow{AB} = 3\hat{i} - \hat{j} + \hat{k}$ and $\overrightarrow{CD} = -3\hat{i} + 2\hat{j} + 4\hat{k}$ are two vectors. The position vectors of the points A and C are $6\hat{i} + 7\hat{j} + 4\hat{k}$ and $-9\hat{j} + 2\hat{k}$, respectively. Find the position vector of a point P on the line AB and a point Q on the line CD such that $\overrightarrow{PQ}$ is perpendicular to $\overrightarrow{AB}$ and $\overrightarrow{CD}$ both.

Sol. Position vector of A is $6\hat{i} + 7\hat{j} + 4\hat{k}$ and $\overrightarrow{AB} = 3\hat{i} - \hat{j} + \hat{k}$

So, equation of any line passing through A and parallel to $\overrightarrow{AB}$

$$\vec{r} = (6\hat{i} + 7\hat{j} + 4\hat{k}) + \lambda(3\hat{i} - \hat{j} + \hat{k}) \qquad ...(i)$$

Now any point P on $\overrightarrow{AB} = (6 + 3\lambda, 7 - \lambda, 4 + \lambda)$

Similarly, position vector of C is $-9\hat{j} + 2\hat{k}$

and $\overrightarrow{CD} = -3\hat{i} + 2\hat{j} + 4\hat{k}$

So, equation of any line passing through C and parallel to $\overrightarrow{CD}$ is

$$\vec{r} = (-9\hat{j} + 2\hat{k}) + \mu(-3\hat{i} + 2\hat{j} + 4\hat{k}) \qquad ...(ii)$$

Any point Q on $\overrightarrow{CD} = (-3\mu, -9 + 2\mu, 2 + 4\mu)$

d'ratios of $\overrightarrow{PQ}$ are

$$(6 + 3\lambda + 3\mu, 7 - \lambda + 9 - 2\mu, 4 + \lambda - 2 - 4\mu)$$

$\Rightarrow$ $(6 + 3\lambda + 3\mu), (16 - \lambda - 2\mu), (2 + \lambda - 4\mu)$

Now $\overrightarrow{PQ}$ is $\perp$ to eq. (*i*), then

$3(6 + 3\lambda + 3\mu) - 1(16 - \lambda - 2\mu) + 1(2 + \lambda - 4\mu) = 0$

$\Rightarrow \quad 18 + 9\lambda + 9\mu - 16 + \lambda + 2\mu + 2 + \lambda - 4\mu = 0$

$\Rightarrow \quad 11\lambda + 7\mu + 4 = 0 \qquad ...(iii)$

$\overrightarrow{PQ}$ is also $\perp$ to eq. (*ii*), then

$-3(6 + 3\lambda + 3\mu) + 2(16 - \lambda - 2\mu) + 4(2 + \lambda - 4\mu) = 0$

$\Rightarrow \quad -18 - 9\lambda - 9\mu + 32 - 2\lambda - 4\mu + 8 + 4\lambda - 16\mu = 0$

$\Rightarrow \quad -7\lambda - 29\mu + 22 = 0$

$\Rightarrow \quad 7\lambda + 29\mu - 22 = 0 \qquad ...(iv)$

Solving eq. (*iii*) and (*iv*) we get

$$\begin{array}{rrrr} 77\lambda & + \; 49\mu & + \; 28 & = 0 \\ 77\lambda & + 319\mu & - 242 & = 0 \\ (-) & (-) & (+) & \\ \hline & -270\mu & + 270 & = 0 \end{array}$$

$\therefore \mu = 1$

Now using $\mu = 1$ in eq. (*iv*) we get

$$7\lambda + 29 - 22 = 0 \quad \Rightarrow \quad \lambda = -1$$

$\therefore$ Position vector of $P = [6 + 3(-1), 7 + 1, 4 - 1] = (3, 8, 3)$

and position vector of $Q = [-3(1), -9 + 2(1), 2 + 4(1)] = (-3, -7, 6)$

Hence, the position vectors of

$$P = 3\hat{i} + 8\hat{j} + 3\hat{k} \text{ and } Q = -3\hat{i} - 7\hat{j} + 6\hat{k}$$

Q27. Show that the straight lines whose direction cosines are given by $2l + 2m - n = 0$ and $mn + nl + lm = 0$ are at right angles.

Sol. Given that $2l + 2m - n = 0$ ...(*i*)

and $mn + nl + lm = 0$ ...(*ii*)

Eliminating m from eq. (*i*) and (*ii*) we get,

$$m = \frac{n-2l}{2} \qquad \text{[from (}i\text{)]}$$

$$\Rightarrow \left(\frac{n-2l}{2}\right)n + nl + l\left(\frac{n-2l}{2}\right) = 0$$

$$\Rightarrow \frac{n^2 - 2nl + 2nl + nl - 2l^2}{2} = 0$$

$$\Rightarrow n^2 + nl - 2l^2 = 0$$

$$\Rightarrow n^2 + 2nl - nl - 2l^2 = 0$$

$$\Rightarrow n(n + 2l) - l(n + 2l) = 0$$

$$\Rightarrow (n - l)(n + 2l) = 0$$

$$\Rightarrow n = -2l \quad \text{and} \quad n = l$$

$$\therefore \quad m = \frac{-2l - 2l}{2}, \quad m = \frac{l - 2l}{2}$$

$$\Rightarrow m = -2\,l, \quad m = \frac{-l}{2}$$

Therefore, the direction ratios are proportional to $l, -2l, -2l$ and $l, \frac{-l}{2}, l$.

$\Rightarrow$ $1, -2, -2$ and $2, -1, 2$

If the two lines are perpendicular to each other then

$$1(2) - 2(-1) - 2 \times 2 = 0$$

$$2 + 2 - 4 = 0$$

$$0 = 0$$

Hence, the two lines are perpendicular.

Q28. If $l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3$ are the direction cosines of three mutually perpendicular lines, prove that the line whose direction cosines are proportional to $l_1 + l_2 + l_3, m_1 + m_2 + m_3, n_1 + n_2 + n_3$, makes equal angles with them.

Sol. Let $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are such that

$$\vec{a} = l_1\hat{i} + m_1\hat{j} + n_1\hat{k}$$

$$\vec{b} = l_2\hat{i} + m_2\hat{j} + n_2\hat{k}$$

$$\vec{c} = l_3\hat{i} + m_3\hat{j} + n_3\hat{k}$$

and $\vec{d} = (l_1 + l_2 + l_3)\hat{i} + (m_1 + m_2 + m_3)\hat{j} + (n_1 + n_2 + n_3)\hat{k}$

Since the given d'cosines are mutually perpendicular then

$$l_1l_2 + m_1m_2 + n_1n_2 = 0$$
$$l_2l_3 + m_2m_3 + n_2n_3 = 0$$
$$l_1l_3 + m_1m_3 + n_1n_3 = 0$$

Let α, β and γ be the angles between $\vec{a}$ and $\vec{d}$, $\vec{b}$ and $\vec{d}$, $\vec{c}$ and $\vec{d}$ respectively.

$\therefore \cos\alpha = l_1(l_1 + l_2 + l_3) + m_1(m_1 + m_2 + m_3) + n_1(n_1 + n_2 + n_3)$

$= l_1^2 + l_1l_2 + l_1l_3 + m_1^2 + m_1m_2 + m_1m_3 + n_1^2 + n_1n_2 + n_1n_3$

$= (l_1^2 + m_1^2 + n_1^2) + (l_1l_2 + m_1m_2 + n_1n_2) + (l_1l_3 + m_1m_3 + n_1n_3)$

$= 1 + 0 + 0 = 1$

$\therefore \cos\beta = l_2(l_1 + l_2 + l_3) + m_2(m_1 + m_2 + m_3) + n_2(n_1 + n_2 + n_3)$

$= l_1l_2 + l_2^2 + l_2l_3 + m_1m_2 + m_2^2 + m_2m_3 + n_1n_2 + n_2^2 + n_2n_3$

$= (l_2^2 + m_2^2 + n_2^2) + (l_1l_2 + m_1m_2 + n_1n_2) + (l_2l_3 + m_2m_3 + n_2n_3)$

$= 1 + 0 + 0 = 1$

Similarly,

$\therefore \cos\gamma = l_3(l_1 + l_2 + l_3) + m_3(m_1 + m_2 + m_3) + n_3(n_1 + n_2 + n_3)$

$= l_1l_3 + l_2l_3 + l_3^2 + m_1m_3 + m_2m_3 + m_3^2 + n_1n_3 + n_2n_3 + n_3^2$

$= (l_3^2 + m_3^2 + n_3^2) + (l_1l_3 + m_1m_3 + n_1n_3) + (l_2l_3 + m_2m_3 + n_2n_3)$

$= 1 + 0 + 0 = 1$

$\therefore \cos\alpha = \cos\beta = \cos\gamma = 1 \Rightarrow \alpha = \beta = \gamma$ which is the required result.

OBJECTIVE TYPE QUESTIONS

Choose the correct answer from the given four options in each of the Exercises from 29 to 36.

Q29. Distance of the point (α, β, γ) from y-axis is

(*a*) β (*b*) $|\beta|$ (*c*) $|\beta| + |\gamma|$ (*d*) $\sqrt{\alpha^2 + \gamma^2}$

Sol. The given point is (α, β, γ)

Any point on y-axis $= (0, \beta, 0)$

$\therefore$ Required distance $= \sqrt{(\alpha - 0)^2 + (\beta - \beta)^2 + (\gamma - 0)^2}$

$= \sqrt{\alpha^2 + \gamma^2}$

Hence, the correct option is (*d*).

Q30. If the direction cosines of a line are k, k, k, then

(a) $k > 0$ (b) $0 < k < 1$ (c) $k = 1$ (d) $k = \frac{1}{\sqrt{3}}$ or $\frac{-1}{\sqrt{3}}$

Sol. If l, m, n are the direction cosines of a line, then

$$l^2 + m^2 + n^2 = 1$$

So, $$k^2 + k^2 + k^2 = 1$$

$$\Rightarrow \quad 3k^2 = 1 \quad \Rightarrow \quad k = \pm\frac{1}{\sqrt{3}}$$

Hence, the correct option is (d).

Q31. The distance of the plane $\vec{r}.\left(\frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} - \frac{6}{7}\hat{k}\right) = 1$ from the origin is

(a) 1 (b) 7 (c) $\frac{1}{7}$ (d) None of these

Sol. Given that: $\vec{r}.\left(\frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} - \frac{6}{7}\hat{k}\right) = 1$

So, the distance of the given plane from the origin is

$$= \left|\frac{-1}{\sqrt{\left(\frac{2}{7}\right)^2 + \left(\frac{3}{7}\right)^2 + \left(\frac{-6}{7}\right)^2}}\right| = \left|\frac{-1}{\sqrt{\frac{4}{49} + \frac{9}{49} + \frac{36}{49}}}\right| = \left|\frac{1}{1}\right| = 1$$

Hence, the correct option is (a).

Q32. The sine of the angle between the straight line $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ and the plane $2x - 2y + z = 5$ is

(a) $\frac{10}{6\sqrt{5}}$ (b) $\frac{4}{5\sqrt{2}}$ (c) $\frac{2\sqrt{3}}{5}$ (d) $\frac{\sqrt{2}}{10}$

Sol. Given that: $l : \frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$

and $\quad P : 2x - 2y + z = 5$

d'ratios of the line are 3, 4, 5

and d'ratios of the normal to the plane are 2, – 2, 1

$$\therefore \quad \sin\theta = \frac{3(2) + 4(-2) + 5(1)}{\sqrt{9+16+25}.\sqrt{4+4+1}}$$

$$\Rightarrow \quad \sin\theta = \frac{6-8+5}{\sqrt{50}.3} \quad \Rightarrow \quad \frac{3}{5\sqrt{2}.3} = \frac{1}{5\sqrt{2}} = \frac{\sqrt{2}}{10}$$

Hence, the correct option is (d).

Q33. The reflection of the point (α, β, γ) in the xy-plane is

(a) $(\alpha,\beta,0)$ (b) $(0,0,\gamma)$ (c) $(-\alpha,-\beta,\gamma)$ (d) $(\alpha,\beta,-\gamma)$

Sol. Reflection of point (α, β, γ) in xy-plane is $(\alpha,\beta,-\gamma)$.

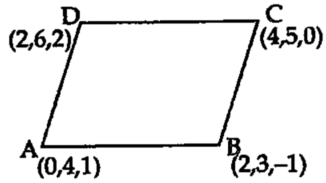
Hence, the correct option is (d).

Q34. The area of the quadrilateral ABCD, where A(0,4,1), B(2,3,–1), C(4,5,0) and D(2,6,2) is equal to

(*a*) 9 sq. units (*b*) 18 sq. units
(*c*) 27 sq. units (*d*) 81 sq. units

Sol. Given points are
A(0, 4, 1), B(2,3,– 1), C(4, 5, 0) and D(2,6,2)

d'ratios of AB $= 2, -1 -2$
and d'ratios of DC $= 2, -1, -2$
$\therefore$ AB || DC
Similarly, d'ratios of AD = 2, 2, 1
and d'ratios of BC = 2, 2, 1
$\therefore$ AD || BC
So □ABCD is a parallelogram.



$$\overline{AB} = 2\hat{i} - \hat{j} - 2\hat{k}$$
$$\overline{AD} = 2\hat{i} + 2\hat{j} + \hat{k}$$

$\therefore$ Area of parallelogram ABCD $= |\overline{AB} \times \overline{AD}|$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{vmatrix} = \hat{i}(-1+4) - \hat{j}(2+4) + \hat{k}(4+2) = 3\hat{i} - 6\hat{j} + 6\hat{k}$$

$$= \sqrt{(3)^2 + (-6)^2 + (6)^2} = \sqrt{9+36+36} = \sqrt{81} = 9 \text{ sq units}$$

Hence, the correct option is (*a*).

Q35. The locus represented by $xy + yz = 0$ is
(*a*) A pair of perpendicular lines
(*b*) A pair of parallel lines
(*c*) A pair of parallel planes
(*d*) A pair of perpendicular planes

Sol. Given that: $xy + yz = 0$
$$y.(x+z) = 0$$
$$y = 0 \text{ or } x + z = 0$$

Here $y = 0$ is one plane and $x + z = 0$ is another plane. So, it is a pair of perpendicular planes.
Hence, the correct option is (*d*).

Q36. The plane $2x - 3y + 6z - 11 = 0$ makes an angle $\sin^{-1}(\alpha)$ with x-axis. The value of α is equal to

(*a*) $\dfrac{\sqrt{3}}{2}$ (*b*) $\dfrac{\sqrt{2}}{3}$ (*c*) $\dfrac{2}{7}$ (*d*) $\dfrac{3}{7}$

Sol. Direction ratios of the normal to the plane $2x - 3y + 6z - 11 = 0$ are $2, -3, 6$
Direction ratios of x-axis are 1, 0, 0

$\therefore$ angle between plane and line is

$$\sin\theta = \frac{2(1)-3(0)+6(0)}{\sqrt{(2)^2+(-3)^2+(6^2)}.\sqrt{(1)^2+(0)^2+(0)^2}}$$

$$= \frac{2}{\sqrt{4+9+36}} = \frac{2}{7}$$

Hence, the correct option is (*c*).

Fill in the blanks in each of the Exercises from 37 to 41.

Q37. A plane passes through the points (2, 0, 0), (0, 3, 0) and (0, 0, 4). The equation of plane is ....

Sol. Given points are (2, 0, 0), (0, 3, 0) and (0, 0, 4).
So, the intercepts cut by the plane on the axes are 2, 3, 4
Equation of the plane (intercept form) is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \quad \Rightarrow \quad \frac{x}{2}+\frac{y}{3}+\frac{z}{4}=1$$

Hence, the equation of plane is $\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=1$.

Q38. The direction cosines of vector $(2\hat{i}+2\hat{j}-\hat{k})$ are ....

Sol. Let $\vec{a} = 2\hat{i}+2\hat{j}-\hat{k}$
direction ratios of $\vec{a}$ are 2, 2, – 1

So, the direction cosines are $\frac{2}{\sqrt{4+4+1}}, \frac{2}{\sqrt{4+4+1}}, \frac{-1}{\sqrt{4+4+1}}$

$$\Rightarrow \frac{2}{3}, \frac{2}{3}, \frac{-1}{3}$$

Hence, the direction cosines of the given vector are $\frac{2}{3}, \frac{2}{3}, \frac{-1}{3}$.

Q39. The vector equation of the line $\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-6}{2}$ is ....

Sol. The given equation is

$$\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-6}{2}$$

Here $\vec{a} = (5\hat{i}-4\hat{j}+6\hat{k})$ and $\vec{b} = (3\hat{i}+7\hat{j}+2\hat{k})$

Equation of the line is $\vec{r}=\vec{a}+\vec{b}\lambda$

Hence, the vector equation of the given line is

$$\vec{r} = (5\hat{i}-4\hat{j}+6\hat{k})+\lambda(3\hat{i}+7\hat{j}+2\hat{k})$$

Q40. The vector equation of the line through the points (3, 4, – 7) and (1, – 1, 6) is ....

Sol. Given the points (3, 4, – 7) and (1, – 1, 6)

Here $\vec{a} = 3\hat{i} + 4\hat{j} - 7\hat{k}$ and $\vec{b} = \hat{i} - \hat{j} + 6\hat{k}$

Equation of the line is $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$

$\Rightarrow \vec{r} = (3\hat{i} + 4\hat{j} - 7\hat{k}) + \lambda\left[(\hat{i} - \hat{j} + 6\hat{k}) - (3\hat{i} + 4\hat{j} - 7\hat{k})\right]$

$\Rightarrow \vec{r} = (3\hat{i} + 4\hat{j} - 7\hat{k}) + \lambda(-2\hat{i} - 5\hat{j} + 13\hat{k})$

$\Rightarrow (x\hat{i} + y\hat{j} + z\hat{k}) = (3\hat{i} + 4\hat{j} - 7\hat{k}) + \lambda(-2\hat{i} - 5\hat{j} + 13\hat{k})$

$\Rightarrow (x-3)\hat{i} + (y-4)\hat{j} + (z+7)\hat{k} = \lambda(-2\hat{i} - 5\hat{j} + 13\hat{k})$

Hence, the vector equation of the line is

$(x-3)\hat{i} + (y-4)\hat{j} + (z+7)\hat{k} = \lambda(-2\hat{i} - 5\hat{j} + 13\hat{k})$

Q41. The Cartesian equation of the plane $\vec{r}.(\hat{i} + \hat{j} - \hat{k}) = 2$ is ....

Sol. Given equation is $\vec{r}.(\hat{i} + \hat{j} - \hat{k}) = 2$

$\Rightarrow \quad (x\hat{i} + y\hat{j} + z\hat{k}).(\hat{i} + \hat{j} - \hat{k}) = 2$

$\Rightarrow \quad x + y - z = 2$

Hence, the Cartesian equation of the plane is $x + y - z = 2$.

State True or False for the statements in each of the Exercises from 42 to 49.

Q42. The unit vector normal to the plane $x + 2y + 3z - 6 = 0$ is $\frac{1}{\sqrt{14}}\hat{i} + \frac{2}{\sqrt{14}}\hat{j} + \frac{3}{\sqrt{14}}\hat{k}$

Sol. Given plane is $x + 2y + 3z - 6 = 0$

Vector normal to the plane $\vec{n} = \hat{i} + 2\hat{j} + 3\hat{k}$

$$\therefore \quad \hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{\hat{i} + 2\hat{j} + 3\hat{k}}{\sqrt{(1)^2 + (2)^2 + (3)^2}} = \frac{1}{\sqrt{14}}\hat{i} + \frac{2}{\sqrt{14}}\hat{j} + \frac{3}{\sqrt{14}}\hat{k}$$

Hence, the given statement is 'true'.

Q43. The intercepts made by the plane $2x - 3y + 5z + 4 = 0$ on the coordinate axes are $-2, \frac{4}{3}, \frac{-4}{5}$.

Sol. Equation of the plane is $2x - 3y + 5z + 4 = 0$

$\Rightarrow \quad 2x - 3y + 5z = -4$

$$\Rightarrow \quad \frac{2}{-4}x - \frac{3y}{-4} + \frac{5z}{-4} = 1$$

$$\Rightarrow \quad \frac{x}{-2} - \frac{y}{4/3} + \frac{z}{-4/5} = 1$$

So, the required intercepts are $-2, \frac{4}{3}$ and $-\frac{4}{5}$

Hence, the given statement is 'true'.

Q44. The angle between the line $\vec{r} = (5\hat{i} - \hat{j} - 4\hat{k}) + \lambda(2\hat{i} - \hat{j} + \hat{k})$ and the plane $\vec{r}.(3\hat{i}-4\hat{j}-\hat{k})+5=0$ is $\sin^{-1}\left(\frac{5}{2\sqrt{91}}\right)$.

Sol. Equation of line is $\vec{r}=(5\hat{i}-\hat{j}-4\hat{k})+\lambda(2\hat{i}-\hat{j}+\hat{k})$ and the equation of the plane is $\vec{r}.(3\hat{i}-4\hat{j}-\hat{k})+5=0$

Here, $\vec{b}_1 = 2\hat{i}-\hat{j}+\hat{k}$ and $\vec{n}_2 = 3\hat{i}-4\hat{j}-\hat{k}$

$$\therefore \quad \sin\theta = \frac{\vec{b}_1 \vec{n}_2}{|\vec{b}_1||\vec{n}_2|}$$

$$\Rightarrow \quad \sin\theta = \frac{(2\hat{i}-\hat{j}+\hat{k}).(3\hat{i}-4\hat{j}-\hat{k})}{\sqrt{4+1+1}.\sqrt{9+16+1}} = \frac{6+4-1}{\sqrt{6}.\sqrt{26}} = \frac{9}{\sqrt{6}.\sqrt{26}}$$

$$\Rightarrow \quad \sin\theta = \frac{9}{2\sqrt{39}}$$ which is false.

Hence, the given statement is 'false'.

Q45. The angle between the planes $\vec{r}.(2\hat{i}-3\hat{j}+\hat{k})=1$ and $\vec{r}.(\hat{i}-\hat{j})=4$ is $\cos^{-1}\left(\frac{-5}{\sqrt{58}}\right)$.

Sol. The given planes are $\vec{r}.(2\hat{i}-3\hat{j}+\hat{k})=1$ and $\vec{r}.(\hat{i}-\hat{j})=4$

Here, $\vec{b}_1 = 2\hat{i}-3\hat{j}+\hat{k}$ and $\vec{b}_2 = (\hat{i}-\hat{j})$

$$\text{So,} \quad \cos\theta = \frac{\vec{b}_1 . \vec{n}_2}{|\vec{b}_1||\vec{n}_2|}$$

$$\Rightarrow \quad \cos\theta = \frac{(2i-3j+\hat{k}).(\hat{i}-\hat{j})}{\sqrt{4+9+1}.\sqrt{1+1}} = \frac{2+3}{\sqrt{14}.\sqrt{2}} = \frac{5}{\sqrt{28}}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{5}{\sqrt{28}}\right)$$ which is false.

Hence, the given statement is 'false'.

Q46. The line $\vec{r}=2\hat{i}-3\hat{j}-\hat{k}+\lambda(\hat{i}-\hat{j}+2\hat{k})$ lies in the plane $r\cdot(3\hat{i}+\hat{j}-\hat{k})+2=0$.

Sol. Direction ratios of the line $(\hat{i}-\hat{j}+2\hat{k})$

Direction ratios of the normal to the plane are $(3\hat{i}+\hat{j}-\hat{k})$

So $(\hat{i}-\hat{j}+2\hat{k}).(3\hat{i}+\hat{j}-\hat{k}) = 3-1-2=0$

Therefore, the line is parallel to the plane.

Now point through which the line is passing

$\vec{a} = 2\hat{i}-3\hat{j}-\hat{k}$

If line lies in the plane then

$(2\hat{i} - 3\hat{j} - \hat{k}).(3\hat{i} + \hat{j} - \hat{k}) + 2 = 0$

$6 - 3 + 1 + 2 \neq 0$

So, the line does not lie in the plane.

Hence, the given statement is 'false'.

Q47. The vector equation of the line $\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}$ is $\vec{r} = 5\hat{i} - 4\hat{j} + 6\hat{k} + \lambda(3\hat{i} + 7\hat{j} + 2\hat{k})$.

Sol. The Cartesian form of the equation is

$\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2} = \lambda$

$\therefore$ Here $x_1 = 5, y_1 = -4, z_1 = 6, a = 3, b = 7, c = 2$

So, the vector equation is $\vec{r} = (5\hat{i} - 4\hat{j} + 6\hat{k}) + \lambda(3\hat{i} + 7\hat{j} + 2\hat{k})$

Hence, the given statement is 'true'.

Q48. The equation of a line, which is parallel to $2\hat{i} + \hat{j} + 3\hat{k}$ and which passes through the point $(5, -2, 4)$ is $\frac{x-5}{2} = \frac{y+2}{-1} = \frac{z-4}{3}$.

Sol. Here, $x_1 = 5, y_1 = -2, z_1 = 4; a = 2, b = 1, c = 3$

We know that the equation of line is $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$

$\Rightarrow \quad \frac{x-5}{2} = \frac{y+2}{1} = \frac{z-4}{3}$

Hence, the given statement is 'false'.

Q49. If the foot of the perpendicular drawn from the origin to a plane is $(5, -3, -2)$, then the equation of plane is $\vec{r}.(5\hat{i} - 3\hat{j} - 2\hat{k}) = 38$.

Sol. The given equation of the plane is $\vec{r}.(5\hat{i} - 3\hat{j} - 2\hat{k}) = 38$

If the foot of the perpendicular to this plane is $(5, -3, -2)$ i.e., $5\hat{i} - 3\hat{j} - 2\hat{k}$ then

$(5\hat{i} - 3\hat{j} - 2\hat{k}).(5\hat{i} - 3\hat{j} - 2\hat{k}) = 38$

$\Rightarrow \quad 25 + 9 + 4 = 38$

$38 = 38$ (satisfied)

Hence, the given statement is 'true'.

12 Linear Programming

12.3 EXERCISE

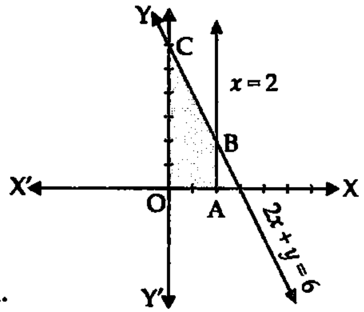
SHORT ANSWER TYPE QUESTIONS

Q1. Determine the maximum value of $Z = 11x + 7y$ subject to the constraints:
$2x + y \leq 6, x \leq 2, x \geq 0, y \geq 0$

Sol. Given that: $Z = 11x + 7y$ and the constraints $2x + y \leq 6$, $x \leq 2$, $x \geq 0, y \geq 0$

Let $2x + y = 6$

x	0	3
y	6	0



The shaded area OABC is the feasible region determined by the constraints
$2x + y \leq 6, x \leq 2, x \geq 0, y \geq 0$
The feasible region is bounded.
So, maximum value will occur at a corner point of the feasible region.
Corner points are (0, 0), (2, 0), (2, 2) and (0, 6).
Now, evaluating the value of Z, we get

Corner points	*Value of Z*	
O(0, 0)	$11(0) + 7(0) = 0$	
A(2, 0)	$11(2) + 7(0) = 22$	
B(2, 2)	$11(2) + 7(2) = 36$	
C(0, 6)	$11(0) + 7(6) = 42$	← Maximum

Hence, the maximum value of Z is 42 at (0, 6).

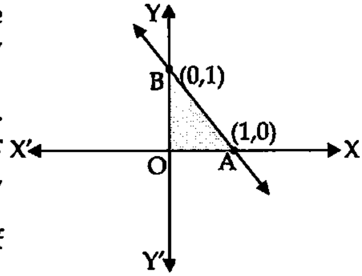
Q2. Maximise $Z = 3x + 4y$, subject to the constraints: $x + y \leq 1, x \geq 0$, $y \geq 0$.

Sol. Given that: $Z = 3x + 4y$ and the constraints $x + y \leq 1, x \geq 0, y \geq 0$

Let $x + y = 1$

x	1	0
y	0	1

The shaded area OAB is the feasible region determined by $x + y \leq 1, x \geq 0, y \geq 0$.
The feasible region is bounded.
So, maximum value will occur at the corner points O(0, 0), A(1, 0), B(0, 1).
Now, evaluating the value of Z, we get

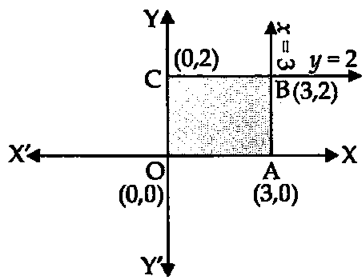


Corner points	*Value of Z*	
O(0, 0)	$3(0) + 4(0) = 0$	
A(1, 0)	$3(1) + 4(0) = 3$	
B(0, 1)	$3(0) + 4(1) = 4$	← Maximum

Hence, the maximum value of Z is 4 at (0, 1).

Q3. Maximise the function $Z = 11x + 7y$ subject to the constraints: $x \leq 3, y \leq 2, x \geq 0, y \geq 0$.

Sol. The shaded region is the feasible region determined by the constraints $x \leq 3, y \leq 2, x \geq 0, y \geq 0$.
The feasible region is bounded with four corners O(0, 0), A(3, 0), B(3, 2) and C(0, 2).
So, the maximum value can occur at any corner.
Let us evaluate the value of Z.



Corner points	*Value of Z*	
O(0, 0)	$11(0) + 7(0) = 0$	
A(3, 0)	$11(3) + 7(0) = 33$	
B(3, 2)	$11(3) + 7(2) = 47$	← Maximum
C(0, 2)	$11(0) + 7(2) = 14$	

Hence, the maximum value of the function Z is 47 at (3, 2).

Q4. Minimise $Z = 13x - 15y$ subject to the constraints: $x + y \leq 7, 2x - 3y + 6 \geq 0, x \geq 0, y \geq 0$.

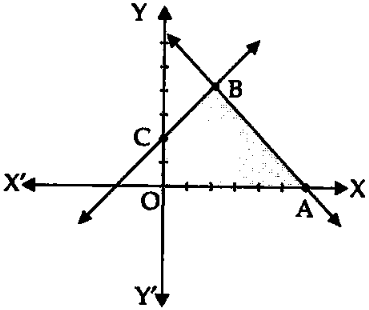
Sol. Given that: $Z = 13x - 15y$ and the constraints $x + y \leq 7, 2x - 3y + 6 \geq 0, x \geq 0, y \geq 0$

Let $x + y = 7$

x	3	4
y	4	3

Let $2x - 3y + 6 = 0$

x	1	-3
y	2	0



The shaded region is the feasible region determined by the constraints $x + y \le 7$, $2x - 3y + 6 \ge 0$, $x \ge 0$, $y \ge 0$

The feasible region is bounded with four corners O(0, 0), A(7, 0), B(3, 4), C(0, 2)

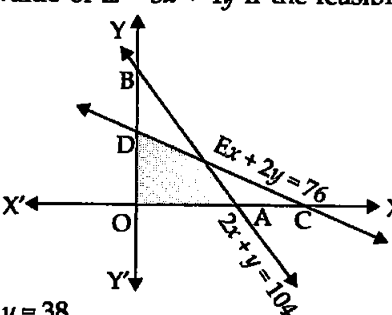
So, the maximum value can occur at any corner.

Let us evaluate the value of Z.

Corner points	*Value of Z*	
O(0, 0)	$13(0) - 15(0) = 0$	
A(7, 0)	$13(7) - 15(0) = 91$	
B(3, 4)	$13(3) - 15(4) = -21$	
C(0, 2)	$13(0) - 15(2) = -30$	$\leftarrow$ Minimum

Hence, the minimum value of Z is -30 at (0, 2).

Q5. Determine the maximum value of $Z = 3x + 4y$ if the feasible region (shaded) for a LPP is shown in figure.



Sol. As shown in the figure, OAED is the feasible region.

At A, $y = 0$ $\therefore$ $2x + y = 104$

$\Rightarrow x = 52$

Which gives corner point $A = (52, 0)$

At D, $x = 0$ $\therefore$ $x + 2y = 76 \Rightarrow y = 38$

Which gives corner point $D = (0, 38)$

Now solving the given equations, we get

$$\begin{array}{rcl}
x + 2y & = & 76 \\
2x + y & = & 104 \\
2x + 4y & = & 152 \\
2x + y & = & 104 \\
(-)\quad (-) & & (-) \\
\hline
3y & = & 48 \quad \Rightarrow y = 16
\end{array}$$

$$x + 2(16) = 76$$
$$\Rightarrow \quad x = 76 - 32 = 44$$

So, the corner point E = (44, 16)

Evaluating the maximum value of Z, we get

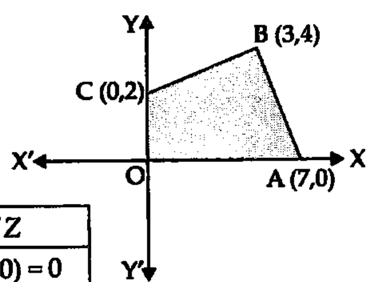
Corner points	$Z = 3x + 4y$	
O(0, 0)	$Z = 3(0) + 4(0) = 0$	
A(52, 0)	$Z = 3(52) + 4(0) = 156$	
E(44, 16)	$Z = 3(44) + 4(16) = 196$	← Maximum
D(0, 38)	$Z = 3(0) + 4(38) = 152$	

Hence, the maximum value of Z is 196 at (44, 16).

Q6. Feasible region (shaded) for a LPP is shown in figure. Maximise $Z = 5x + 7y$.

Sol. OABC is the feasible region whose corner points are O(0, 0), A(7, 0), B(3, 4) and C(0, 2)

Evaluating the value of Z, we get



Corner points	*Value of Z*	
O(0, 0)	$Z = 5(0) + 7(0) = 0$	
A(7, 0)	$Z = 5(7) + 7(0) = 35$	
B(3, 4)	$Z = 5(3) + 7(4) = 43$	← Maximum
C(0, 2)	$Z = 5(0) + 7(2) = 14$	

Hence, the maximum value of Z is 43 at (3, 4).

Q7. The feasible region for a LPP is shown in figure. Find the minimum value of $Z = 11x + 7y$.

Sol. As per the given figure, ABCA is the feasible region. Corner points C(0, 3), B(0, 5) and for A, we have to solve equations

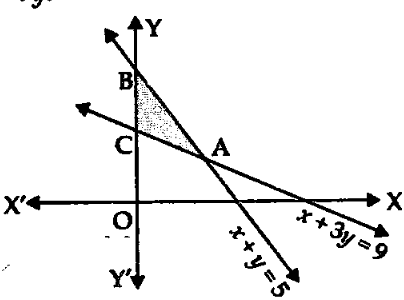
$$x + 3y = 9$$

and $$x + y = 5$$

Which gives $x = 3, y = 2$

i.e., A(3, 2)

Evaluating the value of Z, we get



Corner points	Value of Z
A(3, 2)	$Z = 11(3) + 7(2) = 47$
B(0, 5)	$Z = 11(0) + 7(5) = 35$
C(0, 3)	$Z = 11(0) + 7(3) = 21$ ← Minimum

Hence, the minimum value of Z is 21 at (0, 3).

Q8. Refer to Exercise 7 above. Find the maximum value of Z.

Sol. As per the evaluating table for the value of Z, it is clear that the maximum value of Z is 47 at (3, 2).

Q9. The feasible region for a LPP is shown in figure. Evaluate $Z = 4x + y$ at each of the corner points of this region. Find the minimum value of Z, if it exists.

Sol. As per the given figure, ABC is the feasible region which is open unbounded.

Here, we have

$$x + y = 3 \quad ...(i)$$

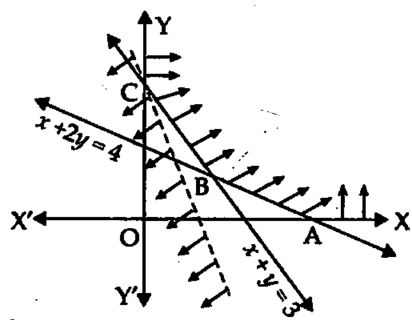
and $$x + 2y = 4 \quad ...(ii)$$

$$Z = 4x + y$$

Solving eq. (*i*) and (*ii*), we get

$x = 2$ and $y = 1$

So, the corner points are A(4, 0), B(2, 1) and C(0, 3)



Let us evaluate the value of Z

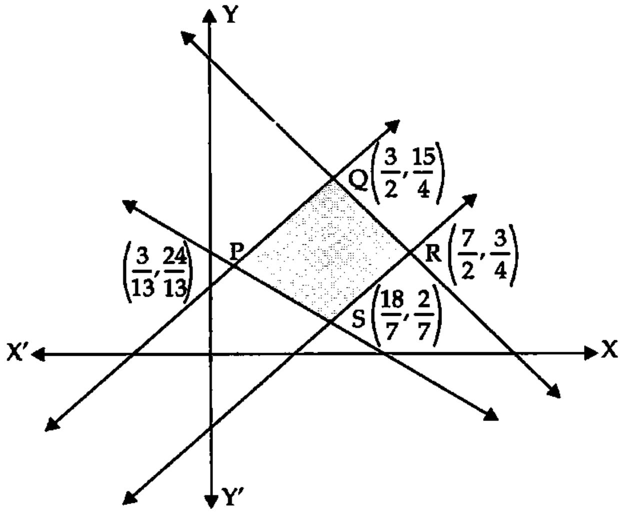
Corner points	$Z = 4x + y$
A(4, 0)	$Z = 4(4) + (0) = 16$
B(2, 1)	$Z = 4(2) + (1) = 9$
C(0, 3)	$Z = 4(0) + (3) = 3$ ← Minimum

Now, the minimum value of Z is 3 at (0, 3) but since, the feasible region is open bounded so it may or may not be the minimum value of Z.

Therefore, to face such situation, we draw a graph of $4x + y < 3$ and check whether the resulting open half plane has no point in common with feasible region. Otherwise Z will have no minimum value. From the graph, we conclude that there is no common point with the feasible region.

Hence, Z has the minimum value 3 at (0, 3).

Q10. In given figure, the feasible region (shaded) for a LPP is shown. Determine the maximum and minimum value of $Z = x + 2y$.



Sol. Here, corner points are given as follows:

$R\left(\frac{7}{2},\frac{3}{4}\right)$, $Q\left(\frac{3}{2},\frac{15}{4}\right)$, $P\left(\frac{3}{13},\frac{24}{13}\right)$ and $S\left(\frac{18}{7},\frac{2}{7}\right)$.

Now, evaluating the value of Z for the feasible region RQPS.

Corner points	*Value of* $Z = x + 2y$	
$R\left(\frac{7}{2},\frac{3}{4}\right)$	$Z=\frac{7}{2}+2\left(\frac{3}{4}\right)=5$	
$Q\left(\frac{3}{2},\frac{15}{4}\right)$	$Z=\frac{3}{2}+2\left(\frac{15}{4}\right)=9$	← Maximum
$P\left(\frac{3}{13},\frac{24}{13}\right)$	$Z=\frac{3}{13}+2\left(\frac{24}{13}\right)=\frac{51}{13}$	
$S\left(\frac{18}{7},\frac{2}{7}\right)$	$Z=\frac{18}{7}+2\left(\frac{2}{7}\right)=\frac{22}{7}$	← Minimum

Hence, the maximum value of Z is 9 at $\left(\frac{3}{2},\frac{15}{4}\right)$ and the minimum value of Z is $\frac{22}{7}$ at $\left(\frac{18}{7},\frac{2}{7}\right)$.

Q11. A manufacturer of electric circuits has a stock of 200 resistors, 120 transistors and 150 capacitors and is required to produce two types of circuits A and B. Type A requires 20 resistors, 10 transistors and 10 capacitors. Type B requires 10 resistors, 20 transistors and 30 capacitors. If the profit on type A circuit is ₹ 50 and that on type B circuit is ₹ 60, formulate this problem as a LPP so that the manufacturer can maximise his profit.

Sol. Let x units of type A and y units of type B electric circuits be produced by the manufacturer.

As per the given information, we construct the following table:

Items	*Type A (x)*	*Type B (y)*	*Maximum stock*
Resistors	20	10	200
Transistors	10	20	120
Capacitors	10	30	150
Profit	₹ 50	₹ 60	$Z = 50x + 60y$

Now, we have the total profit in rupees $Z = 50x + 60y$ to maximise subject to the constraints

$20x + 10y \le 200$ ...(*i*); $10x + 20y \le 120$ ...(*ii*)

$10x + 30y \le 150$ ...(*iii*); $x \ge 0, y \ge 0$ ...(*iv*)

Hence, the required LPP is

Maximise $Z = 50x + 60y$ subject to the constraints

$20x + 10y \le 200 \Rightarrow 2x + y \le 20$; $10x + 20y \le 120 \Rightarrow x + 2y \le 12$

and $10x + 30y \le 150 \Rightarrow x + 3y \le 15, x \ge 0, y \ge 0$

Q12. A firm has to transport 1200 packages using large vans which can carry 200 packages each and small vans which can take 80 packages each. The cost for engaging each large van is ₹ 400 and each small van is ₹ 200. Not more than ₹ 3000 is to be spent on the job and the number of large vans cannot exceed the number of small vans. Formulate this problem as a LPP given that the objective is to minimise cost.

Sol. Let x and y be the number of large and small vans respectively. From the given information, we construct the following corresponding constraints table;

Items	*Large vans (x)*	*Small vans (y)*	*Maximum/ Minimum*
Packages	200	80	1200
Cost	400	200	3000

Now the objective function for minimum cost is

$$Z = 400x + 200y$$

Subject to the constraints;

$$200x + 80y \ge 1200 \Rightarrow 5x + 2y \ge 30 \quad ...(i)$$

$$400x + 200y \le 3000 \Rightarrow 2x + y \le 15 \quad ...(ii)$$

$$x \le y \quad ...(iii)$$

and $x \ge 0, y \ge 0$ (non-negative constraints)

Hence, the required LPP is to minimise $Z = 400x + 200y$

Subject to the constraints $5x + 2y \ge 30$, $2x + y \le 15$, $x \le y$ and $x \ge 0, y \ge 0$.

Q13. A company manufactures two types of screws A and B. All the screws have to pass through a threading machine and a slotting machine. A box of type A screws requires 2 minutes on the threading machine and 3 minutes on the slotting machine. A box of type B screws requires 8 minutes of threading on threading machine and 2 minutes on the slotting machine. In a week, each machine is available for 60 hours.

On selling these screws, the company gets a profit of ₹ 100 per box on type A screws and ₹ 170 per box on type B screws. Formulate this problem as a LPP given that the objective is to maximise profit.

Sol. Let the company manufactures x boxes of type A screws and y boxes of type B screws.

From the given information, we can construct the following table.

Items	*Type A* (x)	*Type B* (y)	*Minimum time available on each machine in a week*
Time required on threading machine	2	8	$60 \times 60 = 3600$ minutes
Time required on slotting machine	3	2	$60 \times 60 = 3600$ minutes
Profit	₹ 100	₹ 170	

As per the information in the above table, the objective function for maximum profit $Z = 100x + 170y$

Subject to the constraints

$$2x + 8y \le 3600 \Rightarrow x + 4y \le 1800 \quad ...(i)$$

$$3x + 2y \le 3600 \quad ...(ii)$$

$x \ge 0, y \ge 0$ (non-negative constraints)

Hence, the required LPP is

Maximise $Z = 100x + 170y$

Subject to the constraints,

$x + 4y \le 1800$, $3x + 2y \le 3600$, $x \ge 0, y \ge 0$.

Q14. A company manufactures two types of sweaters: type A and type B. It costs ₹ 360 to make a type A sweater and ₹ 120 to make a type B sweater. The company can make at most 300 sweaters and spend at most ₹ 72000 a day. The number of sweaters of type B cannot exceed the number of sweaters of type A by more than 100. The company makes a profit of ₹ 200 for each sweater of type A and ₹ 120 for every sweater of type B. Formulate this problem at a LPP to maximise the profit to the company.

Sol. Let x and y be the number of sweaters of type A and type B respectively.

From the given information, we have the following constraints.

$360x + 120y \le 72000 \quad \Rightarrow \quad 3x + y \le 600$ ...(*i*)

$x + y \le 300$ ...(*ii*); $x + 100 \ge y \Rightarrow y \le x + 100$ ...(*iii*)

Profit $(Z) = 200x + 120y$

Hence, the required LPP to maximise the profit is

Maximise $Z = 200x + 120y$ subject to the constraints

$3x + y \le 600,\ x + y \le 300,\ y \le x + 100,\ x \ge 0,\ y \ge 0.$

Q15. A man rides his motorcycle at the speed of 50 km/hr. He has to spend ₹ 2 per km on petrol. If he rides it at a faster speed of 80 km/hr, the petrol cost increases to ₹ 3 per km. He has atmost ₹ 120 to spend on petrol and one hour's time. He wishes to find the maximum distance that he can travel.

Express this problem as a linear programming problem.

Sol. Let the man covers x km on his motorcycle at the speed of 50 km/hr and covers y km at the speed of 80 km/hr.

So, cost of petrol $= 2x + 3y$

The man has to spend ₹ 120 atmost on petrol

$\therefore \quad 2x + 3y \le 120$ ...(*i*)

Now, the man has only 1 hr time

$\therefore \quad \dfrac{x}{50} + \dfrac{y}{80} \le 1 \quad \Rightarrow \quad 8x + 5y \le 400$ ...(*ii*)

$x \ge 0,\ y \ge 0$

To have maximum distance $Z = x + y$.

Hence, the required LPP to travel maximum distance is maximise $Z = x + y$, subject to the constraints

$2x + 3y \le 120,\ 8x + 5y \le 400,\ x \ge 0,\ y \ge 0.$

LONG ANSWER TYPE QUESTIONS

Q16. Refer to Exercise 11. How many of circuits of Type A and of Type B, should be produced by the manufacturer so as to maximise his profit? Determine the maximum profit.

Sol. As per the solution of Question No. 11, we have

Maximise $Z = 50x + 60y$ subject to the contraints

$2x + y \le 20$ ...(*i*); $x + 2y \le 12$ ...(*ii*); $x + 3y \le 15$ ...(*iii*); $x \ge 0, y \ge 0$...(*iv*)

Let us draw the table for the above statements

Table for (*i*)

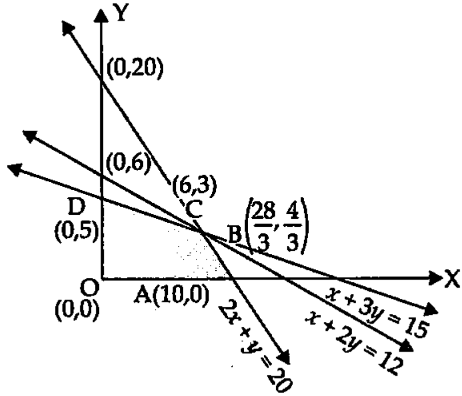
x	0	10
y	20	0

; Table for (*ii*)

x	0	12
y	6	0

Table for (*ii*)

x	0	15
y	5	0



Solving eq. (*i*) and (*ii*) we get,

$x = \frac{28}{3}$, $y = \frac{4}{3}$ $\quad \therefore$ B$\left(\frac{28}{3}, \frac{4}{3}\right)$ is the corner

Solving eq. (*ii*) and (*iii*) we get,

$x = 6, y = 3$ $\quad \therefore$ C(6, 3) is the corner

Solving eq. (*i*) and (*iii*) we get,

$x = 9, y = 2$ (not included in the feasible region)

Here, OABCD is the feasible region.

So, the corner points are O(0, 0), A(10, 0), B$\left(\frac{28}{3}, \frac{4}{3}\right)$, C(6, 3) and D(0, 5).

Let us evaluate the value of Z

Corner points	*Corresponding values of* $Z = 50x + 60y$	
O(0, 0)	$Z = 50(0) + 60(0) = 0$	
A(10, 0)	$Z = 50(10) + 60(0) = 500$	
B$\left(\frac{28}{3}, \frac{4}{3}\right)$	$Z = 50\left(\frac{28}{3}\right) + 60\left(\frac{4}{3}\right) = \frac{1400}{3} + \frac{240}{3}$ $= \frac{1640}{3} = 546.6$	← Maximum
C(6, 3)	$Z = 50(6) + 60(3) = 480$	
D(0, 5)	$Z = 50(0) + 60(5) = 300$	

Here, the maximum profit is ₹ 546.6 which is not possible for number of items in fraction.

Hence, the maximum profit for the manufacturer is ₹ 480 at (6, 3). Type A = 6 and Type B = 3.

Q17. Refer to Exercise 12. What will be the minimum cost?

Sol. As per the solution of Q. 12., we have $Z = 400x + 200y$

Subject to the constraints

$5x + 2y \geq 30$ ...(*i*); $2x + y \leq 15$ ...(*ii*)

$x \leq y, x \geq 0, y \geq 0$

$x - y \leq 0$ ...(*iii*)

Let $5x + 2y = 30$

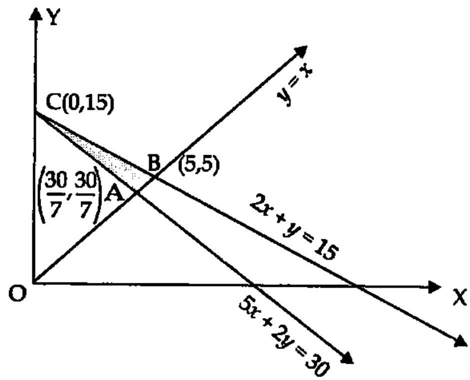
x	0	6
y	15	0

Let $2x + y = 15$

x	0	7.5
y	15	0

Let $x - y = 0$

x	0	1
y	0	1



Solving eq. (*i*) and (*iii*) we get; $x = \frac{30}{7}$ and $y = \frac{30}{7}$

and on solving eq. (*ii*) and (*iii*) we get, $x = 5$ and $y = 5$

Here, ABC is the shaded feasible region whose corner points are $A\left(\frac{30}{7}, \frac{30}{7}\right)$, B(5, 5) and C(0, 15)

Evaluating the value of Z, we have

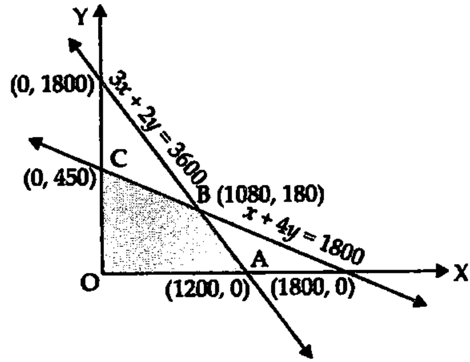
Corner points	*Value of* $Z = 400x + 200y$	
$A\left(\frac{30}{7}, \frac{30}{7}\right)$	$Z = 400\left(\frac{30}{7}\right) + 200\left(\frac{30}{7}\right) = \frac{18000}{7} = 2571.4$	← Minimum
B(5, 5)	$Z = 400(5) + 200(5) = 3000$	
C(0, 15)	$Z = 400(0) + 200(15) = 3000$	

Hence, the required minimum cost is ₹ 2571.4 at $\left(\frac{30}{7}, \frac{30}{7}\right)$.

Q18. Refer to Exercise 13. Solve the linear programming program and determine the maximum profit to the manufacturer.

Sol. As per the solution

of Q. 13, we have:



Let $3x + 2y = 3600$

x	0	1200
y	1800	0

Let $x + 4y = 1800$

x	0	1800
y	450	0

Maximise $Z = 100x + 170y$ subject to the constraints

$$3x + 2y \le 3600 \quad ...(i); \quad x + 4y \le 1800 \quad ...(ii)$$
$$x \ge 0, y \ge 0$$

On solving eq. (*i*) and (*ii*) we get

$x = 1080$ and $y = 180$

OABC is the feasible region whose corner points are O(0, 0), A(1200, 0), B(1080, 180), C(0, 450).

Let us evaluate the value of Z.

Corner points	*Value of* $Z = 100x + 170y$	
O(0, 0)	$Z = 100(0) + 170(0) = 0$	
A(1200, 0)	$Z = 100(1200) + 0 = 120000$	
B(1080, 180)	$Z = 100(1080) + 170(180) = 138600$	← Maximum
C(0, 450)	$Z = 170(450) = 76500$	

Hence, the maximum value of Z is 138600 at (1080, 180).

Q19. Refer to Exercise 14. How many sweaters of each type should the company make in a day so as to get a maximum profit? What is the maximum profit?

Sol. Referring to the solution of Q. 14, we have

Maximise $Z = 200x + 120y$ subject to the constraints

$$x + y \le 300 \quad ...(i); \quad 3x + y \le 600 \quad ...(ii)$$
$$x - y \ge -100 \quad ...(iii)$$
$$x \ge 0, y \ge 0$$

On solving eq. (*i*) and (*iii*) we have

$x = 100, y = 200$

On solving eq. (*i*) and (*ii*) we get

$x = 150, y = 150$

Let $x + y = 300$

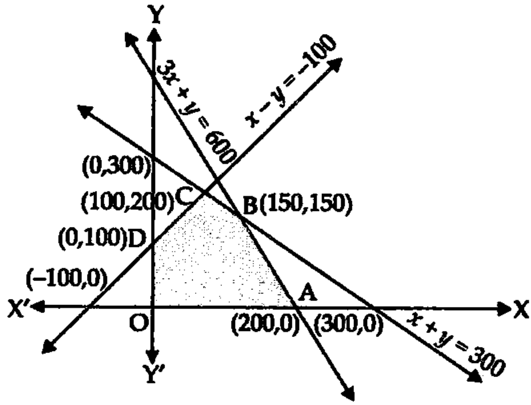
x	0	300
y	300	0

Let $3x + y = 600$

x	0	200
y	600	0

Let $x + y = -100$

x	0	−100
y	100	0



Here, the shaded region is the feasible region whose corner points are O(0, 0), A(200, 0), B(150, 150), C(100, 200), D(0, 100).
Let us evaluate the value of Z.

Corner points	*Value of* $Z = 200x + 120y$	
O(0, 0)	$Z = 200(0) + 120(0) = 0$	
A(200,0)	$Z = 200(200) + 120(0) = 40000$	
B(150, 150)	$Z = 200(150) + 120(150) = 48000$	←Maximum
C(100, 200)	$Z = 200(100) + 120(200) = 44000$	
D(0, 100)	$Z = 200(0) + 120(100) = 12000$	

Hence, the maximum value of Z is 48000 at (150, 150) i.e., 150 sweaters of each type.

Q20. Refer to Exercise 15, Determine the maximum distance that the man can travel.

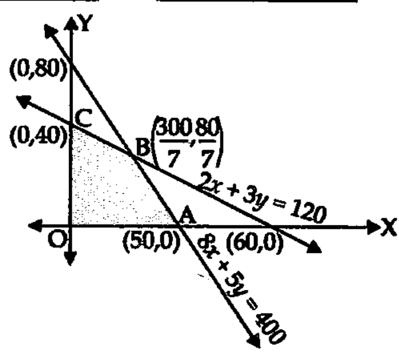
Sol. Referring to the solution of Q. 15, we have
Maximise $Z = x + y$ subject to the constraints

Let $2x + 3y = 120$

x	0	60
y	40	0

Let $8x + 5y = 400$

x	0	50
y	80	0



$$2x + 3y \le 120 \;...(i);\; 8x + 5y \le 400 \;...(ii)$$
$$x \ge 0, y \ge 0$$

On solving eq. (*i*) and (*ii*) we get; $x = \dfrac{300}{7}$ and $y = \dfrac{80}{7}$

Here, OABC is the feasible region whose corner points are O(0, 0), A(50, 0), $B\left(\dfrac{300}{7}, \dfrac{80}{7}\right)$ and C(0, 40).

Let us evaluate the value of Z

Corner points	*Value of* $Z = x + y$	
O(0, 0)	$Z = 0 + 0 = 0$	
A(50, 0)	$Z = 50 + 0 = 50$ km	
$B\left(\dfrac{300}{7}, \dfrac{80}{7}\right)$	$Z = \dfrac{300}{7} + \dfrac{80}{7} = \dfrac{380}{7} = 54.3$ km	← Maximum
C(0, 40)	$Z = 0 + 40 = 40$ km	

Hence, the maximum distance that the man can travel is $54\frac{2}{7}$ km at $\left(\dfrac{300}{7}, \dfrac{80}{7}\right)$.

Q21. Maximise $Z = x + y$ subject to $x + 4y \le 8$, $2x + 3y \le 12$, $3x + y \le 9$, $x \ge 0, y \ge 0$.

Sol. We are given that $Z = x + y$ subject to the constraints

$x + 4y \le 8$ ...(*i*)

x	0	8
y	2	0

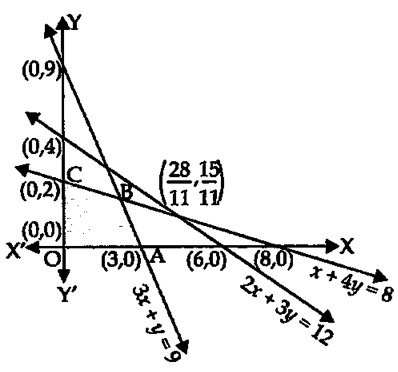
$2x + 3y \le 12$ ...(*ii*)

x	0	6
y	4	0

$3x + y \le 9$ ...(*iii*)

x	0	3
y	9	0

$x \ge 0, y \ge 0$



On solving eq. (*i*) and (*iii*) we get

$x = \frac{28}{11}$ and $y = \frac{15}{11}$

Here, OABC is the feasible region whose corner points are O(0, 0), A(3, 0), $B\left(\frac{28}{11}, \frac{15}{11}\right)$, C(0, 2)

Let us evaluate the value of Z

Corner points	*Value of* $Z = x + y$	
O(0, 0)	$Z = 0 + 0 = 0$	
A(3, 0)	$Z = 3 + 0 = 3$	
$B\left(\frac{28}{11}, \frac{15}{11}\right)$	$Z = \frac{28}{11} + \frac{15}{11} = \frac{43}{11} = 3.9$	← Maximum
C(0, 2)	$Z = 0 + 2 = 2$	

Hence, the maximum value of Z is 3.9 at $\left(\frac{28}{11}, \frac{15}{11}\right)$.

Q22. A manufacturer produces two Models of bikes—Model X and Model Y. Model X takes a 6 man-hours to make per unit, while Model Y takes 10 man-hours per unit. There is a total of 450 man-hour available per week. Handling and marketing costs are ₹ 2,000 and ₹ 1,000 per unit for Models X and Y respectively. The total funds available for these purposes are ₹ 80,000 per week. Profits per unit for Models X and Y are ₹ 1,000 and ₹ 500 respectively. How many bikes of each model should the manufacturer produce so as to yield a maximum profit? Find the maximum profit.

Sol. Let x and y be the number of Models of bike produced by the manufacturer.

Given information is

Model X takes 6 man-hours to make per unit

Model Y takes 10 man-hours to make per unit

Total man-hours available = 450

$\therefore \quad 6x + 10y \le 450 \quad \Rightarrow \quad 3x + 5y \le 225 \qquad ...(i)$

Handling and marketing cost of Model X and Y are ₹ 2,000 and ₹ 1,000 respectively

Total funds available is ₹ 80,000 per week

$\therefore \quad 2000x + 1000y \le 80,000$

$\Rightarrow \quad 2x + y \le 80 \qquad ...(ii)$

and $\qquad x \geq 80, y \geq 0$

Profit (Z) per unit of models X and Y are ₹ 1,000 and ₹ 500 respectively

So, $\qquad Z = 1000x + 500y$

The required LPP is

Maximise Z = 1000x + 500y subject to the constraints

$3x + 5y \leq 225 \qquad ...(i)$

x	0	75
y	45	0

$2x + y \leq 80 \qquad ...(ii)$

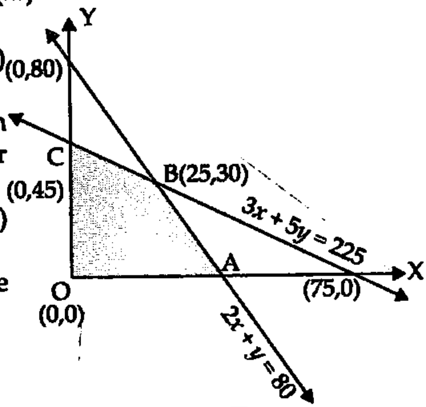
x	0	40
y	80	0

$x \geq 0, y \geq 0 \qquad ...(iii)$

On solving eq. (*i*) and (*ii*) we get, $x = 25, y = 30$

Here, the feasible region is OABC, whose corner points are O(0, 0), A(40, 0), B(25, 30) and C(0, 45).

Let us evaluate the value of Z.



Corner points	*Value of Z = 1000x + 500y*	
O(0, 0)	Z = 0 + 0 = 0	
A(40, 0)	Z = 1000(40) + 0 = 40,000	←Maximum
B(25, 30)	Z = 1000(25) + 500(30) = 40,000	←Maximum
C(0, 45)	Z = 0 + 500(45) = 22500	

Hence, the maximum profit is ₹ 40,000 by producing 25 bikes of Model X and 30 bikes of Model Y.

Q23. In order to supplement daily diet, a person wishes to take some X and some wishes Y tablets. The contents of iron, calcium and vitamins in X and Y (in milligram per tablet) are given as below:

Tablets	Iron	Calcium	Vitamin
X	6	3	2
Y	2	3	4

The person needs atleast 18 milligrams of iron, 21 milligrams of calcium and 16 milligrams of vitamin. The price of each tablet of X and Y is ₹ 2 and ₹ 1 respectively. How many tablets of each should the person take in order to satisfy the above requirement at the minimum cost?

Sol. Let there be x units of tablet X and y units of tablet Y

So, according to the given information, we have

$6x + 2y \geq 18 \Rightarrow 3x + y \geq 9$ ...(*i*)

$3x + 3y \geq 21 \Rightarrow x + y \geq 7$ ...(*ii*)

$2x + 4y \geq 16 \Rightarrow x + 2y \geq 8$ ...(*iii*)

$x \geq 0, y \geq 0$ ...(*iv*)

x	0	3
y	9	0

x	0	7
y	7	0

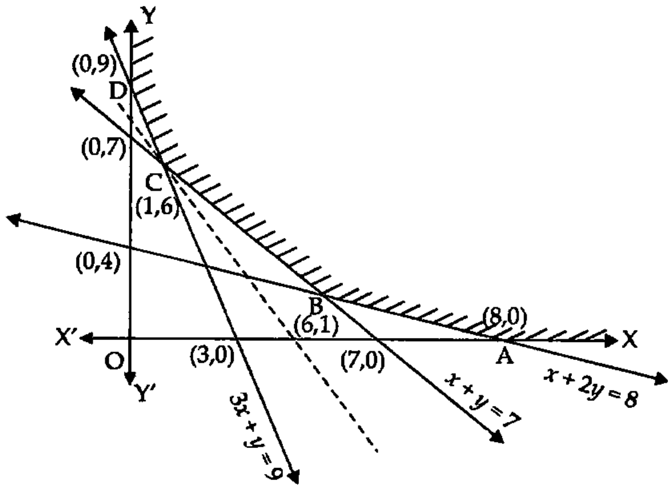
x	0	8
y	4	0

The price of each table of X type is ₹ 2 and that of y is ₹ 1.

So, the required LPP is

Minimise $Z = 2x + y$ subject to the constraints

$3x + y \geq 9, x + y \geq 7, x + 2y \geq 8, x \geq 0, y \geq 0$



On solving (*ii*) and (*iii*) we get

$x = 6$ and $y = 1$

On solving (*i*) and (*ii*) we get $x = 1$ and $y = 6$
From the graph, we see that the feasible region ABCD is unbounded whose corner points are A(8, 0), B(6, 1), C(1, 6) and D(0, 9).
Let us evaluate the value of Z

Corner points	*Value of* $Z = 2x + y$	
A(8, 0)	$Z = 2(8) + 0 = 16$	
B(6, 1)	$Z = 2(6) + 1 = 13$	
C(1, 6)	$Z = 2(1) + 6 = 8$	← Minimum
D(0, 9)	$Z = 2(0) + 9 = 9$	

Here, we see that 8 is the minimum value of Z at (1, 6) but the feasible region is unbounded. So, 8 may or may not be the minimum value of Z.
To confirm it, we will draw a graph of inequality $2x + y < 8$ and check if it has a common point.
We see from the graph that there is no common point on the line.
Hence, the minimum value of Z is 8 at (1, 6).
Tablet X = 1
Table Y = 6.

Q24 A company makes 3 models of calculators: A, B and C at factory I and factory II. The company has orders for atleast 6400 calculators of model A, 4000 calculators of model B and 4800 calculators of model C. At factory I, 50 calculators of model A, 50 of model B and 30 of model C are made everyday; at factory II, 40 calculators of model A, 20 of model B and 40 of model C are made everyday. It costs ₹ 12,000 and ₹ 15000 each day to operate factory I and II respectively. Find the number of days each factory should operate to minimise the operating costs and still meet the demand.

Sol. Let factory I be operated for x days and II for y days
At factory I: 50 calculators of model A and at factory II, 40 calculators of model A are made everyday.
Company has orders of atleast 6400 calculators of model A.
$\therefore\ 50x + 40y \geq 6400 \quad \Rightarrow \quad 5x + 4y \geq 640$
Also, at factory I, 50 calculators of model B and at factory II, 20 calculators of model B are made everyday.
Company has the orders of atleast 4000 of calculators of model B.

$\therefore\ 50x + 20y \geq 4000 \quad \Rightarrow \quad 5x + 2y \geq 4000$

Similarly for model C,

$30x + 40y \geq 4800 \quad \Rightarrow \quad 3x + 4y \geq 480$

and $x \geq 0, y \geq 0$

It costs ₹ 12,000 and ₹ 15000 to operate the factories I and II each day.

$\therefore$ Required LPP is

Minimise $Z = 12000x + 15000y$ subject to the constraints

$$5x + 4y \geq 640 \quad \text{...}(i)$$
$$5x + 2y \geq 400 \quad \text{...}(ii)$$
$$3x + 4y \geq 480 \quad \text{...}(iii)$$
$$x \geq 0, y \geq 0 \quad \text{...}(iv)$$

Table for (*i*) equation $5x + 4y = 640$

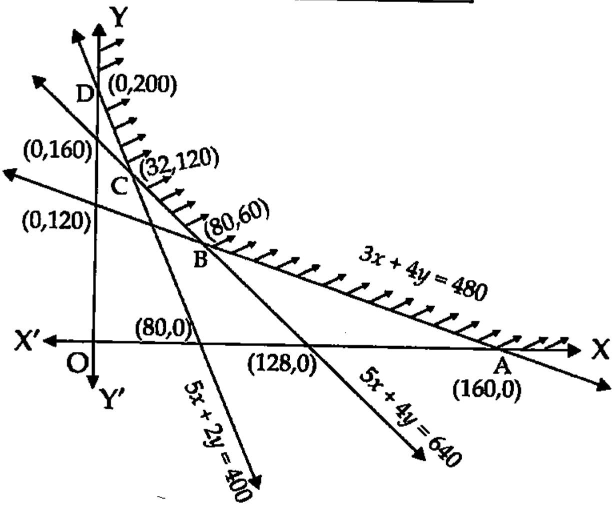
x	0	128
y	160	0

Table for (*ii*) equation $5x + 2y = 400$

x	0	80
y	200	0

Table for (*iii*) equation $3x + 4y = 480$

x	0	160
y	120	0



On solving eq. (*i*) and (*iii*), we get

$x = 80, y = 60$

On solving eq. (*i*) and (*ii*) we get

$x = 32$ and $y = 120$

From the graph, we see that the feasible region ABCD is open unbounded whose corners are A(160, 0), B(80, 60), C(32, 120) and D(0, 200).

Let us find the values of Z.

Corner points	*Value of* $Z = 12000x + 15000y$	
A(160, 0)	$Z = 12000(160) + 0 = 1920000$	
B(80, 60)	$Z = 12000(80) + 15000(60)$ $= 1860000$	← Minimum
C(32, 120)	$Z = 12000(32) + 15000(120)$ $= 2184000$	
D(0, 200)	$Z = 0 + 15000(200) = 3000000$	

From the above table, it is clear that the value of Z = 1860000 may or may not be minimum for an open unbounded region.

Now, to decide this, we draw a graph of

$$12000x + 15000y < 1860000$$

$$\Rightarrow \quad 4x + 5y < 620$$

and we have to check whether there is a common point in this feasible region or not.

So, from the graph, there is no common point.

$\therefore$ $Z = 12000x + 15000y$ has minimum value 1860000 at (80, 60).

Factory I : 80 days

Factory II: 60 days.

Q25. Maximise and minimise $Z = 3x - 4y$ subject to $x - 2y \le 0$, $-3x + y \le 4$, $x - y \le 6$ and $x, y \ge 0$.

Sol. Given LPP is

Maximise and minimise $Z = 3x - 4y$ subject to

$x - 2y \le 0$ ...(*i*)

x	0	2
y	0	1

$-3x + y \le 4$ ...(*ii*)

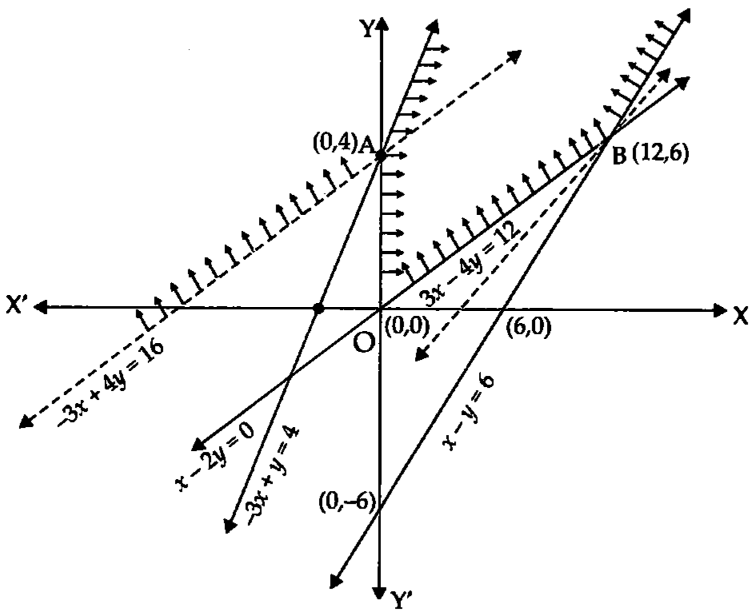
x	0	$-4/3$
y	4	0

$x - y \le 6$ ...(*iii*)

x	0	6
y	-6	0

and $x, y \ge 0$ ...(*iv*)

From the graph, we see that AOB is open unbounded region whose corners are O(0, 0), A(0, 4), B(12, 6).
Let us evaluate the value of Z



Corner points	*Value of Z = 3x – 4y*	
O(0, 0)	$Z = 0$	
A(0, 4)	$Z = 0 - 4(4) = -16$	← Minimum
B(12, 6)	$Z = 3(12) - 4(6) = 12$	← Maximum

For this unbounded region, the value of Z may or may not be -16. So to decide it, we draw a graph of inequality $3x - 4y < -16$ and check whether the open half plane has common points with feasible region or not. But from the graph, we see that it has common points with the feasible region, so it will have not minimum value of Z. Similarly for maximum value, we draw the graph of inequality $3x - 4y > 12$ in which there is no common point with the feasible region.
Hence, the maximum value of Z is 12.

OBJECTIVE TYPE QUESTIONS

Choose the correct answer from the given four options in each of the Exercises 26 to 34.

Q26. The corner points of the feasible region determined by the system of linear constraints are (0, 0), (0, 40), (20, 40), (60, 20), (60, 0). The objective function is $Z = 4x + 3y$. Compare the quantity in Column A and Column B

Column A	*Column B*
Maximum of Z	325

(*a*) The quantity in column A is greater.
(*b*) The quantity in column B is greater
(*c*) The two quantities are equal
(*d*) The relationship cannot be determined on the basis of the information supplied.

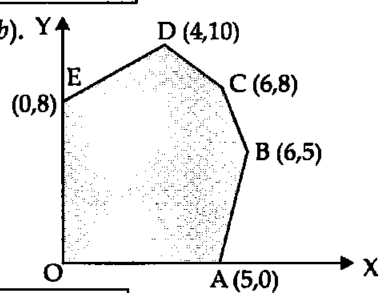
Sol.

Corner points	*Value of* $Z = 4x + 3y$	
(0, 0)	$Z = 0$	
(0, 40)	$Z = 0 + 3(40) = 120$	
(20, 40)	$Z = 4(20) + 3(40) = 200$	
(60, 20)	$Z = 4(60) + 3(20) = 300$	→ Maximum
(60, 0)	$Z = 4(60) + 3(0) = 240$	

Hence, the correct option is (*b*).

Q27. The feasible solution for a LPP is shown in figure. Let $Z = 3x - 4y$ be the objective function. Minimum of Z occurs at

(*a*) (0, 0) (*b*) (0, 8)
(*c*) (5, 0) (*d*) (4, 10)



Sol.

Corner points	*Value of* $Z = 3x - 4y$	
O(0, 0)	$Z = 0$	
A(5, 0)	$Z = 3(5) - 0 = 15$	
B(6, 5)	$Z = 3(6) - 4(5) = -2$	
C(6, 8)	$Z = 3(6) - 4(8) = -14$	
D(4, 10)	$Z = 3(4) - 4(10) = -28$	
E(0, 8)	$Z = 3(0) - 4(8) = -32$	← Minimum

Hence, the correct option is (*b*).

Q28. Refer to Exercise 27. Maximum of Z occurs at

(*a*) (5, 0) (*b*) (6, 5) (*c*) (6, 8) (*d*) (4, 10)

Sol. According to solution of Q. 27, the maximum value of Z is 15 at A (5, 0).

Hence, the correct option is (*a*).

Q29. Refer to Exercise 27. (Maximum value of Z + Minimum value of Z) is equal to

(*a*) 13 (*b*) 1 (*c*) -13 (*d*) -17

Sol. According to the solution of Q. 27, Maximum value of $Z = 15$ and Minimum value of $Z = -32$

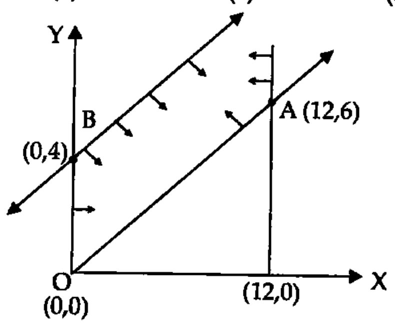
So, the sum of Maximum value and Minimum value of Z

$$= 15 + (-32) = -17$$

Hence, the correct option is (*d*).

Q30. The feasible region for an LPP is shown in the figure. Let $F = 3x - 4y$ be the objective function. Maximum value of F is

(*a*) 0 (*b*) 8 (*c*) 12 (*d*) -18



Sol. The feasible region is shown in the figure for which the objective function $F = 3x - 4y$

Corner point	*Value of $F = 3x - 4y$*	
O(0, 0)	$F = 0$	
A(12, 6)	$F = 3(12) - 4(6) = 12$	← Maximum
B(0, 4)	$F = 0 - 4(4) = -16$	← Minimum

Hence, the correct option is (*c*).

Q31. Refer to Exercise 30. Minimum value of F is

(*a*) 0 (*b*) -16 (*c*) 12 (*d*) does not exist

Sol. According to the solution of Q. 30, the minimum value of F is -16 at (0, 4).

Hence, the correct option is (*b*).

Q32. Corner points of the feasible region for an LPP are (0, 2), (3, 0), (6, 0), (6, 8) and (0, 5).
Let $F = 4x + 6y$ be the objective function.
The minimum value of F occurs at
(*a*) (0, 2) only (*b*) (3, 0) only
(*c*) the mid-point of the line segment joining the points (0, 2) and (3, 0) only
(*d*) any point on the line segment joining the points (0, 2) and (3, 0)

Sol.

Corner points	*Value of* $F = 4x + 6y$	
(0, 2)	$Z = 4(0) + 6(2) = 12$	← Minimum
(3, 0)	$Z = 4(3) + 6(0) = 12$	← Minimum
(6, 0)	$Z = 4(6) + 6(0) = 24$	
(6, 8)	$Z = 4(6) + 6(8) = 72$	← Maximum
(0, 5)	$Z = 4(0) + 6(5) = 30$	

The minimum value of F occurs at any point on the line segment joining the points (0, 2) and (3, 0).
Hence, the correct option is (*d*).

Q33. Refer to Exercise 32, Maximum of F – Minimum of F =
(*a*) 60 (*b*) 48 (*c*) 42 (*d*) 18

Sol. According to the solution of Q. 32,
Maximum value of F – Minimum value of F = 72 – 12 = 60
Hence, the correct option is (*a*).

Q34. Corner points of the feasible region determined by the system of linear constraints are (0, 3), (1, 1) and (3, 0).
Let $Z = px + qy$, where $p, q > 0$. Condition on p and q so that the minimum of Z occurs at (3, 0) and (1, 1) is
(*a*) $p = 2q$ (*b*) $p = \frac{q}{2}$ (*c*) $p = 3q$ (*d*) $p = q$

Sol.

Corner points	*Value of* $Z = px + qy;\ p, q > 0$
(0, 3)	$Z = p(0) + q(3) = 3q$
(1, 1)	$Z = p(1) + q(1) = p + q$
(3, 0)	$Z = p(3) + q(0) = 3p$

So, condition of p and q so that the minimum of Z occurs at (3, 0) and (1, 1) is

$$p + q = 3p \Rightarrow p - 3p + q = 0 \Rightarrow p = \frac{q}{2}.$$

Hence, the correct option is (*b*).

Fill in the blanks in each of the Exercises 35 to 41.

Q35. In a LPP, the linear inequalities or restrictions on the variables are called ....

Sol. constraints.

Q36. In a LPP, the objective function is always ....

Sol. linear.

Q37. If the feasible region for a LPP is .... then the optimal value of the objective function $Z = ax + by$ may or may not exist.

Sol. open unbounded

Q38. In a LPP, if the objective function $Z = ax + by$ has the same maximum value at two corner points of the feasible region, then every point on the line segment joining these two points give the same .... value.

Sol. maximum

Q39. A feasible region of a system of linear inequalities is said to be .... if it can be enclosed within a circle.

Sol. bounded

Q40. A corner point of a feasible region is a point in the region which is the .... of two boundary lines.

Sol. intersection

Q41. The feasible region for an LPP is always a.................. polygon.

Sol. convex

State whether the statements in Exercises 42 to 45 are True or False.

Q42. If the feasible region for a LPP is unbounded, maximum or minimum of the objective function $Z = ax + by$ may or may not exist.

Sol. True.

Q43. Maximum value of the objective function $Z = ax + by$ in a LPP always occurs at only one corner point of the feasible region.

Sol. False.

Q44. In a LPP, the minimum value of the objective function $Z = ax + by$ is always 0 if the origin is one of the corner point of the feasible region.

Sol. False.

Q45. In a LPP, the maximum value of the objective function $Z = ax + by$ is always finite.

Sol. True.

❑❑❑

13 Probability

13.3 EXERCISE

SHORT ANSWER TYPE QUESTIONS

Q1. For a loaded die, the probabilities of outcomes are given as under:
P(1) = P(2) = 0.2, P(3) = P(5) = P(6) = 0.1 and P(4) = 0.3
The die is thrown, two times. Let A and B be the events, 'same number each time'; and 'a total score is 10 or more'; respectively. Determine whether or not A and B are independent.

Sol. A loaded die is thrown such that
P(1) = P(2) = 0.2, P(3) = P(5) = P(6) = 0.1 and P(4) = 0.3 and die is thrown two times. Also given that:
A = Same number each time and
B = Total score is 10 or more.

So, $P(A) = [P(1, 1) + P(2, 2) + P(3, 3) + P(4, 4) + P(5, 5) + P(6, 6)]$
$= P(1).P(1) + P(2).P(2) + P(3).P(3) + P(4).P(4) + P(5).P(5) + P(6).P(6)$
$= 0.2 \times 0.2 + 0.2 \times 0.2 + 0.1 \times 0.1 + 0.3 \times 0.3 + 0.1 \times 0.1 + 0.1 \times 0.1$
$= 0.04 + 0.04 + 0.01 + 0.09 + 0.01 + 0.01 = 0.20$

Now $B = [(4, 6), (6, 4), (5, 5), (5, 6), (6, 5), (6, 6)]$
$P(B) = [P(4).P(6) + P(6).P(4) + P(5).P(5) + P(5).P(6) + P(6).P(5) + P(6).P(6)$
$= 0.3 \times 0.1 + 0.1 \times 0.3 + 0.1 \times 0.1 + 0.1 \times 0.1 + 0.1 \times 0.1 + 0.1 \times 0.1$
$= 0.03 + 0.03 + 0.01 + 0.01 + 0.01 + 0.01 = 0.10$

A and B both events will be independent if
$$P(A \cap B) = P(A).P(B) \quad \text{...}(i)$$
Here, $(A \cap B) = \{(5, 5), (6, 6)\}$
$\therefore$ $P(A \cap B) = P(5, 5) + P(6, 6) = P(5).P(5) + P(6).P(6)$
$= 0.1 \times 0.1 + 0.1 \times 0.1 = 0.02$

From eq. (*i*) we get
$$0.02 = 0.20 \times 0.10$$
$$0.02 = 0.02$$
Hence, A and B are independent events.

Q2. Refer to Exercise 1 above. If the die were fair, determine whether or not, the events A and B are independent.

Sol. According to the solution of Q. 1, we have

$A = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$

$\therefore\ n(A) = 6$ and $n(S) = 6 \times 6 = 36$

So, $\quad P(A) = \dfrac{n(A)}{n(S)} = \dfrac{6}{36} = \dfrac{1}{6}$

and $\quad B = \{(4, 6), (6, 4), (5, 5), (5, 6), (6, 5), (6, 6)\}$

$n(B) = 6$ and $n(S) = 36$

$\therefore\quad P(B) = \dfrac{n(B)}{n(S)} = \dfrac{6}{36} = \dfrac{1}{6}$

$A \cap B = \{(5, 5), (6, 6)\}$

$\therefore P(A \cap B) = \dfrac{2}{36} = \dfrac{1}{18}$

Therefore, if A and B are independent, then

$P(A \cap B) = P(A) . P(B)$

$\Rightarrow \quad \dfrac{1}{18} \neq \dfrac{1}{6} \times \dfrac{1}{6} \quad \Rightarrow \dfrac{1}{18} \neq \dfrac{1}{36}$

Hence, A and B are not independent events.

Q3. The probability that atleast one of the two events A and B occurs is 0.6. If A and B occurs simultaneously with probability 0.3, evaluate $P(\bar{A}) + P(\bar{B})$.

Sol. We know that:

$A \cup B$ denotes that atleast one of the events occurs

and $A \cap B$ denotes that the two events occur simultaneously.

So, $\quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow \quad 0.6 = P(A) + P(B) - 0.3$

$\Rightarrow \quad 0.9 = P(A) + P(B)$

$\Rightarrow \quad 0.9 = 1 - P(\bar{A}) + 1 - P(\bar{B})$

$\Rightarrow \quad P(\bar{A}) + P(\bar{B}) = 2 - 0.9 = 1.1$

Hence, the required answer is 1.1.

Q4. A bag contains 5 red marbles and 3 black marbles. Three marbles are drawn one by one without replacement. What is the probability that atleast one of the three marbles drawn be black, if the first marble is red?

Sol. Let red marbles be represented with R and black marble with B. The following three conditions are possible, if atleast one of the three marbles drawn be black and the first marble is red.

(*i*) E_1 : II ball is black and III is red

(*ii*) E_2 : II ball is black and III is also black

(*iii*) E_3 : II ball is red and III is black

$$\therefore \quad P(E_1) = P(R_1).P(B_1/R_1).P(R_2/R_1B_1) = \frac{5}{8}.\frac{3}{7}.\frac{4}{6} = \frac{60}{336} = \frac{5}{28}$$

$$P(E_2) = P(R_1).P(B_1/R_1).P(B_2/R_1B_1) = \frac{5}{8}.\frac{3}{7}.\frac{2}{6} = \frac{30}{336} = \frac{5}{56}$$

$$\text{and } P(E_3) = P(R_1).P(R_2/R_1).B(B_1/R_1R_2) = \frac{5}{8}.\frac{4}{7}.\frac{3}{6} = \frac{60}{336} = \frac{5}{28}$$

$$\therefore \quad P(E) = P(E_1) + P(E_2) + P(E_3) = \frac{5}{28} + \frac{5}{56} + \frac{5}{28} = \frac{25}{56}$$

Hence the required probability is $\frac{25}{56}$.

Q5. Two dice are thrown together and the total score is noted. The events E, F and G are 'a total of 4' 'a total of 9 or more', and 'a total divisible by 5', respectively. Calculate P(E), P(F) and P(G) and decide which pairs of events, if any, are independent.

Sol. Two dice are thrown together

$\therefore n(S) = 36$

E = A total of 4 = $\{(2, 2), (1, 3), (3, 1)\}$ $\quad \therefore n(E) = 3$

F = A total of 9 or more

$= \{(3, 6), (6, 3), (5, 4), (4, 5), (5, 5), (4, 6), (6, 4), (5, 6), (6, 5), (6, 6)\}$

$\therefore n(F) = 10;\quad G$ = A total divisible by 5

$= \{(1, 4), (4, 1), (2, 3), (3, 2), (4, 6), (6, 4), (5, 5)\} \therefore n(G) = 7$

Here, we see that $(E \cap F) = \phi$ and $(E \cap G) = \phi$

and $\quad (F \cap G) = \{(4, 6), (6, 4), (5, 5)\}$

$\therefore \quad n(F \cap G) = 3$ and $(E \cap F \cap G) = \phi$

$$\therefore \quad P(E) = \frac{n(E)}{n(S)} = \frac{3}{36} = \frac{1}{12}$$

$$P(F) = \frac{n(F)}{n(S)} = \frac{10}{36} = \frac{5}{18};\ P(G) = \frac{n(G)}{n(S)} = \frac{7}{36}$$

$$P(F \cap G) = \frac{3}{36} = \frac{1}{12} \text{ and } P(F).P(G) = \frac{5}{18}\cdot\frac{7}{36} = \frac{35}{648}$$

Since, $P(F \cap G) \neq P(F).P(G)$

Hence, there is no pair of independent events.

Q6. Explain why the experiment of tossing a coin three times is said to have binomial distribution.

Sol. We know that random variable X takes values 0, 1, 2, 3, ..., n is said to be binomial distribution having parameters n and p, if the probability is given by

$P(X = r) = {}^nC_r\, p^r q^{n-r}$, where $q = 1 - p$ and $r = 0, 1, 2, 3, \ldots$

Similarly in case of tossing a coin 3 times,

$n = 3$ and X has the values 0, 1, 2, 3 with $p = \frac{1}{2}, q = \frac{1}{2}$.

Hence, it is said to have a binomial distribution.

Q7. A and B are two events such that $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$ and $P(A \cap B) = \frac{1}{4}$. Find:

(*i*) $P(A/B)$ (*ii*) $P(B/A)$ (*iii*) $P(A'/B)$ (*iv*) $P(A'/B')$

Sol. We have $P(A) = \frac{1}{2}, P(B) = \frac{1}{3}$ and $P(A \cap B) = \frac{1}{4}$

$$P(A') = 1 - \frac{1}{2} = \frac{1}{2}, P(B') = 1 - \frac{1}{3} = \frac{2}{3}$$

$$P(A' \cap B') = 1 - P(A \cup B) = 1 - [P(A) + P(B) - P(A \cap B)]$$

$$= 1 - \left[\frac{1}{2} + \frac{1}{3} - \frac{1}{4}\right] = 1 - \left[\frac{6 + 4 - 3}{12}\right] = 1 - \frac{7}{12} = \frac{5}{12}$$

(*i*) $P(A/B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{1/4}{1/3} = \dfrac{3}{4}$

(*ii*) $P(B/A) = \dfrac{P(A \cap B)}{P(A)} = \dfrac{1/4}{1/2} = \dfrac{1}{2}$

(*iii*) $P(A'/B) = \dfrac{P(A' \cap B)}{P(B)} = \dfrac{P(B) - P(A \cap B)}{P(B)}$

$$= 1 - \frac{P(A \cap B)}{P(B)} = 1 - \frac{1/4}{1/3} = 1 - \frac{3}{4} = \frac{1}{4}$$

(*iv*) $P(A'/B') = \dfrac{P(A' \cap B')}{P(B')} = \dfrac{5/12}{2/3} = \dfrac{5}{12} \times \dfrac{3}{2} = \dfrac{5}{8}$

Q8. Three events A, B and C have probabilities $\frac{2}{5}, \frac{1}{3}$ and $\frac{1}{2}$ respectively. Given that $P(A \cap C) = \frac{1}{5}$ and $P(B \cap C) = \frac{1}{4}$, find the values of $P(C/B)$ and $P(A' \cap C')$.

Sol. We have $P(A) = \frac{2}{5}$, $P(B) = \frac{1}{3}$ and $P(C) = \frac{1}{2}$

$P(A \cap C) = \frac{1}{5}$ and $P(B \cap C) = \frac{1}{4}$

$$\therefore \quad P(C/B) = \frac{P(B \cap C)}{P(B)} = \frac{1/4}{1/3} = \frac{3}{4}$$

$$P(A' \cap C') = 1 - P(A \cup C)$$

$$= 1 - [P(A) + P(C) - P(A \cap C)]$$

$$= 1 - \left[\frac{2}{5} + \frac{1}{2} - \frac{1}{5}\right] = 1 - \frac{7}{10} = \frac{3}{10}$$

Hence, the required probabilities are $\frac{3}{4}$ and $\frac{3}{10}$.

Q9. Let E_1 and E_2 be two independent events such that $P(E_1) = p_1$ and $P(E_2) = p_2$. Describe in words of the events whose probabilities are:

(*i*) p_1p_2 (*ii*) $(1 - p_1)p_2$

(*iii*) $1 - (1 - p_1)(1 - p_2)$ (*iv*) $p_1 + p_2 - 2p_1p_2$

Sol. Here, $P(E_1) = p_1$ and $P(E_2) = p_2$

(*i*) $p_1p_2 = P(E_1) . P(E_2) = P(E_1 \cap E_2)$

So, E_1 and E_2 occur.

(*ii*) $(1 - p_1).p_2 = P(E_1)'.P(E_2) = P(E_1' \cap E_2)$

So, E_1 does not occur but E_2 occurs.

(*iii*) $1 - (1 - p_1)(1 - p_2) = 1 - P(E_1)'P(E_2)' = 1 - P(E_1' \cap E_2')$

$= 1 - [1 - P(E_1 \cup E_2)] = P(E_1 \cup E_2)$

So, either E_1 or E_2 or both E_1 and E_2 occur.

(*iv*) $p_1 + p_2 - 2p_1p_2 = P(E_1) + P(E_2) - 2P(E_1). P(E_2)$

$= P(E_1) + P(E_2) - 2P(E_1 \cap E_2)$

$= P(E_1 \cup E_2) - 2P(E_1 \cap E_2)$

So, either E_1 or E_2 occurs but not both.

Q10. A discrete random variable X has the probability distribution given as below:

X	0.5	1	1.5	2
P(X)	k	k^2	$2k^2$	k

(*i*) Find the value of k.

(*ii*) Determine the mean of the distribution.

Sol. For a probability distribution, we know that if $P_i \geq 0$

(*i*) $\sum_{i=1}^{n} P_i = 1 \Rightarrow k + k^2 + 2k^2 + k = 1$

$\Rightarrow 3k^2 + 2k - 1 = 0 \Rightarrow 3k^2 + 3k - k - 1 = 0$

$\Rightarrow 3k(k + 1) - 1(k + 1) = 0 \Rightarrow (3k - 1)(k + 1) = 0$

$\therefore k = \frac{1}{3}$ and $k = -1$

But $k \geq 0 \quad \therefore k = \frac{1}{3}$

(*ii*) Mean of the distribution

$$E(X) = \sum_{i=1}^{n} X_iP_i = 0.5k + 1.k^2 + 1.5(2k^2) + 2k$$

$$= \frac{k}{2} + k^2 + 3k^2 + 2k = 4k^2 + \frac{5}{2}k$$

$$= 4\left(\frac{1}{3}\right)^2 + \frac{5}{2}\left(\frac{1}{3}\right) = \frac{4}{9} + \frac{5}{6} = \frac{23}{18}$$

Q11. Prove that

(*i*) $P(A) = P(A \cap B) + P(A \cap \bar{B})$

(*ii*) $P(A \cup B) = P(A \cap B) + P(A \cap \bar{B}) + P(\bar{A} \cap B)$

Sol. (*i*) To prove: $P(A) = P(A \cap B) + P(A \cap \bar{B})$

R.H.S. $= P(A \cap B) + P(A \cap \bar{B})$

$= P(A).P(B) + P(A).P(\bar{B}) = P(A)[P(B) + P(\bar{B})]$

$= P(A) . 1$

$= P(A) =$ L.H.S. Hence proved.

(*ii*) To prove: $P(A \cup B) = P(A \cap B) + P(A \cap \bar{B}) + P(\bar{A} \cap B)$

R.H.S. $= P(A).P(B) + P(A).P(\bar{B}) + P(\bar{A}).P(B)$

$= P(A).P(B) + P(A)[1 - P(B)] + [1 - P(A)].P(B)$

$= P(A).P(B) + P(A) - P(A).P(B) + P(B) - P(A).P(B)$

$= P(A) + P(B) - P(A \cap B)$

$= P(A \cup B) =$ L.H.S. Hence proved.

Q12. If X is the number of tails in three tosses of a coin, determine the standard deviation of X.

Sol. Given that: $X = 0, 1, 2, 3$

$\therefore P(X = r) = {}^nC_r p^r q^{n-r}$

where $n = 3, p = \frac{1}{2}, q = \frac{1}{2}$ and $r = 0, 1, 2, 3$

$$P(X = 0) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}; \; P(X = 1) = 3 \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{3}{8}$$

$$P(X = 2) = 3 \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{3}{8}; \; P(X = 3) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$$

Probability distribution table is:

X	0	1	2	3
P(X)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$$E(X) = 0 + 1 \times \frac{3}{8} + 2 \times \frac{3}{8} + 3 \times \frac{1}{8} = \frac{3}{8} + \frac{6}{8} + \frac{3}{8} = \frac{12}{8} = \frac{3}{2}$$

$$E(X^2) = 0 + 1 \times \frac{3}{8} + 4 \times \frac{3}{8} + 9 \times \frac{1}{8} = \frac{3}{8} + \frac{12}{8} + \frac{9}{8} = \frac{24}{8} = 3$$

We know that $Var(X) = E(X^2) - [E(X)]^2 = 3 - \left(\frac{3}{2}\right)^2 = 3 - \frac{9}{4} = \frac{3}{4}$

$\therefore$ Standard deviation $= \sqrt{Var(X)} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$.

Q13. In a dice game, a player pays a stake of Re 1 for each throw of a die. She receives ₹ 5 if the die shows a 3, ₹ 2, if the die shows a 1 or 6, and nothing otherwise. What is the player's expected profit per throw over a long series of throws?

Sol. Let X be the random variable of profit per throw.

X	–1	1	4
P(X)	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

Since, she loses ₹ 1 for getting any of 2, 4, 5.

So, $P(X = -1) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$

$P(X = 1) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$ ($\because$ die showing 1 or 6)

$P(X = 4) = \frac{1}{6}$ ($\because$ die shows only a 3)

Player's expected profit $= \sum p_1 x_i$

$= -1 \times \frac{1}{2} + 1 \times \frac{1}{3} + 4 \times \frac{1}{6} = -\frac{1}{2} + \frac{1}{3} + \frac{2}{3} = \frac{1}{2} =$ ₹ 0.50

Q14. Three dice are thrown at the same time. Find the probability of getting three two's, if it is known that the sum of the numbers on the dice was six.

Sol. The dice is thrown three times

$\therefore$ Sample space $n(S) = (6)^3 = 216$

Let E_1 be the event when the sum of numbers on the dice was six and E_2 be the event when three two's occur.

$\Rightarrow E_1 = \{(1, 1, 4), (1, 2, 3), (1, 3, 2), (1, 4, 1), (2, 1, 3), (2, 2, 2), (2, 3, 1), (3, 1, 2), (3, 2, 1), (4, 1, 1)\}$

$\Rightarrow n(E_1) = 10$ and $n(E_2) = 1$ $[\because E_2 = \{2, 2, 2\}]$

$\therefore P(E_2/E_1) = \frac{P(E_1 \cap E_2)}{P(E_1)} = \frac{1/216}{10/216} = \frac{1}{10}$.

Q15. Suppose 10,000 tickets are sold in a lottery each for Re. 1. First prize is of ₹ 3,000 and the second prize is of ₹ 2,000. There are three third prizes of ₹ 500 each. If you buy one ticket, what is your expectation?

Sol. Let X be the random variable where X = 0, 500, 2000 and 3000

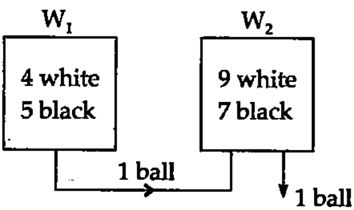
X	0	500	2000	3000
P(X)	$\frac{9995}{10000}$	$\frac{3}{10000}$	$\frac{1}{10000}$	$\frac{1}{10000}$

$$E(X) = 0 \times \frac{9995}{10000} + 500 \times \frac{3}{10000} + 2000 \times \frac{1}{10000} + 3000 \times \frac{1}{10000}$$

$$= 0 + \frac{1500}{10000} + \frac{2000}{10000} + \frac{3000}{10000} = \frac{6500}{10000} = \frac{13}{20} = ₹\,0.65$$

Hence, expectation is ₹ 0.65.

Q16. A bag contains 4 white and 5 black balls. Another bag contains 9 white and 7 black balls. A ball is transferred from the first bag to the second and then a ball is drawn at random from the second bag. Find the probability that the ball drawn is white.



Sol. Let W_1 and W_2 be two bags containing (4 W, 5 B) and (9 W, 7 B) balls respectively.

Let E_1 be the event that the transferred ball from the bag W_1 to W_2 is white and E_2 the event that the transferred ball is black.

And E be the event that the ball drawn from the second bag is white.

$$\therefore\ P(E/E_1) = \frac{10}{17}, \quad P(E/E_2) = \frac{9}{17}$$

$$P(E_1) = \frac{4}{9} \quad \text{and} \quad P(E_2) = \frac{5}{9}$$

$$\therefore \quad P(E) = P(E_1).P(E/E_1) + P(E_2).P(E/E_2)$$

$$= \frac{4}{9} \times \frac{10}{17} + \frac{5}{9} \times \frac{9}{17} = \frac{40}{153} + \frac{45}{153} = \frac{85}{153} = \frac{5}{9}$$

Hence, the required probability is $\frac{5}{9}$.

Q17. Bat I contains 3 black and 2 white balls, bag II contains 2 black and 4 white balls. A bag and a ball is selected at random. Determine the probability of selecting a black ball.

Sol. Given that bag I = {3 B, 2 W}

and bag II = {2 B, 4 W}

Let E_1 = The event that bag I is selected

E_2 = The event that bag II is selected

and E = The event that a black ball is selected

$\therefore \quad P(E_1) = \frac{1}{2},\ P(E_2) = \frac{1}{2},\ P(E/E_1) = \frac{3}{5}$ and $P(E/E_2) = \frac{1}{3}$

$$P(E) = P(E_1).P(E/E_1) + P(E_2).P(E/E_2)$$

$$= \frac{1}{2} \times \frac{3}{5} + \frac{1}{2} \times \frac{1}{3} = \frac{3}{10} + \frac{1}{6} = \frac{9+5}{30} = \frac{14}{30} = \frac{7}{15}$$

Hence, the required probability is $\frac{7}{15}$.

Q18. A box has 5 blue and 4 red balls. One ball is drawn at random and not replaced. Its colour is also not noted. Then another ball is drawn at random. What is the probability of second ball being blue?

Sol. Given that the box has 5 blue and 4 red balls.

Let E_1 be the event that first ball drawn is blue

E_2 be the event that first ball drawn is red

and E is the event that second ball drawn is blue.

$\therefore \quad P(E) = P(E_1).P(E/E_1) + P(E_2).P(E/E_2)$

$$= \frac{5}{9} \times \frac{4}{8} + \frac{4}{9} \times \frac{5}{8} = \frac{20}{72} + \frac{20}{72} = \frac{40}{72} = \frac{5}{9}$$

Hence, the required probability is $\frac{5}{9}$.

Q19. Four cards are successively drawn without replacement from a deck of 52 playing cards. What is the probability that all the four cards are Kings?

Sol. Let E_1, E_2, E_3 and E_4 be the events that first, second, third and fourth card is King respectively.

$\therefore P(E_1 \cap E_2 \cap E_3 \cap E_4)$

$$= P(E_1).P(E_2/E_1).P\left[\frac{E_3}{(E_1 \cap E_2)}\right].P\left[\frac{E_4}{(E_1 \cap E_2 \cap E_3 \cap E_4)}\right]$$

$$= \frac{4}{52} \times \frac{3}{51} \times \frac{2}{50} \times \frac{1}{49} = \frac{24}{52 \cdot 51 \cdot 50 \cdot 49} = \frac{1}{13 \cdot 17 \cdot 25 \cdot 49} = \frac{1}{27075}$$

Hence, the required probability is $\frac{1}{27075}$.

Q20. A die is thrown 5 times. Find the probability that an odd number will come up exactly three times.

Sol. Here, $\quad p = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2} \Rightarrow q = 1 - \frac{1}{2} = \frac{1}{2}$ and $\quad n = 5$

$$\therefore \quad P(x = r) = {}^nC_r\, p^r\, q^{n-r}$$

$$= {}^5C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^{5-3} = \frac{5!}{3!2!} \cdot \left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right)^2 = 10 \cdot \frac{1}{8} \cdot \frac{1}{4} = \frac{5}{16}$$

Hence, the required probability is $\frac{5}{16}$.

Q21. Ten coins are tossed. What is the probability of getting atleast 8 heads?

Sol. Here, $n = 10$, $p = \frac{1}{2}$, $q = \quad -\frac{1}{2} = \frac{1}{2}$

$$P(X \geq 8) = P(x = 8) + P(x = 9) + P(x = 10)$$

$$= {}^{10}C_8 \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^{10-8} + {}^{10}C_9 \left(\frac{1}{2}\right)^9 \left(\frac{1}{2}\right)^{10-9} + {}^{10}C_{10} \left(\frac{1}{2}\right)^{10} \left(\frac{1}{2}\right)^0$$

$$= \frac{10!}{8!2!} \cdot \left(\frac{1}{2}\right)^8 \cdot \left(\frac{1}{2}\right)^2 + \frac{10!}{9!1!} \left(\frac{1}{2}\right)^9 \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^{10}$$

$$= 45 \cdot \left(\frac{1}{2}\right)^{10} + 10 \cdot \left(\frac{1}{2}\right)^{10} + \left(\frac{1}{2}\right)^{10} = \left(\frac{1}{2}\right)^{10} \cdot (45 + 10 + 1)$$

$$= 56 \left(\frac{1}{2}\right)^{10} = 56 \times \frac{1}{1024} = \frac{7}{128}$$

Hence, the required probability is $\frac{7}{128}$.

Q22. The probability of a man hitting a target is 0.25. He shoots 7 times. What is the probability of his hitting at least twice?

Sol. Here $n = 7$, $\quad p = 0.25 = \frac{25}{100} = \frac{1}{4}$ and $q = 1 - \frac{1}{4} = \frac{3}{4}$

$$P(X \geq 2) = 1 - [P(X = 0) + P(X = 1)]$$

$$= 1 - \left[{}^7C_0 \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^7 + {}^7C_1 \left(\frac{1}{4}\right)^1 \left(\frac{3}{4}\right)^6\right]$$

$$= 1 - \left[\left(\frac{3}{4}\right)^7 + \frac{7}{4}\left(\frac{3}{4}\right)^6\right] = 1 - \left(\frac{3}{4}\right)^6 \left(\frac{3}{4} + \frac{7}{4}\right)$$

$$= 1 - \left(\frac{3}{4}\right)^6 \left(\frac{10}{4}\right) = 1 - \frac{729}{4096} \times \frac{10}{4} = 1 - \frac{7290}{16384}$$

$$= \frac{16384 - 7290}{16384} = \frac{9094}{16384} = \frac{4547}{8192}$$

Hence, the required probability is $\frac{4547}{8192}$.

Q23. A lot of 100 watches is known to have 10 defective watches. If 8 watches are selected (one by one with replacement) at random, what is the probability that there will be atleast one defective watch?

Sol. Probability of defective watch out of 100 watches $= \frac{10}{100} = \frac{1}{10}$.

Here, $n = 8$, $p = \frac{1}{10}$ and $q = 1 - \frac{1}{10} = \frac{9}{10}$ and $r \geq 1$

$$P(X \geq 1) = 1 - P(x = 0) = 1 - {}^8C_0 \left(\frac{1}{10}\right)^0 \left(\frac{9}{10}\right)^{8-0} = 1 - \left(\frac{9}{10}\right)^8$$

Hence, the required probability is $1 - \left(\frac{9}{10}\right)^8$.

Q24. Consider the probability distribution of a random variable X:

X	0	1	2	3	4
P(X)	0.1	0.25	0.3	0.2	0.15

Calculate: (*i*) $V\left(\frac{X}{2}\right)$ (*ii*) Variance of X.

Sol. Here, we have

X	0	1	2	3	4
P(X)	0.1	0.25	0.3	0.2	0.15

We know that: $\text{Var}(X) = E(X^2) - [E(X)]^2$

where $E(X) = \sum_{i=1}^{n} x_i p_i$ and $E(X^2) = \sum_{i=1}^{n} p_i x_i^2$

$$\therefore \quad E(X) = 0 \times 0.1 + 1 \times 0.25 + 2 \times 0.3 + 3 \times 0.2 + 4 \times 0.15$$
$$= 0 + 0.25 + 0.6 + 0.6 + 0.6 = 2.05$$
$$E(X^2) = 0 \times 0.1 + 1 \times 0.25 + 4 \times 0.3 + 9 \times 0.2 + 16 \times 0.15$$
$$= 0 + 0.25 + 1.2 + 1.8 + 2.40 = 5.65$$

(*i*) $$V\left(\frac{X}{2}\right) = \frac{1}{4}V(X) = \frac{1}{4}[5.65 - (2.05)^2] = \frac{1}{4}[5.65 - 4.2025]$$
$$= \frac{1}{4} \times 1.4475 = 0.361875$$

$$\left[\because V\left(\frac{X}{2}\right) = \frac{1}{4}V(X)\right]$$

(*ii*) $\text{Var}(X) = 1.4475$

Q25. The probability distribution of a random variable X is given below:

X	0	1	2	3
P(X)	k	$\frac{k}{2}$	$\frac{k}{4}$	$\frac{k}{8}$

(*i*) Determine the value of k.
(*ii*) Determine $P(X \le 2)$ and $P(X > 2)$
(*iii*) Find $P(X \le 2) + P(X > 2)$.

Sol. (*i*) We know that $P(0) + P(1) + P(2) + P(3) = 1$

$$\Rightarrow \quad k + \frac{k}{2} + \frac{k}{4} + \frac{k}{8} = 1$$

$$\Rightarrow \quad \frac{8k + 4k + 2k + k}{8} = 1 \quad \Rightarrow 15k = 8$$

$$\therefore \quad k = \frac{}{15}$$

(*ii*)
$$P(X \le 2) = P(X = 0) + P(X = 1) + P(X = 2)$$
$$= k + \frac{k}{2} + \frac{k}{4} = \frac{7k}{4} = \frac{7}{4} \times \frac{8}{15} = \frac{14}{15}$$

and $P(X > 2) = P(X = 3) = \frac{k}{8} = \frac{1}{8} \times \frac{8}{15} = \frac{1}{15}$

(*iii*) $P(X \le 2) + P(X > 2) = \frac{14}{15} + \frac{1}{15} = \frac{14+1}{15} = \frac{15}{15} = 1.$

Q26. For the following probability distribution, determine standard deviation of the random variable X.

X	2	3	4
P(X)	0.2	0.5	0.3

Sol. We know that: Standard deviation (S.D.) = $\sqrt{\text{Variance}}$

$\therefore \quad \text{Var}(X) = E(X^2) - [E(X)]^2$

$E(X) = 2 \times 0.2 + 3 \times 0.5 + 4 \times 0.3 = 0.4 + 1.5 + 1.2 = 3.1$

$E(X^2) = 4 \times 0.2 + 9 \times 0.5 + 16 \times 0.3 = 0.8 + 4.5 + 4.8 = 10.1$

So $\quad V(X) = 10.1 - (3.1)^2 = 10.1 - 9.61 = 0.49$

$\therefore \quad \text{S.D.} = \sqrt{\text{Var}(X)} = \sqrt{0.49} = 0.7$

Q27. A biased die is such that $P(4) = \frac{1}{10}$ and other scores being equally likely. The die is tossed twice. If X is the 'number of fours seen', find the variance of the random variable X.

Sol. Here, random variable $X = 0, 1, 2$

$$P(4) = \frac{1}{10}, \quad P(\bar{4}) = 1 - \frac{1}{10} = \frac{9}{10}$$

$$P(X = 0) = P(\bar{4}).P(\bar{4}) = \frac{9}{10} \times \frac{9}{10} = \frac{81}{100}$$

$$P(X = 1) = P(\bar{4}).P(4) + P(4).P(\bar{4}) = \frac{9}{10} \times \frac{1}{10} + \frac{1}{10} \times \frac{9}{10} = \frac{18}{100}$$

$$P(X=2) = P(4).P(4) = \frac{1}{10} \times \frac{1}{10} = \frac{1}{100}$$

X	0	1	2
P(X)	$\frac{81}{100}$	$\frac{18}{100}$	$\frac{1}{100}$

We know that $V(X) = E(X^2) - [E(X)]^2$

$$E(X) = 0 \times \frac{81}{100} + 1 \times \frac{18}{100} + \frac{2}{100} = \frac{20}{100} = \frac{1}{5}$$

$$E(X^2) = 0 \times \frac{81}{100} + 1 \times \frac{18}{100} + 4 \times \frac{1}{100} = \frac{22}{100} = \frac{11}{50}$$

$$\therefore \quad Var(X) = \frac{11}{50} - \left(\frac{1}{5}\right)^2 = \frac{11}{50} - \frac{1}{25} = \frac{9}{50} = 0.18$$

Hence, the required variance = 0.18.

Q28. A die is thrown three times. Let X be the 'number of twos seen', find the expectation of X.

Sol. Here, we have $X = 0, 1, 2, 3$ [$\because$ die is thrown 3 times]

and $p = \frac{1}{6}, q = \frac{5}{6}$

$$\therefore \; P(X=0) = P(\text{not } 2).P(\text{not } 2).P(\text{not } 2) = \frac{5}{6}.\frac{5}{6}.\frac{5}{6} = \frac{125}{216}$$

$$P(X=1) = P(2).P(\text{not } 2).P(\text{not } 2) + P(\text{not } 2).P(2).P(\text{not } 2) + P(\text{not } 2).P(\text{not } 2).P(2)$$

$$= \frac{1}{6}.\frac{5}{6}.\frac{5}{6} + \frac{5}{6}.\frac{1}{6}.\frac{5}{6} + \frac{5}{6}.\frac{5}{6}.\frac{1}{6} = \frac{25}{216} + \frac{25}{216} + \frac{25}{216} = \frac{75}{216}$$

$$P(X=2) = P(2).P(2).P(\text{not } 2) + P(2).P(\text{not } 2).P(2) + P(\text{not } 2).P(2).P(2)$$

$$= \frac{1}{6}.\frac{1}{6}.\frac{5}{6} + \frac{1}{6}.\frac{5}{6}.\frac{1}{6} + \frac{5}{6}.\frac{1}{6}.\frac{1}{6} = \frac{5}{216} + \frac{5}{216} + \frac{5}{216} = \frac{15}{216}$$

$$P(X=3) = P(2).P(2).P(2) = \frac{1}{6}.\frac{1}{6}.\frac{1}{6} = \frac{1}{216}$$

Now $E(X) = \sum_{i=1}^{n} p_i x_i$

$$= 0 \times \frac{125}{216} + 1 \times \frac{75}{216} + 2 \times \frac{15}{216} + 3 \times \frac{1}{216}$$

$$= 0 + \frac{75}{216} + \frac{30}{216} + \frac{3}{216} = \frac{75+30+3}{216} = \frac{108}{216} = \frac{1}{2}$$

Hence, the required expectation is $\frac{1}{2}$.

Q29. Two biased dice are thrown together. For the first die $P(6) = \frac{1}{2}$, the other scores being equally likely while for the second die, $P(1) = \frac{2}{5}$ and the other scores are equally likely. Find the probability distribution of 'the number of ones seen'.

Sol. Given that: for the first die, $P(6) = \frac{1}{2}$ and $P(\bar{6}) = 1 - \frac{1}{2} = \frac{1}{2}$

$\Rightarrow P(1) + P(2) + P(3) + P(4) + P(5) = \frac{1}{2}$

But $P(1) = P(2) = P(3) = P(4) = P(5)$

$\therefore 5.P(1) = \frac{1}{2} \Rightarrow P(1) = \frac{1}{10}$ and $P(\bar{1}) = 1 - \frac{1}{10} = \frac{9}{10}$

For the second die, $P(1) = \frac{2}{5}$ and $P(\bar{1}) = 1 - \frac{2}{5} = \frac{3}{5}$

Let X be the number of one's seen

$\therefore X = 0, 1, 2$

$$\Rightarrow \quad P(X=0) = \underset{\text{I}}{P(\bar{1})}.\underset{\text{II}}{P(\bar{1})} = \frac{9}{10}.\frac{3}{5} = \frac{27}{50} = 0.54$$

$$P(X=1) = \underset{\text{I}}{P(\bar{1})}\cdot\underset{\text{II}}{P(1)} + \underset{\text{I}}{P(1)}\cdot\underset{\text{II}}{P(\bar{1})}$$

$$= \frac{9}{10}.\frac{2}{5} + \frac{1}{10}.\frac{3}{5} = \frac{18+3}{50} = \frac{21}{50} = 0.42$$

$$P(X=2) = \underset{\text{I}}{P(1)}.\underset{\text{II}}{P(1)} = \frac{1}{10}.\frac{2}{5} = \frac{2}{50} = 0.04$$

Hence, the required probability distribution is

X	0	1	2
P(X)	0.54	0.42	0.04

Q30. Two probability distributions of the discrete random variable X and Y are given below.

X	0	1	2	3
P(X)	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

(Y)	0	1	2	3
P(Y)	$\frac{1}{5}$	$\frac{3}{10}$	$\frac{2}{5}$	$\frac{1}{10}$

Prove that: $E(Y^2) = 2E(X)$.

Sol. First probability distribution is given by

X	0	1	2	3
P(X)	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

We know that, $E(X) = \sum_{i=1}^{n} P_i X_i$

$$\Rightarrow \quad E(X) = 0 \cdot \frac{1}{5} + 1 \cdot \frac{2}{5} + 2 \cdot \frac{1}{5} + 3 \cdot \frac{1}{5} = 0 + \frac{2}{5} + \frac{2}{5} + \frac{3}{5} = \frac{7}{5}$$

For the second probability distribution,

Y	0	1	2	3
P(Y)	$\frac{1}{5}$	$\frac{3}{10}$	$\frac{2}{5}$	$\frac{1}{10}$

$$E(Y^2) = 0 \cdot \frac{1}{5} + 1 \cdot \frac{3}{10} + 4 \cdot \frac{2}{5} + 9 \cdot \frac{1}{10}$$

$$= 0 + \frac{3}{10} + \frac{8}{5} + \frac{9}{10} = \frac{28}{10} = \frac{14}{5}$$

Now $E(Y^2) = \frac{14}{5}$ and $2\,E(X) = 2.\frac{7}{5} = \frac{14}{5}$

Hence, $E(Y^2) = 2E(X)$.

Q31. A factory produces bulbs. The probability that any one bulb is defective is $\frac{1}{50}$ and they are packed in boxes of 10. From the single box, find the probability that (*i*) none of the bulbs is defective (*ii*) exactly two bulbs are defective (*iii*) more than 8 bulbs work properly.

Sol. Let X be the random variable denoting a bulb to be defective.

Here, $n = 10, p = \frac{1}{50}, q = 1 - \frac{1}{50} = \frac{49}{50}$

We know that $P(X = r) = {}^nC_r p^r q^{n-r}$

(*i*) None of the bulbs is defective, i.e., $r = 0$

$$P(x = 0) = {}^{10}C_0 \left(\frac{1}{50}\right)^0 \left(\frac{49}{50}\right)^{10-0} = \left(\frac{49}{50}\right)^{10}$$

(*ii*) Exactly two bulbs are defective

$$\therefore \quad P(x = 2) = {}^{10}C_2 \left(\frac{1}{50}\right)^2 \left(\frac{49}{50}\right)^{10-2}$$

$$= 45.\frac{(49)^8}{(50)^{10}} = 45 \times \left(\frac{1}{50}\right)^{10} \times (49)^8$$

(*iii*) More than 8 bulbs work properly

We can say that less than 2 bulbs are defective

$$P(x < 2) = P(x = 0) + P(x = 1)$$

$$= {}^{10}C_0 \left(\frac{1}{50}\right)^0 \left(\frac{49}{50}\right)^{10} + {}^{10}C_1 \left(\frac{1}{50}\right)^1 \left(\frac{49}{50}\right)^9 = \left(\frac{49}{50}\right)^{10} + \frac{1}{5}\left(\frac{49}{50}\right)^9$$

$$= \left(\frac{49}{50}\right)^9 \left(\frac{49}{50} + \frac{1}{5}\right) = \left(\frac{49}{50}\right)^9 \left(\frac{59}{50}\right) = \frac{59\,(49)^9}{(50)^{10}}.$$

Q32. Suppose you have two coins which appear identical in your pocket. You know that one is fair and one is 2-headed coin. If you take one out, toss it and get a head, what is the probability that it was a fair coin?

Sol. Let E_1 = Event that the coin is fair

E_2 = Event that the coin is 2-headed

and H = Event that the tossed coin gets head.

$$P(E_1) = \frac{1}{2}, \quad P(E_2) = \frac{1}{2}, \quad P(H/E_1) = \frac{1}{2}, \quad P(H/E_2) = 1$$

$\therefore$ Using Bayes' Theorem, we get

$$P(E_1/H) = \frac{P(E_1).P(H/E_1)}{P(E_1).P(H/E_1) + P(E_2).P(H/E_2)}$$

$$= \frac{\frac{1}{2}.\frac{1}{2}}{\frac{1}{2}.\frac{1}{2} + \frac{1}{2}.1} = \frac{\frac{1}{4}}{\frac{1}{4} + \frac{1}{2}} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

Hence the required probability is $\frac{1}{3}$.

Q33. Suppose that 6% of the people with blood group O are left handed and 10% of those with other blood groups are left handed, 30% of the people have blood group O. If a left handed person is selected at random, what is the probability that he/she will have blood group O?

Sol. Let E_1 = The event that a person selected is of blood group O

E_2 = The event that the person selected is of other group

and H = The event that selected person is left handed

$\therefore \quad P(E_1) = 0.30$ and $P(E_2) = 0.70$

$P(H/E_1) = 0.06 \qquad P(H/E_2) = 0.10$

So, from Bayes' Theorem

$$P\left(\frac{E_1}{H}\right) = \frac{P(E_1).P(H/E_1)}{P(E_1).P(H/E_1) + P(E_2).P(H/E_2)}$$

$$= \frac{0.30 \times 0.06}{0.30 \times 0.06 + 0.70 \times 0.10} = \frac{0.018}{0.018 + 0.070} = \frac{0.018}{0.088} = \frac{9}{44}$$

Hence, the required probability is $\frac{9}{44}$.

Q34. Two natural numbers r and s are drawn one at a time, without replacement from the set $S = \{1, 2, 3, ..., n\}$. Find $P(r \le p/s \le p)$, where $p \in S$.

Sol. Given that: $S = \{1, 2, 3, ..., n\}$

$$\therefore \quad P(r \le p/s \le p) = \frac{P(P \cap S)}{P(S)} = \frac{p-1}{n} \times \frac{n}{n-1} = \frac{p-1}{n-1}$$

Hence, the required probability is $\frac{p-1}{n-1}$.

Q35. Find the probability distribution of the maximum of the two scores obtained when a die is thrown twice. Determine also the mean of the distribution.

Sol. Let X be the random variable scores when a die is thrown twice.

$$X = 1, 2, 3, 4, 5, 6$$

and $\quad S = \{(1, 1), (1, 2), (2, 1), (2, 2), (1, 3), (2, 3), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), ..., (6, 6)\}$

So, $\quad P(X = 1) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$

$$P(X = 2) = \frac{1}{6} \cdot \frac{1}{6} + \frac{1}{6} \cdot \frac{1}{6} + \frac{1}{6} \cdot \frac{1}{6} = \frac{3}{36}$$

$$P(X = 3) = \frac{1}{6} \cdot \frac{1}{6} + \frac{1}{6} \cdot \frac{1}{6} + \frac{1}{6} \cdot \frac{1}{6} + \frac{1}{6} \cdot \frac{1}{6} + \frac{1}{6} \cdot \frac{1}{6} = \frac{5}{36}$$

Similarly

$$P(X = 4) = \frac{7}{36}; \; P(X = 5) = \frac{9}{36} \text{ and } P(X = 6) = \frac{11}{36}$$

So, the required distribution is

X	1	2	3	4	5	6
P(X)	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

Now, the mean $E(X) = \sum_{i=1}^{n} x_i p_i$

$$= 1 \times \frac{1}{36} + 2 \times \frac{3}{36} + 3 \times \frac{5}{36} + 4 \times \frac{7}{36} + 5 \times \frac{9}{36} + 6 \times \frac{11}{36}$$

$$= \frac{1}{36} + \frac{6}{36} + \frac{15}{36} + \frac{28}{36} + \frac{45}{36} + \frac{66}{36} = \frac{161}{36}$$

Hence, the required mean $= \frac{161}{36}$.

Q36. The random variable X can take only the values 0, 1, 2. If $P(X=0)=P(X=1)=p$ and $E(X^2)=E(X)$, then find the value of p.

Sol. Given that: $X = 0, 1, 2$

and P(X) at $X=0$ and 1 is p. Let P(X) at $X=2$ is x

$$\Rightarrow \quad p+p+x=1 \quad \Rightarrow \quad x=1-2p$$

Now we have the following distributions.

X	0	1	2
P(X)	p	p	$1-2p$

$$\therefore \quad E(X) = 0.p+1.p+2(1-2p)=p+2-4p=2-3p$$

and $$E(X^2) = 0.p+1.p+4(1-2p)=p+4-8p=4-7p$$

Given that: $E(X^2) = E(X)$

$$\therefore \quad 4-7p=2-3p \quad \Rightarrow \quad 4p=2 \quad \Rightarrow \quad p=\frac{1}{2}$$

Hence, the required value of p is $\frac{1}{2}$.

Q37. Find the variance of the distribution:

X	0	1	2	3	4	5
P(X)	$\frac{1}{6}$	$\frac{5}{18}$	$\frac{2}{9}$	$\frac{1}{6}$	$\frac{1}{9}$	$\frac{1}{18}$

Sol. We know that:

$$\text{Variance }(X) = E(X^2)-[E(X)]^2$$

$$E(X) = \sum_{i=1}^{n} p_i x_i$$

$$= 0\times\frac{1}{6}+1\times\frac{5}{18}+2\times\frac{2}{9}+3\times\frac{1}{6}+4\times\frac{1}{9}+5\times\frac{1}{18}$$

$$= 0+\frac{5}{18}+\frac{4}{9}+\frac{3}{6}+\frac{4}{9}+\frac{5}{18}=\frac{5+8+9+8+5}{18}=\frac{35}{18}$$

$$E(X^2) = 0\times\frac{1}{6}+1\times\frac{5}{18}+4\times\frac{2}{9}+9\times\frac{1}{6}+16\times\frac{1}{9}+25\times\frac{1}{18}$$

$$= \frac{5}{18}+\frac{8}{9}+\frac{9}{6}+\frac{16}{9}+\frac{25}{18}=\frac{5+16+27+32+25}{18}=\frac{105}{18}$$

$$\therefore \text{Var }(X) = \frac{105}{18}-\frac{35}{18}\times\frac{35}{18}=\frac{1890-1225}{324}=\frac{665}{324}$$

Hence, the required variance is $\frac{665}{324}$.

Q38. A and B throw a pair of dice alternately. A wins the game if he gets a total of 6 and B wins if she gets a total of 7. If A starts the game, find the probability of winning the game by A in third throw of the pair of dice.

Sol. Let A_1 be the event of getting a total of 6

$$= \{(2, 4), (4, 2), (1, 5), (5, 1), (3, 3)\}$$

and B_1 be the event of getting a total of 7

$$= \{(2, 5), (5, 2), (1, 6), (6, 1), (3, 4), (4, 3)\}$$

Let $P(A_1)$ is the probability, if A wins in a throw $= \frac{5}{36}$

and $P(B_1)$ is the probability, if B wins in a throw $= \frac{1}{6}$

$\therefore$ The required probability of winning A in his third throw

$$= P(\bar{A}_1) \, . \, P(\bar{B}_1) \, . \, P(A_1) = \frac{31}{36} \cdot \frac{5}{6} \cdot \frac{5}{36} = \frac{775}{7776}.$$

Q39. Two dice are tossed. Find whether the following two events A and B are independent. $A = \{(x, y) : x + y = 11\}$ and $B = \{(x, y) : x \neq 5\}$ where (x, y) denotes a typical sample point.

Sol. Given that:

$$A = \{(x, y) : x + y = 11\} \text{ and } B = \{(x, y) : x \neq 5\}$$

$\therefore \quad A = \{(5, 6), (6, 5)\},$

$B = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}$

$\Rightarrow n(A) = 2, n(B) = 30$ and $n(A \cap B) = 1$

$\therefore \quad P(A) = \frac{2}{36} = \frac{1}{18}$ and $P(B) = \frac{30}{36} = \frac{5}{6}$

$\Rightarrow \quad P(A).\, P(B) = \frac{1}{18} \cdot \frac{5}{6} = \frac{5}{108}$ and $P(A \cap B) = \frac{1}{36}$

Since $P(A).\, P(B) \neq P(A \cap B)$

Hence, A and B are not independent.

Q40. An urn contains m white and n black balls. A ball is drawn at random and is put back into the urn along with k additional balls of the same colour as that of the ball drawn. A ball is again drawn at random. Show that the probability of drawing a white ball now does not depend on k.

Sol. Let A be the event having m white and n black balls

E_1 = {first ball drawn of white colour}

E_2 = {first ball drawn of black colour}

E_3 = {second ball drawn of white colour}

$$\therefore \quad P(E_1) = \frac{m}{m+n} \text{ and } P(E_2) = \frac{n}{m+n}$$

$$P(E_3/E_1) = \frac{m+k}{m+n+k} \text{ and } P(E_3/E_2) = \frac{m}{m+n+k}$$

Now $P(E_3) = P(E_1) \cdot P(E_3/E_1) + P(E_2) \cdot P(E_3/E_2)$

$$= \frac{m}{m+n} \times \frac{m+k}{m+n+k} + \frac{n}{m+n} \times \frac{m}{m+n+k}$$

$$= \frac{m}{m+n+k}\left[\frac{m+k}{m+n} + \frac{n}{m+n}\right]$$

$$= \frac{m}{m+n+k}\left[\frac{m+n+k}{m+n}\right] = \frac{m}{m+n}$$

Hence, the probability of drawing a white ball does not depend upon k.

LONG ANSWER TYPE QUESTIONS

Q41. Three bags contain a number of red and white balls as follows, Bag I: 3 red balls, Bag II: 2 red balls and 1 white ball and Bag III: 3 white balls. The probability that bag i will be chosen and a ball is selected from it is $\frac{i}{6}$, where $i = 1, 2, 3$. What is the probability that (*i*) a red ball will be selected (*ii*) a white ball is selected?

Sol. Given that:

Bag I: 3 red balls and no white ball

Bag II: 2 red balls and 1 white ball

Bag III: no red ball and 3 white balls

Let E_1, E_2 and E_3 be the events of choosing Bag I, Bag II and Bag III respectively and a ball is drawn from it.

$$\therefore P(E_1) = \frac{1}{6},\ P(E_2) = \frac{2}{6} \text{ and } P(E_3) = \frac{3}{6}$$

(*i*) Let E be the event that red ball is selected

$$\therefore P(E) = P(E_1).P(E/E_1) + P(E_2).P(E/E_2) + P(E_3).P(E/E_3)$$

$$= \frac{1}{6}.\frac{3}{3} + \frac{2}{6}.\frac{2}{3} + \frac{3}{6}.0 = \frac{3}{18} + \frac{4}{18} = \frac{7}{18}$$

(*ii*) Let F be the event that a white ball is selected

$\therefore P(F) = 1 - P(E)$ $\qquad$ $[P(E) + P(F) = 1]$

$$= 1 - \frac{7}{18} = \frac{11}{18}$$

Hence, the required probabilities are $\frac{7}{18}$ and $\frac{11}{18}$.

Q42. Refer to Exercise Q.41 above. If a white ball is selected, what is the probability that it came from

(i) Bag II (ii) Bag III?

Sol. Referring to Exercise Q.41, we will use here Bayes' Theorem

$$(i)\ P(E_2/F) = \frac{P(E_2).P(F/E_2)}{P(E_1).P(F/E_1)+P(E_2).P(F/E_2)+P(E_3).P(F/E_3)}$$

$$= \frac{\frac{2}{6}.\frac{1}{3}}{\frac{1}{6}.0+\frac{2}{6}.\frac{1}{3}+\frac{3}{6}.1} = \frac{\frac{2}{18}}{\frac{2}{18}+\frac{3}{6}} = \frac{2}{11}$$

$$(ii)\ P(E_3/F) = \frac{P(E_3).P(F/E_3)}{P(E_1).P(F/E_1)+P(E_2).P(F/E_2)+P(E_3).P(F/E_3)}$$

$$= \frac{\frac{3}{6}.1}{\frac{1}{6}.0+\frac{2}{6}.\frac{1}{3}+\frac{3}{6}.1} = \frac{\frac{3}{6}}{\frac{2}{18}+\frac{3}{6}} = \frac{3}{6}\times\frac{18}{11} = \frac{9}{11}$$

Hence, the required probabilities are $\frac{2}{11}$ and $\frac{9}{11}$.

Q43. A shopkeeper sells three types of flower seeds A_1, A_2 and A_3. They are sold as a mixture, where the proportions are 4 : 4 : 2, respectively. The germination rates of the three types of seeds are 45%, 60% and 35%. Calculate the probability

(i) of a randomly chosen seed to germinate

(ii) that it will not germinate given that the seed is of type A_3

(iii) that it is of the type A_2 given that a randomly chosen seed does not germinate.

Sol. Given that $A_1 : A_2 : A_3 = 4 : 4 : 2$

$$\therefore P(A_1) = \frac{4}{10},\ P(A_2) = \frac{4}{10} \text{ and } P(A_3) = \frac{2}{10}$$

where A_1, A_2 and A_3 are the three types of seeds.

Let E be the event that a seed germinates and $\bar{E}$ be the event that a seed does not germinate

$$\therefore P\left(\frac{E}{A_1}\right) = \frac{45}{100},\ P\left(\frac{E}{A_2}\right) = \frac{60}{100} \text{ and } P\left(\frac{E}{A_3}\right) = \frac{35}{100}$$

$$\text{and } P\left(\frac{\bar{E}}{A_1}\right) = \frac{55}{100},\ P\left(\frac{\bar{E}}{A_2}\right) = \frac{40}{100} \text{ and } P\left(\frac{\bar{E}}{A_3}\right) = \frac{65}{100}$$

(*i*) $P(E) = P(A_1).P\left(\frac{E}{A_1}\right) + P(A_2).P\left(\frac{E}{A_2}\right) + P(A_3).P\left(\frac{E}{A_3}\right)$

$$= \frac{4}{10}\cdot\frac{45}{100} + \frac{4}{10}\cdot\frac{60}{100} + \frac{2}{10}\cdot\frac{35}{100}$$

$$= \frac{180}{1000} + \frac{240}{1000} + \frac{70}{1000} = \frac{490}{1000} = 0.49$$

(*ii*) $P(\bar{E}/A_3) = 1 - P(E/A_3) = 1 - \frac{35}{100} = \frac{65}{100} = 0.65$

(*iii*) Using Bayes' Theorem, we get

$$P(A_2/\bar{E}) = \frac{P(A_2).P(\bar{E}/A_2)}{P(A_1).P(\bar{E}/A_1)+P(A_2).P(\bar{E}/A_2)+P(A_3).P(\bar{E}/A_3)}$$

$$= \frac{\frac{4}{10}\cdot\frac{40}{100}}{\frac{4}{10}\cdot\frac{55}{100} + \frac{4}{10}\cdot\frac{40}{100} + \frac{2}{10}\cdot\frac{65}{100}}$$

$$= \frac{\frac{160}{1000}}{\frac{220}{1000} + \frac{160}{1000} + \frac{130}{1000}} = \frac{160}{220+160+130} = \frac{160}{510} = \frac{16}{51} = 0.314$$

Hence, the required probability is $\frac{16}{51}$ or 0.314.

Q44. A letter is known to have come either from TATA NAGAR or from CALCUTTA. On the envelope, just two consecutive letters TA are visible. What is the probability that the letter came from TATA NAGAR?

Sol. Let E_1 : The event that the letter comes from TATA NAGAR

and E_2 : The event that the letter comes from CALCUTTA

Also E_3 : The event that on the letter, two consecutive letters TA are visible

$\therefore P(E_1) = \frac{1}{2}$ and $P(E_2) = \frac{1}{2}$ and $P\left(\frac{E_3}{E_1}\right) = \frac{2}{8}$ and $P\left(\frac{E_3}{E_2}\right) = \frac{1}{7}$

[$\because$ For TATA NAGAR, the two consecutive letters visible are TA, AT, TA, AN, NA, AG, GA, AR] $\therefore$ $P(E_3/E_1) = \frac{2}{8}$

and [For CALCUTTA, the two consecutive letters visible are CA, AL, LC, CU, UT, TT and TA] So, $P(E_3/E_2) = \frac{1}{7}$

Now using Bayes' Theorem, we have

$$P(E_1/E_3) = \frac{P(E_1).P(E_3/E_1)}{P(E_1).P(E_3/E_1)+P(E_2).P(E_3/E_2)}$$

$$= \frac{\frac{1}{2}\cdot\frac{2}{8}}{\frac{1}{2}\cdot\frac{2}{8}+\frac{1}{2}\cdot\frac{1}{7}} = \frac{\frac{1}{8}}{\frac{1}{8}+\frac{1}{14}} = \frac{\frac{1}{8}}{\frac{7+4}{56}} = \frac{7}{11}$$

Hence, the required probability is $\frac{7}{11}$.

Q45. There are two bags, one of which contains 3 black and 4 white balls while the other contains 4 black and 3 white balls. A die is thrown. If it shows up 1 or 3, a ball is taken from the first bag but if it shows up any other number, a ball is chosen from the second bag. Find the probability of choosing a black ball.

Sol. Let E_1 be the event of selecting Bag I

and E_2 be the event of selecting Bag II

Let E_3 be the event that black ball is selected

$\therefore \quad P(E_1) = \frac{2}{6} = \frac{1}{3}$ and $P(E_2) = 1 - \frac{1}{3} = \frac{2}{3}$

$P(E_3/E_1) = \frac{3}{7}$ and $P(E_3/E_2) = \frac{4}{7}$

$\therefore \quad P(E_3) = P(E_1).P(E_3/E_1) + P(E_2).P(E_3/E_2)$

$$= \frac{1}{3}\cdot\frac{3}{7}+\frac{2}{3}\cdot\frac{4}{7} = \frac{3+8}{21} = \frac{11}{21}$$

Hence, the required probability is $\frac{11}{21}$.

Q46. There are three urns containing 2 white and 3 black balls, 3 white and 2 black balls and 4 white and 1 black balls, respectively. There is an equal probability of each urn being chosen. A ball is drawn at random from the chosen urn and it is found to be white. Find the probability that the ball drawn was from the second urn.

Sol. We have 3 urns:

2 W 3 B	3 W 2 B	4 W 1 B
U_1	U_2	U_3

$\therefore$ Probabilities of choosing either of the urns are

$$P(U_1) = P(U_2) = P(U_3) = \frac{1}{3}$$

Let H be the event of drawing white ball from the chosen urn.

$\therefore P(H/U_1) = \frac{2}{5}, P(H/U_2) = \frac{3}{5}$ and $P(H/U_3) = \frac{4}{5}$

$$\therefore P(U_2/H) = \frac{P(U_2).P(H/U_2)}{P(U_1).P(H/U_1)+P(U_2).P(H/U_2)+P(U_3).P(H/U_3)}$$

$$= \frac{\frac{1}{3}.\frac{3}{5}}{\frac{1}{3}.\frac{2}{5}+\frac{1}{3}.\frac{3}{5}+\frac{1}{3}.\frac{4}{5}} = \frac{\frac{3}{5}}{\frac{2}{5}+\frac{3}{5}+\frac{4}{5}} = \frac{3}{9} = \frac{1}{3}$$

Hence, the required probability is $\frac{1}{3}$.

Q47. By examining the chest X-ray, the probability that TB is detected when a person is actually suffering is 0.99. The probability of an healthy person diagnosed to have TB is 0.001. In a certain city, 1 in 1000 people suffer from TB. A person is selected at random and is diagnosed to have TB. What is the probability that he actually has TB?

Sol. Let E_1: Event that a person has TB

E_2: Event that a person does not have TB

and H: Event that the person is diagnosed to have TB.

$$\therefore \quad P(E_1) = \frac{1}{1000} = 0.001, \; P(E_2) = 1 - \frac{1}{1000} = \frac{999}{1000} = 0.999$$

$$P(H/E_1) = 0.99, \; P(H/E_2) = 0.001$$

$$\therefore P(E_1/H) = \frac{P(E_1).P(H/E_1)}{P(E_1).P(H/E_1) + P(E_2).P(H/E_2)}$$

$$= \frac{0.001 \times 0.99}{0.001 \times 0.99 + 0.999 \times 0.001} = \frac{0.99}{0.99 + 0.999}$$

$$= \frac{0.990}{0.990 + 0.999} = \frac{990}{1989} = \frac{110}{221}$$

Hence, the required probability is $\frac{110}{221}$.

Q48. An item is manufactured by three machines A, B and C. Out of the total number of items manufactured during a specified period, 50% are manufactured on machine A, 30% on B and 20% on C. 2% of items produced on A and 2% of items produced on B are defective and 3% of these produced on machine C are defective. All the items are stored at one godown. One item is drawn at random and is found to be defective. What is the probability that it was manufactured on machine A?

Sol. Let E_1 : The event that the item is manufactured on machine A

E_2 : The event that the item is manufactured on machine B

E_3 : The event that the item is manufactured on machine C

Let H be the event that the selected item is defective.

$\therefore$ Using Bayes' Theorem,

$$P(E_1) = \frac{50}{100}, \; P(E_2) = \frac{30}{100}, \; P(E_3) = \frac{20}{100}$$

$$P(H/E_1) = \frac{2}{100}, \; P(H/E_2) = \frac{2}{100} \text{ and } P(H/E_3) = \frac{3}{100}$$

$$\therefore P(E_1/H) = \frac{P(E_1).P(H/E_1)}{P(E_1).P(H/E_1) + P(E_2).P(H/E_2) + P(E_3).P(H/E_3)}$$

$$= \frac{\frac{50}{100} \times \frac{2}{100}}{\frac{50}{100} \times \frac{2}{100} + \frac{30}{100} \times \frac{2}{100} + \frac{20}{100} \times \frac{3}{100}}$$

$$= \frac{100}{100 + 60 + 60} = \frac{100}{220} = \frac{10}{22} = \frac{5}{11}$$

Hence, the required probability is $\frac{5}{11}$.

Q49. Let X be a discrete random variable whose probability distribution is defined as follows:

$$P(X = x) = \begin{cases} k(x+1) & \text{for } x = 1, 2, 3, 4 \\ 2kx & \text{for } x = 5, 6, 7 \\ 0, & \text{otherwise} \end{cases}$$

where k is a constant—Calculate:

(*i*) the value of k (*ii*) E(X) (*iii*) Standard deviation of X.

Sol. (*i*) Here, $P(X = x) = k(x + 1)$ for $x = 1, 2, 3, 4$

So, $P(X = 1) = k(1 + 1) = 2k$; $P(X = 2) = k(2 + 1) = 3k$

$P(X = 3) = k(3 + 1) = 4k$; $P(X = 4) = k(4 + 1) = 5k$

Also, $P(X = x) = 2kx$ for $x = 5, 6, 7$

$P(X = 5) = 2(5)k = 10k$; $P(X = 6) = 2(6)k = 12k$

$P(X = 7) = 2(7)k = 14k$

and for otherwise it is 0.

$\therefore$ The probability distribution is given by

X	1	2	3	4	5	6	7	otherwise
P(X)	$2k$	$3k$	$4k$	$5k$	$10k$	$12k$	$14k$	0

We know that $\sum_{i=1}^{n} P(X_i) = 1$

So, $2k + 3k + 4k + 5k + 10k + 12k + 14k = 1$

$$\Rightarrow \quad 50k = 1 \quad \Rightarrow \quad k = \frac{1}{50}$$

Hence, the value of k is $\frac{1}{50}$

(*ii*) Now the probability distribution is

X	1	2	3	4	5	6	7
P(X)	$\frac{2}{50}$	$\frac{3}{50}$	$\frac{4}{50}$	$\frac{5}{50}$	$\frac{10}{50}$	$\frac{12}{50}$	$\frac{14}{50}$

$$E(X) = 1\times\frac{2}{50}+2\times\frac{3}{50}+3\times\frac{4}{50}+4\times\frac{5}{50}+5\times\frac{10}{50}+6\times\frac{12}{50}+7\times\frac{14}{50}$$

$$= \frac{2}{50}+\frac{6}{50}+\frac{12}{50}+\frac{20}{50}+\frac{50}{50}+\frac{72}{50}+\frac{98}{50} = \frac{260}{50} = \frac{26}{5} = 5.2$$

(*iii*) We know that Standard deviation (SD) = $\sqrt{\text{Variance}}$

Variance= $E(X^2) - [E(X)]^2$

$$E(X^2) = 1\times\frac{2}{50}+4\times\frac{3}{50}+9\times\frac{4}{50}+16\times\frac{5}{50}+25\times\frac{10}{50}$$
$$36\times\frac{12}{50}+49\times\frac{14}{50}$$

$$= \frac{2}{50}+\frac{12}{50}+\frac{36}{50}+\frac{80}{50}+\frac{250}{50}+\frac{432}{50}+\frac{686}{50} = \frac{1498}{50}$$

$$\therefore \text{ Variance (X)} = \frac{1498}{50} - \left(\frac{26}{5}\right)^2$$

$$= \frac{1498}{50} - \frac{676}{25} = \frac{1498-1352}{50} = \frac{146}{50} = 2.92$$

$$\therefore \quad \text{S.D} = \sqrt{2.92} = 1.7 \text{ (approx.)}$$

Q50. The probability distribution of a discrete random variable X is given as under:

X	1	2	4	2A	3A	5A
P(X)	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{3}{25}$	$\frac{1}{10}$	$\frac{1}{25}$	$\frac{1}{25}$

Calculate: (*i*) The value of A if E(X) = 2.94; (*ii*) Variance of X

Sol. (*i*) We know that: $E(X) = \sum_{i=1}^{n} P_i X_i$

$$\therefore \quad E(X) = 1\times\frac{1}{2}+2\times\frac{1}{5}+4\times\frac{3}{25}+2A\times\frac{1}{10}+3A\times\frac{1}{25}+5A\times\frac{1}{25}$$

$$2.94 = \frac{1}{2}+\frac{2}{5}+\frac{12}{25}+\frac{A}{5}+\frac{3A}{25}+\frac{A}{5}$$

$$\Rightarrow \quad 2.94 = 0.5+0.4+0.48+\frac{13A}{25} = 1.38+\frac{13A}{25}$$

$$\Rightarrow \quad 2.94-1.38 = \frac{13A}{25} \Rightarrow 1.56 = \frac{13A}{25}$$

$$\Rightarrow \quad A = \frac{1.56\times 25}{13} = 0.12\times 25$$

$$\therefore \quad A = 3$$

(*ii*) Now the distribution becomes

X	1	2	4	6	9	15
P(X)	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{3}{25}$	$\frac{1}{10}$	$\frac{1}{25}$	$\frac{1}{25}$

$$E(X^2) = 1\times\frac{1}{2}+4\times\frac{1}{5}+16\times\frac{3}{25}+36\times\frac{1}{10}+81\times\frac{1}{25}+225\times\frac{1}{25}$$

$$= \frac{1}{2}+\frac{4}{5}+\frac{48}{25}+\frac{36}{10}+\frac{81}{25}+\frac{225}{25}$$

$$= 0.5+0.8+1.92+3.6+3.24+9.00 = 19.06$$

$$\text{Variance }(X) = E(X^2)-[E(X)]^2$$

$$= 19.06-(2.94)^2 = 19.06-8.64 = 10.42$$

Q51. The probability distribution of a random variable X is given as under:

$$P(X=x) = \begin{cases} kx^2 & \text{for } x = 1, 2, 3 \\ 2kx & \text{for } x = 4, 5, 6 \\ 0 & \text{otherwise} \end{cases}$$

where k is a constant. Calculate:

(*i*) E(X) (*ii*) $E(3X^2)$ (*iii*) $P(X \geq 4)$

Sol. Given that: $P(X=x) = \begin{cases} kx^2 & \text{for } x = 1, 2, 3 \\ 2kx & \text{for } x = 4, 5, 6 \\ 0 & \text{otherwise} \end{cases}$

$\therefore$ Probability distribution of random variable X is

X	1	2	3	4	5	6	otherwise
P(X)	k	$4k$	$9k$	$8k$	$10k$	$12k$	0

We know that $\sum_{i=1}^{n} P(X_i) = 1$

$\therefore\ k+4k+9k+8k+10k+12k = 1 \Rightarrow 44k = 1 \Rightarrow k = \frac{1}{44}$

(i) $E(X) = \sum_{i=1}^{n} P_i X_i = 1\times k + 2\times 4k + 3\times 9k + 4\times 8k + 5\times 10k + 6\times 12k$

$= k + 8k + 27k + 32k + 50k + 72k = 190k$

$= 190 \times \frac{1}{44} = \frac{95}{22} = 4.32$ (approx.)

(ii) $E(3X^2) = 3[k + 4\times 4k + 9\times 9k + 16\times 8k + 25\times 10k + 36\times 12k]$

$= 3[k + 16k + 81k + 128k + 250k + 432k] = 3[908k]$

$= 3\times 908 \times \frac{1}{44} = \frac{2724}{44} = 61.9$ (approx.)

(iii) $P(X \geq 4) = P(X=4) + P(X=5) + P(X=6)$

$= 8k + 10k + 12k = 30k$

$= 30 \times \frac{1}{44} = \frac{15}{22}$.

Q52. A bag contains $(2n+1)$ coins. It is known that n of these coins have a head on both sides whereas the rest of the coins are fair. A coin is picked up at random from the bag and is tossed. If the probability that the toss results in a head is $\frac{31}{42}$, determine the value of n.

Sol. Given that n coins are two headed coins and the remaining $(n+1)$ coins are fair.

Let E_1 : the event that unfair coin is selected

E_2 : the event that the fair coin is selected

E : the event that the toss results in a head

$\therefore\quad P(E_1) = \frac{n}{2n+1}$ and $P(E_2) = \frac{n+1}{2n+1}$

$P(E/E_1) = 1$ (sure event) and $P(E/E_2) = \frac{1}{2}$

$\therefore\quad P(E) = P(E_1).P(E/E_1) + P(E_2).P(E/E_2)$

$= \frac{n}{2n+1}.1 + \frac{n+1}{2n+1}.\frac{1}{2} = \frac{1}{2n+1}\left(n + \frac{n+1}{2}\right)$

$= \frac{1}{2n+1}\left(\frac{2n+n+1}{2}\right) = \frac{3n+1}{2(2n+1)}$

But $\quad P(E) = \dfrac{31}{42}$ (given)

$$\therefore \frac{3n+1}{2(2n+1)} = \frac{31}{42} \quad \Rightarrow \quad \frac{3n+1}{2n+1} = \frac{31}{21}$$

$$\Rightarrow \quad 63n + 21 = 62n + 31$$

$$\Rightarrow \quad n = 10$$

Hence, the required value of n is 10.

Q53. Two cards are drawn successively without replacement from a well shuffled deck of cards. Find the mean and standard deviation of the random variable X where X is the number of aces.

Sol. Let X be the random variable such that X = 0, 1, 2

and E = the event of drawing an ace

and F = the event of drawing non-ace.

$$\therefore P(E) = \frac{4}{52} \text{ and } P(\bar{E}) = \frac{48}{52}$$

$$\text{Now } P(X=0) = P(\bar{E}).P(\bar{E}) = \frac{48}{52}.\frac{47}{51} = \frac{188}{221}$$

$$P(X=1) = P(E).P(\bar{E}) + P(\bar{E}).P(E) = \frac{4}{52}\times\frac{48}{51} + \frac{48}{52}\times\frac{4}{51} = \frac{32}{221}$$

$$P(X=2) = P(E).P(E) = \frac{4}{52}.\frac{3}{51} = \frac{1}{221}$$

We have Distribution Table:

X	0	1	2
P(X)	$\frac{188}{221}$	$\frac{32}{221}$	$\frac{1}{221}$

$$\text{Now, Mean } E(X) = 0\times\frac{188}{221} + 1\times\frac{32}{221} + 2\times\frac{1}{221} = \frac{32}{221} + \frac{2}{221} = \frac{34}{221} = \frac{2}{13}$$

$$E(X^2) = 0\times\frac{188}{221} + 1\times\frac{32}{221} + 4\times\frac{1}{221} = \frac{32}{221} + \frac{4}{221} = \frac{36}{221}$$

$$\therefore \text{ Variance} = E(X^2) - [E(X)]^2$$

$$= \frac{36}{221} - \left(\frac{2}{13}\right)^2 = \frac{36}{221} \quad \frac{4}{169} = \frac{468-68}{13\times 221} = \frac{400}{2873}$$

$$\text{Standard deviation} = \sqrt{\frac{400}{2873}} = 0.377 \text{ (approx.)}$$

Q54. A die is tossed twice. A 'success' is getting an even number on a toss. Find the variance of the number of successes.

Sol. Let E be the event of getting even number on tossing a die.

$\therefore\ P(E) = \frac{3}{6} = \frac{1}{2}$ and $P(\bar{E}) = 1 - \frac{1}{2} = \frac{1}{2}$

Here X = 0, 1, 2

$$P(X=0) = P(\bar{E}).P(\bar{E}) = \frac{1}{2}.\frac{1}{2} = \frac{1}{4}$$

$$P(X=1) = P(E).P(\bar{E}) + P(\bar{E}).P(E) = \frac{1}{2}.\frac{1}{2} + \frac{1}{2}.\frac{1}{2} = \frac{1}{4} + \frac{1}{4} = \frac{2}{4}$$

$$P(X=2) = P(E).P(E) = \frac{1}{2}.\frac{1}{2} = \frac{1}{4}$$

$\therefore$ Probability distribution table is

X	0	1	2
P(X)	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$

$$E(X) = 0 \times \frac{1}{4} + 1 \times \frac{2}{4} + 2 \times \frac{1}{4} = \frac{2}{4} + \frac{2}{4} = 1$$

$$E(X^2) = 0 \times \frac{1}{4} + 1 \times \frac{2}{4} + 4 \times \frac{1}{4} = \frac{3}{2}$$

$\therefore$ Variance (X) $= E(X^2) - [E(X)]^2 = \frac{3}{2} - 1 = \frac{1}{2} = 0.5$

Q55. There are 5 cards numbered 1 to 5, one number on one card. Two cards are drawn at random without replacement. Let X denotes the sum of the numbers on two cards drawn. Find the mean and variance of X.

Sol. Here, sample space S = {(1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2), (1, 4), (4, 1), (1, 5), (5, 1), (2, 4), (4, 2), (2, 5), (5, 2), (3, 4), (4, 3), (3, 5), (5, 3), (5, 4), (4, 5)}

$\therefore\quad n(S) = 20$

Let X be the random variable denoting the sum of the numbers on two cards drawn.

$\therefore\qquad X = 3, 4, 5, 6, 7, 8, 9$

So, $P(X=3) = \frac{2}{20}$ $\qquad P(X=4) = \frac{2}{20}$

$P(X=5) = \frac{4}{20}$ $\qquad P(X=6) = \frac{4}{20}$

$P(X=7) = \frac{4}{20}$ $\qquad P(X=8) = \frac{2}{20}$

$$P(X=9) = \frac{2}{20}$$

$$\therefore \text{ Mean, } E(X) = 3\times\frac{2}{20}+4\times\frac{2}{20}+5\times\frac{4}{20}+6\times\frac{4}{20}+7\times\frac{4}{20}+8\times\frac{2}{20}+9\times\frac{2}{20}$$

$$= \frac{6}{20}+\frac{8}{20}+\frac{20}{20}+\frac{24}{20}+\frac{28}{20}+\frac{16}{20}+\frac{18}{20} = \frac{120}{20} = 6$$

$$E(X^2) = 9\times\frac{2}{20}+16\times\frac{2}{20}+25\times\frac{4}{20}+36\times\frac{4}{20}+49\times\frac{4}{20}+64\times\frac{2}{20}+81\times\frac{2}{20}$$

$$= \frac{18}{20}+\frac{32}{20}+\frac{100}{20}+\frac{144}{20}+\frac{196}{20}+\frac{128}{20}+\frac{162}{20} = \frac{780}{20} = 39$$

$$\therefore \text{Variance } (X) = E(X^2)-[E(X)]^2 = 39-(6)^2 = 39-36 = 3$$

OBJECTIVE TYPE QUESTIONS

Choose the correct answer from the given four options in each of the Exercises from Q. 56 to 93.

Q56. If $P(A) = \frac{4}{5}$ and $P(A\cap B) = \frac{7}{10}$, then $P(B/A)$ is equal to

(*a*) $\frac{1}{10}$ (*b*) $\frac{1}{8}$ (*c*) $\frac{7}{8}$ (*d*) $\frac{17}{20}$

Sol. Given that: $P(A) = \frac{4}{5}$ and $P(A\cap B) = \frac{7}{10}$

$$\therefore \quad P(B/A) = \frac{P(A\cap B)}{P(A)} = \frac{7/10}{4/5} = \frac{7}{8}$$

Hence, the correct option is (*c*).

Q57. If $P(A\cap B) = \frac{7}{10}$ and $P(B) = \frac{17}{20}$, then $P(A/B)$ equals

(*a*) $\frac{14}{17}$ (*b*) $\frac{17}{20}$ (*c*) $\frac{7}{8}$ (*d*) $\frac{1}{8}$

Sol. Given that: $P(A\cap B) = \frac{7}{10}$ and $P(B) = \frac{17}{20}$

$$\therefore \quad P(A/B) = \frac{P(A\cap B)}{P(B)} = \frac{7/10}{17/20} = \frac{14}{17}$$

Hence, the correct option is (*a*).

Q58. If $P(A) = \frac{3}{10}$, $P(B) = \frac{2}{5}$ and $P(A\cup B) = \frac{3}{5}$, then $P(B/A)+P(A/B)$ equals to (*a*) $\frac{1}{4}$ (*b*) $\frac{1}{3}$ (*c*) $\frac{5}{12}$ (*d*) $\frac{7}{12}$

Sol. Here, $P(A) = \frac{3}{10}$, $P(B) = \frac{2}{5}$ and $P(A \cup B) = \frac{3}{5}$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\Rightarrow \quad \frac{3}{5} = \frac{3}{10} + \frac{2}{5} - P(A \cap B)$$

$$\Rightarrow \quad P(A \cap B) = \frac{3}{10} + \frac{2}{5} - \frac{3}{5} = \frac{3+4-6}{10} = \frac{1}{10}$$

$$\text{Now } P(A/B) + P(B/A) = \frac{P(A \cap B)}{P(B)} + \frac{P(A \cap B)}{P(A)}$$

$$= \frac{1/10}{2/5} + \frac{1/10}{3/10} = \frac{1}{4} + \frac{1}{3} = \frac{7}{12}$$

Hence, the correct option is (*d*).

Q59. If $P(A) = \frac{2}{5}$, $P(B) = \frac{3}{10}$ and $P(A \cap B) = \frac{1}{5}$, then $P(A'/B').\,P(B'/A')$ is equal to

(*a*) $\frac{5}{6}$ (*b*) $\frac{5}{7}$ (*c*) $\frac{25}{42}$ (*d*) 1

Sol. Given that: $P(A) = \frac{2}{5}$, $P(B) = \frac{3}{10}$ and $P(A \cap B) = \frac{1}{5}$

$$P(A') = 1 - \frac{2}{5} = \frac{3}{5}, \; P(B') = 1 - \frac{3}{10} = \frac{7}{10}$$

$$\text{and } P(A' \cap B') = 1 - P(A \cup B) = 1 - [P(A) + P(B) - P(A \cap B)]$$

$$= 1 - \left[\frac{2}{5} + \frac{3}{10} - \frac{1}{5}\right] = 1 - \left[\frac{1}{5} + \frac{3}{10}\right] = 1 - \frac{5}{10} = \frac{1}{2}$$

$$\therefore \quad P(A'/B') = \frac{P(A' \cap B')}{P(B')} = \frac{1/2}{7/10} = \frac{5}{7}$$

$$\text{and } \quad P(B'/A') = \frac{P(A' \cap B')}{P(A')} = \frac{1/2}{3/5} = \frac{5}{6}$$

$$\therefore P(A'/B').P(B'/A') = \frac{5}{7} \times \frac{5}{6} = \frac{25}{42}$$

Hence, the correct option is (*c*).

Q60. If A and B are two events such that $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$ and $P(A/B) = \frac{1}{4}$, then $P(A' \cap B')$ equals

(a) $\frac{1}{12}$ (b) $\frac{3}{4}$ (c) $\frac{1}{4}$ (d) $\frac{3}{16}$

Sol. Given that: $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$ and $P(A/B) = \frac{1}{4}$

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$\frac{1}{4} = \frac{P(A \cap B)}{1/3} \Rightarrow P(A \cap B) = \frac{1}{4} \times \frac{1}{3} = \frac{1}{12}$$

Now $\quad P(A' \cap B') = 1 - P(A \cup B)$

$$= 1 - [P(A) + P(B) - P(A \cap B)]$$

$$= 1 - \left[\frac{1}{2} + \frac{1}{3} - \frac{1}{12}\right] = 1 - \left[\frac{5}{6} - \frac{1}{12}\right] = 1 - \frac{9}{12} = \frac{3}{12} = \frac{1}{4}$$

Hence, the correct option is (c).

Q61. If $P(A) = 0.4$, $P(B) = 0.8$ and $P(B/A) = 0.6$ then $P(A \cup B)$ is equal to

(a) 0.24 (b) 0.3 (c) 0.48 (d) 0.96

Sol. Given that: $P(A) = 0.4$, $P(B) = 0.8$ and $P(B/A) = 0.6$

$$P(B/A) = \frac{P(A \cap B)}{P(A)} \Rightarrow 0.6 = \frac{P(A \cap B)}{0.4}$$

$\therefore \quad P(A \cap B) = 0.6 \times 0.4 = 0.24$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= 0.4 + 0.8 - 0.24 = 1.20 - 0.24 = 0.96$$

Hence, the correct option is (d).

Q62. If A and B are two events and $A \neq \phi$, $B \neq \phi$, then

(a) $P(A/B) = P(A).P(B)$ (b) $P(A/B) = \frac{P(A \cap B)}{P(B)}$

(c) $P(A/B).P(B/A) = 1$ (d) $P(A/B) = \frac{P(A)}{P(B)}$

Sol. Given that: $A = \phi$ and $B \neq \phi$,

then $\quad P(A/B) = \frac{P(A \cap B)}{P(B)}$

Hence, the correct option is (b).

Q63. A and B are events such that $P(A) = 0.4$, $P(B) = 0.3$ and $P(A \cup B) = 0.5$. Then $P(B' \cap A)$ equals

(a) $\frac{2}{3}$ (b) $\frac{1}{2}$ (c) $\frac{3}{10}$ (d) $\frac{1}{5}$

Sol. Given that: $P(A) = 0.4$, $P(B) = 0.3$ and $P(A \cup B) = 0.5$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$0.5 = 0.4 + 0.3 - P(A \cap B)$$
$$P(A \cap B) = 0.4 + 0.3 - 0.5 = 0.2$$
$$\therefore \quad P(B' \cap A) = P(A) - P(A \cap B)$$
$$= 0.4 - 0.2 = 0.2 = \frac{1}{5}$$

Hence, the correct option is (*d*).

Q64. If A and B are two events such that $P(B) = \frac{3}{5}$, $P(A/B) = \frac{1}{2}$ and $P(A \cup B) = \frac{4}{5}$, then P(A) equals

(*a*) $\frac{3}{10}$ (*b*) $\frac{1}{5}$ (*c*) $\frac{1}{2}$ (*d*) $\frac{3}{5}$

Sol. Given that: $P(B) = \frac{3}{5}$, $P(A/B) = \frac{1}{2}$ and $P(A \cup B) = \frac{4}{5}$

We know that $P(A/B) = \frac{P(A \cap B)}{P(B)} \Rightarrow \frac{1}{2} = \frac{P(A \cap B)}{3/5}$

$$\therefore \quad P(A \cap B) = \frac{3}{10}$$

Now $\quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$\frac{4}{5} = P(A) + \frac{3}{5} - \frac{3}{10}$$

$$\Rightarrow \quad P(A) = \frac{4}{5} - \frac{3}{5} + \frac{3}{10} = \frac{1}{5} + \frac{3}{10} = \frac{5}{10} = \frac{1}{2}$$

Hence, the correct option is (*c*).

Q65. In Exercise 64 above, $P(B/A')$ is equal to

(*a*) $\frac{1}{5}$ (*b*) $\frac{3}{10}$ (*c*) $\frac{1}{2}$ (*d*) $\frac{3}{5}$

Sol. According to Exercise 64, we have

$P(B) = \frac{3}{5}$, $P(A/B) = \frac{1}{2}$, $P(A \cup B) = \frac{4}{5}$

$$P(B/A') = \frac{P(B \cap A')}{P(A')} = \frac{P(B) - P(A \cap B)}{1 - P(A)} = \frac{\frac{3}{5} - \frac{3}{10}}{1 - \frac{1}{2}} = \frac{\frac{3}{10}}{\frac{1}{2}} = \frac{3}{5}$$

Hence, the correct option is (*d*).

Q66. If $P(B) = \frac{3}{5}$, $P(A/B) = \frac{1}{2}$ and $P(A \cup B) = \frac{4}{5}$, then

$P(A \cup B)' + P(A' \cup B) =$

(*a*) $\frac{1}{5}$ (*b*) $\frac{4}{5}$ (*c*) $\frac{1}{2}$ (*d*) 1

Sol. Given that: $P(B) = \frac{3}{5}$, $P(A/B) = \frac{1}{2}$ and $P(A \cup B) = \frac{4}{5}$

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$\Rightarrow \quad \frac{1}{2} = \frac{P(A \cap B)}{3/5} \Rightarrow P(A \cap B) = \frac{3}{10}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\frac{4}{5} = P(A) + \frac{3}{5} - \frac{3}{10}$$

$$\therefore \quad P(A) = \frac{4}{5} - \frac{3}{5} + \frac{3}{10} = \frac{1}{5} + \frac{3}{10} = \frac{5}{10} = \frac{1}{2}$$

Now $P(A \cup B)' + P(A' \cup B)$

$$= 1 - P(A \cup B) + 1 - P(A \cap B')$$

$$= 2 - \frac{4}{5} - P(A).P(B')$$

$$= \frac{6}{5} - \frac{1}{2}.\left(1 - \frac{3}{5}\right) = \frac{6}{5} - \frac{1}{2} \times \frac{2}{5} = \frac{6}{5} - \frac{1}{5} = \frac{5}{5} = 1$$

Hence, the correct option is (*d*).

Q67. Let $P(A) = \frac{7}{13}$, $P(B) = \frac{9}{13}$ and $P(A \cap B) = \frac{4}{13}$. Then $P(A'/B)$ is equal to

(*a*) $\frac{6}{13}$ (*b*) $\frac{4}{13}$ (*c*) $\frac{4}{9}$ (*d*) $\frac{5}{9}$

Sol. Given that: $P(A) = \frac{7}{13}$, $P(B) = \frac{9}{13}$ and $P(A \cap B) = \frac{4}{13}$

$$P(A'/B) = \frac{P(A' \cap B)}{P(B)} = \frac{P(B) - P(A \cap B)}{P(B)} = \frac{\frac{9}{13} - \frac{4}{13}}{\frac{9}{13}} = \frac{\frac{5}{13}}{\frac{9}{13}} = \frac{5}{9}$$

Hence, the correct option is (*d*).

Q68. If A and B are such events that $P(A) > 0$ and $P(B) \neq 1$ then $P(A'/B')$ equals

(*a*) $1 - P(A/B)$ (*b*) $1 - P(A'/B)$

(*c*) $\frac{1 - P(A \cup B)}{P(B')}$ (*d*) $\frac{P(A')}{P(B')}$

Sol. Given that: $P(A) > 0$ and $P(B) \neq 1$

$$\therefore \quad P(A'/B') = \frac{P(A' \cap B')}{P(B')} = \frac{1 - P(A \cup B)}{P(B')}$$

Hence, the correct option is (*c*).

Q69. If A and B are two independent events with $P(A) = \frac{3}{5}$ and $P(B) = \frac{4}{9}$, then $P(A' \cap B')$ equals

(*a*) $\frac{4}{15}$ (*b*) $\frac{8}{45}$ (*c*) $\frac{1}{3}$ (*d*) $\frac{2}{9}$

Sol. Given that: A and B are independent events

such that $P(A) = \frac{3}{5} \quad \therefore P(A') = 1 - \frac{3}{5} = \frac{2}{5}$

$P(B) = \frac{4}{9} \quad \therefore P(B') = 1 - \frac{4}{9} = \frac{5}{9}$

$$\therefore \quad P(A' \cap B') = P(A') \,.\, P(B') = \frac{2}{5} . \frac{5}{9} = \frac{2}{9}$$

Hence, the correct option is (*d*).

Q70. If two events are independent, then

(*a*) they must be mutually exclusive
(*b*) the sum of their probabilities must be equal to 1
(*c*) (*a*) and (*b*) both are correct
(*d*) none of the above is correct

Sol. For independent events A and B, $P(A).P(B) = P(A \cap B)$

So, they will not be mutually exclusive.

If $P(A) + P(B) = 1$, they are exhaustive events and for independent events A and $P(A \cap B) \neq 0$.

Hence, the correct option is (*d*).

Q71. Let A and B be two events such that $P(A) = \frac{3}{8}$, $P(B) = \frac{5}{8}$ and $P(A \cup B) = \frac{3}{4}$. Then $P(A/B).P(A'/B)$ is equal to

(*a*) $\frac{2}{5}$ (*b*) $\frac{3}{8}$ (*c*) $\frac{3}{20}$ (*d*) $\frac{6}{25}$

Sol. Given that: $P(A) = \frac{3}{8}$, $P(B) = \frac{5}{8}$ and $P(A \cup B) = \frac{3}{4}$

$$\therefore \quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\frac{3}{4} = \frac{3}{8} + \frac{5}{8} - P(A \cap B)$$

$$\Rightarrow \quad P(A \cap B) = \frac{3}{8} + \frac{5}{8} - \frac{3}{4} = \frac{1}{4}$$

$$\text{Now } P(A/B).P(A'/B) = \frac{P(A \cap B)}{P(B)}.\frac{P(A' \cap B)}{P(B)}$$

$$= \frac{P(A \cap B)}{P(B)}.\frac{P(B) - P(A \cap B)}{P(B)}$$

$$= \frac{\frac{1}{4}}{\frac{5}{8}}.\frac{\left(\frac{5}{8} - \frac{1}{4}\right)}{\frac{5}{8}} = \frac{2}{5}.\frac{3}{5} = \frac{6}{25}$$

Hence, the correct option is (*d*).

Q72. If the events A and B are independent, then $P(A \cap B)$ is equal to

(*a*) $P(A) + P(B)$ (*b*) $P(A) - P(B)$ (*c*) $P(A).P(B)$ (*d*) $\frac{P(A)}{P(B)}$

Sol. Since A and B are two independent events

$\therefore$ $P(A \cap B) = P(A).P(B)$

Hence, the correct option is (*c*).

Q73. Two events E and F are independent. If $P(E) = 0.3$ and $P(E \cup F) = 0.5$, then $P(E/F) - P(F/E)$ equals

(*a*) $\frac{2}{7}$ (*b*) $\frac{3}{35}$ (*c*) $\frac{1}{70}$ (*d*) $\frac{1}{7}$

Sol. Given that: E and F are independent events such that $P(E) = 0.3$ and $P(E \cup F) = 0.5$

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

$$0.5 = 0.3 + P(F) - P(E).P(F)$$

$\Rightarrow$ $0.5 - 0.3 = P(F)\,[1 - P(E)] \Rightarrow 0.2 = P(F)\,(1 - 0.3)$

$\Rightarrow$ $0.2 = P(F).(0.7)$

$\therefore$ $P(F) = \frac{0.2}{0.7} = \frac{2}{7}$

$$\text{Now } P(E/F) - P(F/E) = \frac{P(E \cap F)}{P(F)} - \frac{P(E \cap F)}{P(E)}$$

$$= \frac{P(E).P(F)}{P(F)} - \frac{P(E).P(F)}{P(E)} = P(E) - P(F) = \frac{3}{10} - \frac{2}{7} = \frac{1}{70}$$

Hence, the correct option is (*c*).

Q74. A bag contains 5 red and 3 blue balls. If 3 balls are drawn at random without replacement, then the probability of getting exactly one red ball is

(*a*) $\frac{45}{196}$ (*b*) $\frac{135}{392}$ (*c*) $\frac{15}{56}$ (*d*) $\frac{15}{29}$

Sol. Given that: Bag contains 5 red and 3 blue balls.

Probability of getting exactly one red ball if 3 balls are randomly drawn without replacement

$= P(R) \cdot P(B) \cdot P(B) + P(B) \cdot P(R) \cdot P(B) + P(B) \cdot P(B) \cdot P(R)$

$= \frac{5}{8} \cdot \frac{3}{7} \cdot \frac{2}{6} + \frac{3}{8} \cdot \frac{5}{7} \cdot \frac{2}{6} + \frac{3}{8} \cdot \frac{2}{7} \cdot \frac{5}{6} = \frac{30}{336} + \frac{30}{336} + \frac{.30}{336} = \frac{90}{336} = \frac{15}{56}$

Hence, the correct option is (*c*).

Q75. Refer to Question 74 above. The probability that exactly two of the three balls were red, the first ball being red is

(*a*) $\frac{1}{3}$ (*b*) $\frac{4}{7}$ (*c*) $\frac{15}{28}$ (*d*) $\frac{5}{28}$

Sol. According to Question 74,

Let E_1 be the event that first ball is red.

E_2 be the event that exactly two of the three balls are red.

$\therefore$ $P(E_1) = P(R).P(R).P(B) + P(R).P(R).P(R) + P(R).P(B).P(R) + P(R).P(B).P(B)$

$= \frac{5}{8} \cdot \frac{4}{7} \cdot \frac{3}{6} + \frac{5}{8} \cdot \frac{4}{7} \cdot \frac{3}{6} + \frac{5}{8} \cdot \frac{3}{7} \cdot \frac{4}{6} + \frac{5}{8} \cdot \frac{3}{7} \cdot \frac{2}{6}$

$= \frac{60}{336} + \frac{60}{336} + \frac{60}{336} + \frac{30}{336} = \frac{210}{336}$

$P(E_1 \cap E_2) = P(R).P(B).P(R) + P(R).P(R).P(B)$

$= \frac{5}{8} \cdot \frac{3}{7} \cdot \frac{4}{6} + \frac{5}{8} \cdot \frac{4}{7} \cdot \frac{3}{6} = \frac{60}{336} + \frac{60}{336} = \frac{120}{336}$

$\therefore P(E_2/E_1) = \frac{P(E_1 \cap E_2)}{P(E_1)} = \frac{120/336}{210/336} = \frac{4}{7}$

Hence, the correct option is (*b*).

Q76. Three persons A, B and C fire at a target in turn, starting with A. Their probabilities of hitting the target are 0.4, 0.3 and 0.2 respectively. The probabilities of two hits is

(*a*) 0.024 (*b*) 0.188 (*c*) 0.336 (*d*) 0.452

Sol. Given that: $P(A) = 0.4$, $P(B) = 0.3$ and $P(C) = 0.2$

Also $P(\bar{A}) = 1 - 0.4 = 0.6$, $P(\bar{B}) = 1 - 0.3 = 0.7$

and $P(\bar{C}) = 1 - 0.2 = 0.8$

$\therefore$ Probabilities of two hits

$= P(A).P(B).P(\bar{C}) + P(A).P(\bar{B}).P(C) + P(\bar{A}).P(B).P(C)$

$= 0.4 \times 0.3 \times 0.8 + 0.4 \times 0.7 \times 0.2 + 0.6 \times 0.3 \times 0.2$

$= 0.096 + 0.056 + 0.036 = 0.188$

Hence, the correct option is (*b*).

Q77. Assume that in a family, each child is equally likely to be a boy or a girl. A family with three children is chosen at random. The probability that the eldest child is a girl given that the family has atleast one girl is

(*a*) $\frac{1}{2}$ (*b*) $\frac{1}{3}$ (*c*) $\frac{2}{3}$ (*d*) $\frac{4}{7}$

Sol. Let G denotes the girl and B denotes the boy of the given family.

So, $n(S)$ = {(BGG), (GBG), (GGB), (GBB), (BGB), (BBG), (BBB), (GGG)}

Let E_1 be the event that the family has alteast one girl.

$\therefore$ E_1 = {(BGG), (GBG), (GGB), (GBB), (BGB), (BBG), (GGG)}

Let E_2 be the event that the eldest child is a girl.

$\therefore$ E_2 = {(GBG), (GGB), (GBB), (GGG)}

$(E_1 \cap E_2)$ = {(GBB), (GGB), (GBG), (GGG)}

$$\therefore P(E_2/E_1) = \frac{P(E_1 \cap E_2)}{P(E_1)} = \frac{4/8}{7/8} = \frac{4}{7}$$

Hence, the correct option is (*d*).

Q78. If a die is thrown and a card is selected at random from a deck of 52 playing cards. The probability of getting an even number on the die and a spade card is

(*a*) $\frac{1}{2}$ (*b*) $\frac{1}{4}$ (*c*) $\frac{1}{8}$ (*d*) $\frac{3}{4}$

Sol. Let E_1 be the event of getting even number on the die. E_2 be the event of selecting a spade card.

$$\therefore P(E_1) = \frac{3}{6} = \frac{1}{2} \text{ and } P(E_2) = \frac{13}{52} = \frac{1}{4}$$

So $\qquad P(E_1 \cap E_2) = P(E_1).P(E_2) = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$

Hence, the correction option is (c).

Q79. A box contains 3 orange balls, 3 green balls and 2 blue balls. Three balls are drawn at random from the box without replacement. The probability of drawing 2 green balls and one blue ball is

(*a*) $\frac{3}{28}$ (*b*) $\frac{2}{21}$ (*c*) $\frac{1}{28}$ (*d*) $\frac{167}{168}$

Sol. Probability of drawing 2 green and 1 blue balls

= P(G) . P(G) . P(B) + P(G) . P(B) . P(G) + P(B) . P(G) . P(G)

$$= \frac{3}{8} \cdot \frac{2}{7} \cdot \frac{2}{6} + \frac{3}{8} \cdot \frac{2}{7} \cdot \frac{2}{6} + \frac{2}{8} \cdot \frac{3}{7} \cdot \frac{2}{6} = \frac{12}{336} + \frac{12}{336} + \frac{12}{336} = \frac{36}{336} = \frac{3}{28}$$

Hence, the correct option is (*a*).

Q80. A flashlight has 8 batteries out of which 3 are dead. If two batteries are selected without replacement and tested then the probability that both are dead is

(*a*) $\frac{33}{56}$ (*b*) $\frac{9}{64}$ (*c*) $\frac{1}{14}$ (*d*) $\frac{3}{28}$

Sol. Required probability = P(dead).P(dead)

$$= \frac{3}{8}.\frac{2}{7} = \frac{3}{28}$$

Hence, the correct option is (*d*).

Q81. If eight coins are tossed together, then the probability of getting exactly 3 heads is

(*a*) $\frac{1}{256}$ (*b*) $\frac{7}{32}$ (*c*) $\frac{5}{32}$ (*d*) $\frac{3}{32}$

Sol. Here, $n = 8, p = \frac{1}{2}, q = 1 - \frac{1}{2} = \frac{1}{2}$ and $r = 3$

We know that $P(x = r) = {}^nC_r\, p^r.q^{n-r}$

$$\therefore \quad P(x = 3) = {}^8C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^{8-3}$$

$$= \frac{8!}{3!.5!}.\left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^5 = 56.\left(\frac{1}{2}\right)^8 = 56.\frac{1}{256} = \frac{7}{32}$$

Hence, the correct option is (*b*).

Q82. Two dice are thrown. If it is known that the sum of numbers on the dice was less than 6, the probability of getting a sum 3, is

(*a*) $\frac{1}{18}$ (*b*) $\frac{5}{18}$ (*c*) $\frac{1}{5}$ (*d*) $\frac{2}{5}$

Sol. Let E_1 be the event showing the sum of the numbers on the two dice was less than 6 and E_2 be the event that the sum of the numbers is 3.

$$\therefore \quad E_1 = \{(1, 1), (1, 2), (2, 1), (1, 3), (3, 1), (1, 4), (4, 1), (2, 2), (2, 3), (3, 2)\}$$

$$n(E_1) = 10$$

and $E_2 = \{(1, 2), (2, 1)\} \Rightarrow n(E_2) = 2$ and $n(E_1 \cap E_2) = 2$

$\therefore$ Required probability

$$P(E_2/E_1) = \frac{n(E_1 \cap E_2)}{n(E_1)} = \frac{2}{10} = \frac{1}{5}$$

Hence, the correct option is (*c*).

Q83. Which one is not a requirement of a Binomial distribution?

(*a*) There are two outcomes for each trial.

(b) There is a fixed number of trials.
(c) The outcomes must be dependent on each other.
(d) The probability of success must be the same for all trials.

Sol. We know that for a Binomial distribution, the outcomes must not be dependent on each other.
Hence, the correct option is (c).

Q84. If two cards are drawn from a well shuffled deck of 52 playing cards with replacement, then the probability that both cards are queens is

(a) $\frac{1}{13} \times \frac{1}{13}$ (b) $\frac{1}{13} + \frac{1}{13}$ (c) $\frac{1}{13} \times \frac{1}{17}$ (d) $\frac{1}{13} \times \frac{4}{51}$

Sol. Probability of getting Queen = $\frac{4}{52}$
So, the required probability

$$= P(\text{Queen}).P(\text{Queen})$$
$$= \frac{4}{52} \cdot \frac{4}{52} = \frac{1}{13} \cdot \frac{1}{13} \quad \text{(with replacement)}$$

Hence, the correct option is (a).

Q85. The probability of guessing correctly atleast 8 out of 10 answers on a true false type examination is

(a) $\frac{7}{64}$ (b) $\frac{7}{128}$ (c) $\frac{45}{1024}$ (d) $\frac{7}{41}$

Sol. Here, $n = 10$, $p = \frac{1}{2}$ and $q = \frac{1}{2}$ (for true/false questions)

and $r \geq 8$ i.e. 8, 9, 10

$$\therefore \; P(X \geq 8) = P(x = 8) + P(x = 9) + P(x = 10)$$
$$= {}^{10}C_8 \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^2 + {}^{10}C_9 \left(\frac{1}{2}\right)^9 \left(\frac{1}{2}\right) + {}^{10}C_{10} \left(\frac{1}{2}\right)^{10} \left(\frac{1}{2}\right)^0$$
$$= 45.\left(\frac{1}{2}\right)^{10} + 10.\left(\frac{1}{2}\right)^{10} + \left(\frac{1}{2}\right)^{10} = \left(\frac{1}{2}\right)^{10} (45 + 10 + 1)$$
$$= 56 \times \frac{1}{1024} = \frac{7}{128}$$

Hence, the correct option is (b).

Q86. The probability that a person is not a swimmer is 0.3. The probability that out of 5 persons 4 are swimmers is

(a) ${}^5C_4 (0.7)^4 (0.3)$ (b) ${}^5C_1 (0.7) (0.3)^4$
(c) ${}^5C_4 (0.7) (0.3)^4$ (d) $(0.7)^4 (0.3)$

Sol. Given that: $\overline{P} = 0.3 \;\; \therefore \; p = 0.7$ and $q = 1 - 0.7 = 0.3$

$n = 5$ and $r = 4$

We know that

$$P(X = r) = {}^nC_r\,(p)^r.(q)^{n-r}$$

$$\therefore \quad P(x = 4) = {}^5C_4\,(0.7)^4\,(0.3)^{5-4} = {}^5C_4\,(0.7)^4\,(0.3)$$

Hence, the correct option is (*a*).

Q87. The probability distribution of a discrete random variable X is given below:

X	2	3	4	5
P(X)	$\frac{5}{k}$	$\frac{7}{k}$	$\frac{9}{k}$	$\frac{11}{k}$

The value of k is

(*a*) 8 (*b*) 16

(*c*) 32 (*d*) 48

Sol. We know that $\sum_{i=1}^{n} P(X_i) = 1$

$$\therefore \quad \frac{5}{k} + \frac{7}{k} + \frac{9}{k} + \frac{11}{k} = 1$$

$$\frac{32}{k} = 1 \Rightarrow k = 32.$$

Hence, the correct option is (*c*).

Q88. For the following probability distribution:

X	−4	−3	−2	−1	0
P(X)	0.1	0.2	0.3	0.2	0.2

E(X) is equal to:

(*a*) 0 (*b*) −1

(*c*) −2 (*d*) −1.8

Sol. We know that:

$$E(X) = \sum_{i=1}^{n} X_i P_i$$

$$= (-4)(0.1) + (-3)(0.2) + (-2)(0.3) + (-1)(0.2) + 0(0.2)$$

$$= -0.4 - 0.6 - 0.6 - 0.2 = -1.8$$

Hence, the correct option is (*d*).

Q89. For the following probability distribution

X	1	2	3	4
P(X)	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{3}{10}$	$\frac{2}{5}$

$E(X^2)$ is equal to

(*a*) 3 (*b*) 5

(*c*) 7 (*d*) 10

Sol. We know that

$$E(X^2) = \sum_{i=1}^{n} P_i X_i^2$$

$$= 1 \times \frac{1}{10} + 4 \times \frac{1}{5} + 9 \times \frac{3}{10} + 16 \times \frac{2}{5}$$

$$= \frac{1}{10}+\frac{4}{5}+\frac{27}{10}+\frac{32}{5} = \frac{28}{10}+\frac{36}{5}=\frac{100}{10}=10$$

Hence, the correct option is (*d*).

Q90. Suppose a random variable X follows the Binomial distribution with parameters n and p, where $0 < p < 1$. If $\frac{P(x=r)}{P(x=n-r)}$ is independent of n and r, then p equals

(*a*) $\frac{1}{2}$ (*b*) $\frac{1}{3}$ (*c*) $\frac{1}{5}$ (*d*) $\frac{1}{7}$

Sol. $P(X=r) = {}^nC_r\, p^r q^{n-r} = \frac{n!}{r!(n-r)!} p^r \cdot (1-p)^{n-r}$

$$P(X=n-r) = {}^nC_{n-r}\, p^{n-r} \cdot (q)^{n-(n-r)} = {}^nC_{n-r}\, p^{n-r} \cdot q^r$$

$$= \frac{n!}{(n-r)!(n-n+r)!} p^{n-r} q^r = \frac{n!}{(n-r)!r!} \cdot p^{n-r} \cdot q^r$$

Now $$\frac{P(x=r)}{P(x=n-r)} = \frac{\frac{n!}{r!(n-r)!} \cdot p^r \cdot (1-p)^{n-r}}{\frac{n!}{r!(n-r)!} \cdot p^{n-r} \cdot (1-p)^r} = \frac{\left(\frac{1-p}{p}\right)^{n-r}}{\left(\frac{1-p}{p}\right)^r}$$

The above expression will be independent of n and r if

$$\left(\frac{1-p}{p}\right) = 1 \Rightarrow \frac{1}{p} = 2 \Rightarrow p = \frac{1}{2}$$

Hence, the correct option is (*a*).

Q91. In a college, 30% students fail in Physics, 25% fail in Mathematics and 10% fail in both. One student is chosen at random. The probability that she fails in Physics if she has failed in Mathematics is

(*a*) $\frac{1}{10}$ (*b*) $\frac{2}{5}$ (*c*) $\frac{9}{20}$ (*d*) $\frac{1}{3}$

Sol. Let E_1 be the event that the student fails in Physics and E_2 be the event that she fails in Mathematics.

$$\therefore \quad P(E_1) = \frac{30}{100},\ P(E_2) = \frac{25}{100} \text{ and } P(E_1 \cap E_2) = \frac{10}{100}$$

$$\therefore P(E_1/E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)} = \frac{10/100}{25/100} = \frac{2}{5}$$

Hence, the correct option is (*b*).

Q92. A and B are two students. Their chances of solving a problem correctly are $\frac{1}{3}$ and $\frac{1}{4}$ respectively. If the probability of their making a common error is $\frac{1}{20}$ and they obtain the same answer, then the probability of their answer to be correct is

(*a*) $\frac{1}{12}$ (*b*) $\frac{1}{40}$ (*c*) $\frac{13}{120}$ (*d*) $\frac{10}{13}$

Sol. Let E_1 be the event that both of them solve the problem.

$\therefore \quad P(E_1) = \frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$

and E_2 be the event that both of them same incorrectly the problem.

$\therefore \quad P(E_2) = \left(1 - \frac{1}{3}\right) \times \left(1 - \frac{1}{4}\right) = \frac{2}{3} \times \frac{3}{4} = \frac{1}{2}$

Let H be the event that both of them get the same answer.

Here, $P(H/E_1) = 1,\ P(H/E_2) = \frac{1}{20}$

$$\therefore P(E_1/H) = \frac{P(E_1).P(H/E_1)}{P(E_1).P(H/E_1) + P(E_2).P(H/E_2)}$$

$$= \frac{\frac{1}{12} \times 1}{\frac{1}{12} \times 1 + \frac{1}{2} \times \frac{1}{20}} = \frac{\frac{1}{12}}{\frac{1}{12} + \frac{1}{40}} = \frac{\frac{1}{12}}{\frac{10+3}{120}} = \frac{1/12}{13/120} = \frac{10}{13}$$

Hence, the correct option is (*d*).

Q93. A box has 100 pens of which 10 are defective. What is the probability that out of a sample of 5 pens drawn one by one with replacement atmost one is defective?

(*a*) $\left(\frac{9}{10}\right)^5$ (*b*) $\frac{1}{2}\left(\frac{9}{10}\right)^4$

(*c*) $\frac{1}{2}\left(\frac{9}{10}\right)^5$ (*d*) $\left(\frac{9}{10}\right)^5 + \frac{1}{2}\left(\frac{9}{10}\right)^4$

Sol. Here, $n = 5,\ p = \frac{10}{100} = \frac{1}{10}$ and $q = 1 - \frac{1}{10} = \frac{9}{10}$ and $r \leq 1$

We know that

$$P(X = r) = {}^nC_r\, p^r q^{n-r}$$

So $\quad P(x \leq 1) = P(x = 0) + P(x = 1)$

$$= {}^5C_0 \left(\frac{1}{10}\right)^0 \left(\frac{9}{10}\right)^5 + {}^5C_1 \left(\frac{1}{10}\right)\left(\frac{9}{10}\right)^4$$

$$= \left(\frac{9}{10}\right)^5 + \frac{5}{10}\left(\frac{9}{10}\right)^4 = \left(\frac{9}{10}\right)^5 + \frac{1}{2}\left(\frac{9}{10}\right)^4$$

Hence, the correct option is (*d*).

State True or False for the statements in each of the Exercises 94 to 103.

Q94. Let P(A) > 0 and P(B) > 0. Then A and B can be both mutually exclusive and independent.

Sol. False

Q95. If A and B are independent events, then A′ and B′ are also independent.

Sol. True

Q96. If A and B are mutually exclusive events, then they will be independent also.

Sol. False

Q97. Two independent events are always mutually exclusive.

Sol. False

Q98. If A and B are two independent events, then P(A and B) = P(A).P(B).

Sol. True

Q99. Another name for the mean of a probability distribution is expected value.

Sol. True $\left[\because \; E(X) = \sum X_i P(X_i)\right]$

Q100. If A and B′ are independent events, then $P(A' \cup B) = 1 - P(A).P(B')$

Sol. True $[\because \; P(A' \cup B) = 1 - P(A \cap B') = 1 - P(A).P(B')]$

Q101. If A and B are independent events, then P (exactly one of A, B occurs) = P(A).P(B′) + P(B).P(A′)

Sol. True

Q102. If A and B are two events such that P(A) > 0 and P(A) + P(B) > 1, then $P(B/A) \geq 1 - \frac{P(B')}{P(A)}$

Sol. False $\left[\because \; P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A) + P(B) - P(A \cup B)}{P(A)} > \frac{1 - P(A \cup B)}{P(A)}\right]$

Q103. If A, B and C are three independent events such that P(A) = P(B) = P(C) = p, then P (atleast two of A, B and C occur) = $3p^2 - 2p^3$

Sol. True

Since P (atleast two of A, B and C occur)

$$= p \times p \times (1-p) + (1-p).p.p + p(1-p).p + p.p.p$$
$$= 3p^2(1-p) + p^3 = 3p^2 - 3p^3 + p^3 = 3p^2 - 2p^3$$

Fill in the blanks in each of the following questions.

Q104. If A and B are two events such that $P(A/B) = p$, $P(A) = p$, $P(B) = \frac{1}{3}$ and $P(A \cup B) = \frac{5}{9}$, then p is equal to ....

Sol. Given that: $P(A) = p$, $P(B) = \frac{1}{3}$ and $P(A \cup B) = \frac{5}{9}$

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = p \Rightarrow P(A \cap B) = p.P(B) = p.\frac{1}{3}$$

and $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$\frac{5}{9} = p + \frac{1}{3} - \frac{p}{3} \Rightarrow \frac{5}{9} - \frac{1}{3} = \frac{2p}{3} \Rightarrow \frac{2}{9} = \frac{2p}{3} \quad \Rightarrow p = \frac{1}{3}$$

Hence, p is equal to $\frac{1}{3}$.

Q105. If A and B are such that $P(A' \cup B') = \frac{2}{3}$ and $P(A \cup B) = \frac{5}{9}$, then $P(A') + P(B')$ is equal to ....

Sol. Here, $P(A' \cup B') = \frac{2}{3}$ and $P(A \cup B) = \frac{5}{9}$

$\therefore \quad 1 - P(A \cap B) = \frac{2}{3}$

$\Rightarrow \quad P(A \cap B) = 1 - \frac{2}{3} = \frac{1}{3}$

Now $\quad P(A') + P(B') = 1 - P(A) + 1 - P(B) = 2 - [P(A) + P(B)]$

$$= 2 - [P(A \cup B) + P(A \cap B)]$$
$$= 2 - \left[\frac{5}{9} + \frac{1}{3}\right] = 2 - \frac{8}{9} = \frac{10}{9}$$

Hence, the value of the filler is $\frac{10}{9}$.

Q106. If X follows Binomial distribution with parameters $n = 5$, p and $P(X = 2) = 9P(X = 3)$, then p is equal to ....

Sol. Given that: $P(X = 2) = 9P(X = 3)$

$\Rightarrow \quad {}^5C_2\, p^2 q^3 = 9.\, {}^5C_3\, p^3 q^2$

$$\Rightarrow \quad \frac{1}{9} = \frac{{}^5C_3 p^3 q^2}{{}^5C_2 p^2 q^3} \Rightarrow \frac{1}{9} = \frac{p}{q} \qquad [\because {}^5C_3 = {}^5C_2]$$

$$\Rightarrow \quad 9p = q$$

$$\Rightarrow \quad 9p = 1 - p \Rightarrow 9p + p = 1 \Rightarrow 10p = 1 \therefore p = \frac{1}{10}$$

Hence, the value of the filler is $\frac{1}{10}$.

Q107. Let X be a random variable taking values $x_1, x_2, x_3, \dots, x_n$ with probabilities $p_1, p_2, p_3, \dots, p_n$ respectively. Then Var(X) is equal to ....

Sol. $\text{Var}(X) = E(X^2) - [E(X)]^2$

$$= \sum X^2 P(X) - \left[\sum X.P(X)\right]^2 = \sum p_i x_i^2 - \left(\sum p_i x_i\right)^2$$

Hence, Var (X) is equal to $\sum p_i x_i^2 - \left(\sum p_i x_i\right)^2$.

Q108. Let A and B be two events. If $P(A/B) = P(A)$, then A is .... of B.

Sol. $\because \quad P(A/B) = \frac{P(A \cap B)}{P(B)}$

$$\Rightarrow \quad P(A) = \frac{P(A \cap B)}{P(B)}$$

$$\Rightarrow \quad P(A \cap B) = P(A).P(B)$$

So, A is **independent** of B.

❑❑❑